Charged-particle kinetics in a stochastic medium with long-wavelength fluctuations

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We consider the effects of the spatial transport and the change of energy of charged particles interacting with MHD-type large-scale electromagnetic fluctuations. By averaging the transport equation we evaluate the coefficients for the particle diffusion in coordinate and in momentum space. We pay special attention to the strong turbulence case when it is necessary to renormalize the kinetic coefficients obtained from perturbation theory. We propose a method for calculating the renormalized kinetic coefficients which is similar to the self-consistent field method. We calculate the kinetic coefficients by solving transcendental equations. The calculations were carried out for various kinds of turbulent spectra. The method does not require a Gaussian distribution for the probabilities of the random quantities, but assumes that only spectral harmonics with neighboring wave vectors are correlated. We show that the Fourier transforms of the kernels of the integral operators satisfy transcendental equations and can be calculated when the turbulent spectra are given.

1. INTRODUCTION

The problems of turbulent transport of a passive impurity, either of a scalar type (particles) or of a vector type (electromagnetic fields) often crop up under laboratory or natural conditions. As an example we mention just two cases: the anomalous particle transport in tokamaks and other laboratory plasma installations,[1,2] and also the problems of the generation and transport of magnetic fields in galaxies.[3] Various theoretical approaches to the problem have been developed.[4–6] A method was proposed in Ref. 4 for calculating the diffusion coefficient of a passive impurity in a turbulent flow with a Gaussian ensemble of realizations. The method is based on a diagram technique with expansion in the amplitude of the turbulent velocity pulsations of the medium. Particle transport was recently simulated by a single-scale velocity experiment in computer experiments.[7] Their results agree quite well with a calculation method[4] for isotropic turbulence. Interaction of charged particles with MHD turbulence leads, in addition to particle transport in space, also to a change of the particle energy. In a number of cases this effect reduces to particle diffusion in momentum space; this raises the question of calculating the corresponding diffusion coefficient. This problem was solved by perturbation theory in Refs. 8–11. We have previously[9] considered particle acceleration by MHD fluctuations having scales L exceeding the particle mean free path Λ with respect to scattering by small-scale fluctuations of a magnetic field (or of Coulomb collisions). The acceleration effectiveness depends substantially, in particular, on the parameter $\beta = uL_0/\varkappa$, where u is the amplitude of the velocity-fluctuation spectrum, L_0 is the main scale of the turbulence, and $\varkappa = v\Lambda/3$ is the particle diffusion coefficient governed by the small-scale field fluctuations. If $\beta \ll 1$, it is convenient to solve the problem by perturbation theory (details in Sec. 2). In the case $\beta \gg 1$ it is possible to obtain expressions in closed form for the kinetic coefficients[9] if the random velocity field is delta-correlated in time. Actually, however MHD turbulence is frequently not delta-correlated, and the condition $\beta \gg 1$ is met, for example, for charged particles of moderate energy ($\varepsilon \lesssim 10$ GeV). For diffusion of passive particles in the atmosphere and in the ocean,the condition $\beta \gg 1$ is as a rule met with sufficient margin (in which case $\varkappa$ must be taken to mean the diffusion coefficient governed by particle collisions. Bearing these circumstances in mind, we pay principal attention in this paper to the case $\beta > 1$.

We consider here the evolution of a system of charged particles (passive impurities) in a turbulent plasma with large-scale ($L > \Lambda$) velocity fluctuations. The particle distribution function $N(r, p, t)$ averaged over the small-scale weak fluctuations of the magnetic field, satisfies the transport equation[12]

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial r_\alpha}\varkappa_{\alpha\beta}\frac{\partial N}{\partial r_\beta} - u_\alpha\frac{\partial N}{\partial r_\alpha} + \frac{p}{3}\frac{\partial N}{\partial p}\frac{\partial u_\alpha}{\partial r_\alpha}. \tag{1}$$

Here $u_\alpha(\mathbf{r}, t)$ is the random large-scale velocity field, which we assume to be statistically homogeneous and isotropic, and which we describe by the tensor correlation function

$$K_{\alpha\beta}(\rho, \tau) = \langle u_\alpha(\mathbf{r}, t)u_\beta(\mathbf{r}', t')\rangle = \int \tilde{K}_{\alpha\beta}(k, \omega)e^{i(\mathbf{k}\rho-\omega t)}\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}, \tag{2}$$

where

$$\tilde{K}_{\alpha\beta}(\mathbf{k}, \omega) = T(\mathbf{k}, \omega)(\delta_{\alpha\beta} - k_\alpha k_\beta/k^2) + S(\mathbf{k}, \omega)k_\alpha k_\beta/k^2,$$
$$\rho = \mathbf{r} - \mathbf{r}', \quad \tau = t - t'. \tag{3}$$

Equation (1) describes the transport of charged particles in space and the adiabatic changes introduced into their energy by the compressibility of the medium ($\operatorname{div}\mathbf{u} \neq 0$). As to the change of the particle energy in an incompressible medium with $\operatorname{div}\mathbf{u} = 0$, the contribution of such motion to the acceleration is smaller by a factor $(\Lambda/L)^2 \ll 1$ than the contribution of the compressibility,[13] and is disregarded in Eq. (1).

We average next Eq. (1) over a random field of long-wave velocity fluctuations. As already mentioned, the averaging result depends substantially on the parameter β. In Sec. 2 we analyze the dependence of the rate of charge-parti-

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cle acceleration on the diffusion coefficient $\varkappa$ and on the correlation time $\tau_c(k)$ of the spectral harmonics of the velocity field. We show, in particular, that at a finite correlation time τ_c particle acceleration occurs also in the case of an arbitrarily small diffusion coefficient $\varkappa$. If the time $\tau_c \to \infty$, particles are accelerated only in systems with a nonzero coefficient $\varkappa$, in agreement with the result of Ref. 10.

Perturbation theory cannot be used for systems with strong turbulence if $\beta > 1$. We calculate in this case the renormalized diffusion coefficients in real (χ) and momentum (D) spaces, by a method similar to the self-consistent field method. We note here that renormalization of particle diffusion coefficients in a plasma is used quite extensively.[14,15] The idea of the method developed in Sec. 3 for renormalizing the kinetic coefficients reduces to the following: We select arbitrarily a small section, of width Δk, of the random-velocities spectrum and calculate the contributions Δ_χ and ΔD from the harmonics contained in this spectrum section. Since Δk is a macroscopically small quantity, the contributions Δ_χ and ΔD can be calculated exactly. The entire random-velocity spectrum is accounted for in the corresponding Green's function by the complete diffusion coefficients that are assumed at this stage to be unknown parameters. The integration over the entire spectrum of $\Delta\chi$ and ΔD yields closed algebraic transcendental equations for the coefficients χ and D. We present the results of numerical computations of the transcendental equations for various turbulence spectra.

2. TRANSPORT AND ACCELERATION OF CHARGED PARTICLES BY WEAK LONG-WAVELENGTH TURBULENCE

The problem of describing the interaction of fast particles with weak MHD turbulence can be solved by perturbation theory methods. In particular, it is also convenient to use in this case the quasilinear approximation which is widely applied in plasma theory.[16] We write the fast-particle distribution function in the form

$$N(\mathbf{r}, p, t) = F(\mathbf{r}, p, t) + \delta F(\mathbf{r}, p, t) \tag{4}$$

where $F = \langle N \rangle$; using (4) to average Eq. (1) over the ensemble of random velocities in the medium we get in the first non-vanishing approximation an equation for F (Refs. 9 and 10)

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}\frac{\partial F}{\partial r_\beta} + \frac{1}{p^2}\frac{\partial}{\partial p}D(p)p^2\frac{\partial F}{\partial p}. \tag{5}$$

The coefficients for the diffusion in coordinate and momentum space have, respectively, the form

$$\chi_{\alpha\beta} = \varkappa\delta_{\alpha\beta} - \int d\rho \int_0^\infty d\tau\, \rho_\alpha K_{\beta\gamma}(\rho, \tau)\frac{\partial G}{\partial \rho_\gamma}, \tag{6}$$

and

$$D(p) = -\frac{p^2}{9}\int d\rho \int_0^\infty d\tau \frac{\partial^2 K_{\alpha\beta}}{\partial\rho_\alpha\,\partial\rho_\beta}G. \tag{7}$$

The correlation functions $K_{\alpha\beta}$ in Eqs. (6) and (7) are defined by Eqs. (2) and (3), and $G(\rho,\tau,p)$ is the Green function of the diffusion equation with the small-scale diffusion coefficient $\varkappa(p)$ which is given and is not averaged here. The condition for the applicability of the approximate relations (5)–(7) for hydrodynamical kind of turbulence (phase velocities of the order of the hydrodynamic velocity of the various Fourier components) is given by the inequality $\beta \ll 1$.[9] However, according to Ptuskin's results,[10] Eqs. (6), (7) are also applicable for turbulence which is a superposition of modes with phase velocities v_0 satisfying the condition

$$u \ll v_0 \tag{8}$$

(weak acoustic or magnetosonic turbulence). The latter inequality does not contain $\varkappa$ and hence, when (8) is satisfied, one can use (7) to study the acceleration effect also in systems with an arbitrarily small diffusion coefficient $\varkappa \to 0$. To analyze this case we rewrite the coefficients (6) and (7) in terms of the Fourier transforms of the correlators and as a result we get

$$\chi_{\alpha\beta} = \chi\delta_{\alpha\beta},$$

where

$$\chi = \varkappa + \frac{1}{3}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\left[\frac{2T+S}{i\omega+\varkappa k^2} - \frac{2\varkappa k^2 S}{(i\omega+\varkappa k^2)^2}\right], \tag{9}$$

$$D(p) = \frac{p^2}{9}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\frac{k^2 S(k,\omega)}{i\omega+\varkappa k^2}. \tag{10}$$

It follows from Eq. (9) that the correction to the spatial diffusion coefficient is caused by comparable contributions from the rotational and the potential components of the velocity field. At the same time, in the conditions considered the acceleration is, according to (10), caused exclusively by the potential component of the field. We specify the form of the spectral function $S(k,\omega)$, introducing the correlation time of the harmonics $\tau_c(k) = \Gamma_k^{-1}$ in explicit form:

$$S(k,\omega) = S(k)\frac{\Gamma_k}{4\pi}\left[\frac{1}{(\omega-\omega_0)^2+\Gamma_k^2/4} + \frac{1}{(\omega+\omega_0)^2+\Gamma_k^2/4}\right]. \tag{11}$$

We have chosen here the Lorentz or dispersion form for the frequency dependence of the spectral function in view of its universality and simplicity. Using the spectrum (11) and integrating by parts in (10) we find

$$D(p) = \frac{p^2}{9}\int d\mathbf{k}\, k^2 S(k)\frac{\varkappa k^2+\Gamma_k/2}{\omega_0^2+(\varkappa k^2+\Gamma_k/2)^2}. \tag{12}$$

Expression (12) shows that D depends significantly both on the correlation time (or the resonance broadening Γ_k) and on the coefficient $\varkappa$. When $\Gamma_k \to 0$ and $\omega_0 = kv_0$ we get from (12) the result of Ref. 10:

$$D(p) = \frac{p^2}{9}\int d\mathbf{k}\, k^2 S(k)\frac{\varkappa}{v_0^2+\varkappa^2 k^2}. \tag{13}$$

In the case of weak turbulence, which is a superposition of monochromatic waves with random phases, the acceleration effect vanishes in the limit as $\varkappa \to 0$. For (13) to be applicable in the limit as $\varkappa \to 0$, the condition

$$\langle u_{pot}^2\rangle = 4\pi\int_0^\infty k^2 S(k)\,dk \ll v_0^2 \tag{14}$$

must be satisfied.

If the resonance broadening Γ_k is finite, particle acceleration occurs also in a system with very strong particle scattering ($\varkappa \to 0$):

$$D(p)=\frac{p^2}{9}\int d\mathbf{k}\,k^2S(k)\frac{\Gamma_k/2}{\omega_0{}^2+\Gamma_k{}^2/4}. \tag{15}$$

Qualitative conclusions about the role of diffusion and of the finite correlation time of the turbulence are general in character and are not connected with a specific form of the function $S(k,\omega)$. For instance, we choose instead of (11) a Gaussian form for the frequency dependence of the spectrum:

$$\frac{S(k,\omega)}{(2\pi)^4}=S(k)\frac{\tau_c(k)}{2\pi^{1/2}}[\exp(-\tau_c{}^2(\omega-\omega_0)^2)$$
$$+\exp(-\tau_c{}^2(\omega+\omega_0)^2)]. \tag{16}$$

In the case of weak turbulence $\omega_0\tau_c \gg 1$ we can use for all k which are important in the integration the approximation

$$(\tau_c/\pi^{1/2})\exp[-\tau_c{}^2(\omega\pm\omega_0)^2]\to\delta(\omega\pm\omega_0).$$

This leads to the result (13) with two different acceleration regimes:

$$D(p)\approx\frac{p^2}{9}\begin{cases}\dfrac{\langle u_{pot}^2\rangle}{\varkappa}, & \varkappa k_0\gg v_0,\\[2ex] \dfrac{\varkappa}{v_0{}^2}\displaystyle\int S(k)\,k^2\,d\mathbf{k}, & \varkappa k_{\max}\ll v_0.\end{cases} \tag{17}$$

Here $2\pi/k_0=L_0$ is the main turbulence scale and $2\pi/k_{\max}$ the internal scale at which the spectrum is cut off.

If, however, the correlation time τ_c is finite and the scattering strong we can use in (10) the approximation

$$\frac{1}{i\omega+\varkappa k^2}\to\pi\delta(\omega)-i\,\mathrm{VP}\frac{1}{\omega}.$$

In that limit we get the result

$$D(p)=\frac{p^2\pi^{1/2}}{9}\int d\mathbf{k}\,S(k)\tau_c(k)\exp(-\omega_0{}^2\tau_c{}^2), \tag{17$'$}$$

which is similar to Eq. (15).

The presence of the acceleration effect in the strong-scattering limit ($\varkappa\to 0$) with a finite correlation time τ_c is caused by the presence in that case of a stochastic velocity field which is necessary for Fermi acceleration. As $\tau_c\to\infty$ the velocity field consists of standard modes in which there are no "frozen-in" particles ($\varkappa\to 0$), and thus no acceleration.

The value of the product $\omega_0\tau_c$ may be determined by the nonlinear interaction of the harmonics as is usually assumed in turbulence theory. This quantity then depends on the strength of the turbulence and in the case of weak turbulence $\omega_0\tau_c\gg 1$. Strong turbulence corresponds to $\omega_0(k)\tau_c(k)\sim 1$. In that case the function $\tau_c(k)$ has usually a power-law form, $\tau_c\propto k^{(\nu-3)/2}$, where ν is the exponent of the turbulence spectrum (in Kolomogorov turbulence $\nu=\frac{5}{3}$), in the inertial range of scales. Outside the inertial range, or in systems where it does not exist at all, the quantity $\omega_0\tau_c$ may not be connected with the amplitude of the spectrum but is determined by the properties of the source and the dissipative characteristics of the medium.

One should note that when we have strong turbulence the criteria of the applicability of perturbation theory which we have indicated in this section may be violated, since the effective phase velocity $v_0\approx u$. In the next section we therefore evaluate the kinetic coefficients without using the conditions $\beta\ll 1$ or $u\ll v_0$.

3. RENORMALIZATION OF THE KINETIC COEFFICIENTS FOR A SYSTEM WITH STRONG TURBULENCE

We obtained in Sec. 2 in the perturbation theory framework an equation for the transport of particles (5), averaged over an ensemble of turbulent pulsations. One sees easily that under well defined conditions it retains its form (even though the expression for the kinetic coefficients changes) also in the case of strong long-wavelength fluctuations with $uL_0\gg\varkappa$ where the quasilinear perturbation theory is inapplicable. Indeed, in order that the required equation for the average distribution function have the form of a Fokker-Planck equation we must assume F to be sufficiently smooth. This condition will be satisfied if the averaging is carried out over spatial regions with dimensions which are larger than the main turbulence scale L_0. In order to guarantee the differential form of the acceleration term in the averaged transport equation it is necessary that the change in the particle momentum Δp over the correlation length L be small, $\Delta p\ll p$ (see, e.g., Ref. 17).

When these conditions are satisfied the averaged transport equation retains also in the case of strong turbulence the form (5) and the problem reduces merely to finding the diffusion coefficients χ and D, which now are no longer described by Eqs. (10). The calculation of the coefficients χ and D can be carried out as follows.

We consider, together with the completely averaged Eq. (5) (with the correct coefficients), another equation for the distribution function, in which the averaging is over all harmonics of the velocity field except those which refer to a narrow angle of wave numbers, with width Δk:

$$\frac{\partial\tilde F}{\partial t}=\frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}'\frac{\partial\tilde F}{\partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D'(p)\frac{\partial\tilde F}{\partial p}$$
$$-\left(\delta u_\alpha\frac{\partial}{\partial r_\alpha}\right)\tilde F+\frac{p}{3}\frac{\partial\tilde F}{\partial p}\frac{\partial\delta u_\alpha}{\partial r_\alpha}. \tag{18}$$

Here

$$\delta\mathbf{u}(\mathbf{r},t)=\int_{-\infty}^{\infty}d\omega\int_{\Delta k}d\mathbf{k}\,\mathbf{u}(\mathbf{k},\omega)\exp i(\mathbf{kr}-\omega t) \tag{19}$$

is the unaveraged part of the velocity; integration over $d\mathbf{k}$ is performed within the confines of a spherical layer of thickness Δk, $\tilde F$ is the distribution function which is not averaged over the random velocity $\delta\mathbf{u}$, and χ' and D' are diffusion coefficients due to the turbulent velocity field after subtracting $\delta\mathbf{u}$. The averaging of (18) over the ensemble of the velocities $\delta\mathbf{u}$ must lead to Eq. (5) with the complete diffusion coefficients χ and D.

The averaging of Eq. (18) can be carried out using perturbation theory, taking into account that $\delta\mathbf{u}$ is small. As the choice of $\Delta k\ll k$ is arbitrary, such an approach does not restrict the accuracy of the results obtained. However, we assume here that the Fourier harmonics of the velocity field from the interval Δk are not correlated with those outside that range. Such an assumption agrees with the model of a Kolmogorov-like turbulence and with the possibility of describing it by specifying the energy spectral density.

Writing

$$\tilde{F}=F+\delta F, \quad \langle \delta F\rangle=0, \tag{20}$$

where the angle brackets indicate averaging over the ensemble of the $\delta\mathbf{u}$ and averaging Eq. (18) we find

$$\frac{\partial F}{\partial t}=\frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}'\frac{\partial F}{\partial r_\beta}+\frac{1}{p^2}\frac{\partial}{\partial p}p^2D'\frac{\partial F}{\partial p}-\left\langle \delta u_\alpha\frac{\partial}{\partial r_\alpha}\delta F\right\rangle + \frac{p}{3}\frac{\partial}{\partial p}\left\langle \delta F\frac{\partial \delta u_\alpha}{\partial r_\alpha}\right\rangle. \tag{21}$$

The correction δF to the distribution function must here be calculated from the equation

$$\frac{\partial}{\partial t}\delta F-\frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}'\frac{\partial}{\partial r_\beta}\delta F-\frac{1}{p^2}\frac{\partial}{\partial p}p^2D'\frac{\partial \delta F}{\partial p} =-\delta u_\alpha\frac{\partial}{\partial r_\alpha}F+\frac{p}{3}\frac{\partial F}{\partial p}\frac{\partial}{\partial r_\alpha}\delta u_\alpha, \tag{22}$$

in which terms quadratic in Δk have been dropped. We express the solution of this equation in terms of the Green function of the operator on the left-hand side. As this solution must be constructed for distances which do not exceed the correlation length L_0 over which the particle acceleration is small by assumption, we can drop in the equation for the Green function the acceleration operator

$$\frac{\partial G}{\partial t}-\frac{\partial}{\partial r_\alpha}\chi_{\alpha\beta}\frac{\partial G}{\partial r_\beta}=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{23}$$

As to the diffusion coefficient χ', it differs little from the complete diffusion coefficient χ, and with the same accuracy with which we wrote down Eq. (22) we can replace in G the quantity χ' by χ.

Writing down the solution of Eq. (22) and substituting it into (21) we get after some simple transformations Eq. (5) in which $\chi=\chi'+\Delta\chi$, $D=D'+\Delta D$ with

$$\Delta\chi_{\alpha\beta}=-\int d\rho\int_0^\infty d\tau\langle\delta u_\beta(\mathbf{r},t)\,\delta u_\gamma(\mathbf{r}',t')\rangle\rho_\alpha\frac{\partial G}{\partial\rho_\gamma},$$
$$\Delta D=\frac{p^2}{9}\int d\rho\int_0^\infty d\tau\langle \operatorname{div}\delta\mathbf{u}(\mathbf{r},t)\operatorname{div}\delta\mathbf{u}(\mathbf{r}',t')\rangle G(\rho,\tau,p). \tag{24}$$

These expressions differ from (6) and (7) in two important respects: they contain the small part δu of the velocity field and a Green function $G(\mathbf{r},t,p)$ in which the complete (χ) and not the small-scale ($\varkappa$) diffusion coefficient occurs.

Changing in (24) to the Fourier representation, using Eqs. (3), (4), and (19), and taking the Green function in the form

$$G_{\mathbf{k},\omega}=\frac{1}{-i\omega+\chi k^2},$$

we get

$$\Delta\chi_{\alpha\beta}=\frac{\delta_{\alpha\beta}}{3}4\pi k^2\Delta k\int_{-\infty}^{\infty}\frac{d\omega}{(2\pi)^4}\left[\frac{2T+S}{i\omega+\chi k^2}-\frac{2k^2\chi S}{(i\omega+\chi k^2)^2}\right],$$

$$\Delta D=\frac{p^2}{9}4\pi k^4\Delta k\int_{-\infty}^{\infty}\frac{d\omega}{(2\pi)^4}\frac{S(k,\omega)}{i\omega+\chi k^2}. \tag{25}$$

Integrating these expressions over the wave numbers and using the fact that the spectral functions T and S are even functions of the argument ω, we can rewrite (25) in the following form:

$$\chi=\varkappa+\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\left[\frac{2T+S}{i\omega+\chi k^2}-\frac{2k^2\chi S}{(i\omega+\chi k^2)^2}\right], \tag{26}$$

$$D(p)=\frac{p^2}{9}\chi\int\frac{d\mathbf{k}\,\delta\omega}{(2\pi)^4}\frac{k^4S(k,\omega)}{\omega^2+\chi^2k^4}. \tag{27}$$

Equation (26) determines the spatial diffusion coefficient χ as a solution of a transcendental equation. We recall that when evaluating the contribution from the velocity field through the form of the Green function we took into account the particle transport by flows of all scales, i.e., we carried out a renormalization of χ. The diffusion coefficient $D(p)$ in momentum space is evaluated by using Eq. (27) once we know χ. In the case when the acceleration is insignificant and we have only turbulent diffusion, the turbulent diffusion coefficient is found from Eq. (26). If, however, the acceleration over a length L_0 is strong ($\Delta p\gg p$), Eqs. (26) and (27) lose their meaning and the acceleration term in Eq. (5) takes on an integral form. It is remarkable that if then the condition that the spatial gradients of F over the correlation length L_0 of the random field be small remains true, one can significantly generalize the renormalization scheme considered above.

4. STRONG ACCELERATION OVER A CORRELATION LENGTH

When the acceleration is strong we write the equation which is averaged over the ensemble of fluctuations in integral form

$$\frac{\partial F}{\partial t}=\int_{-\infty}^{\infty}d\eta'\chi_{\alpha\beta}(\eta-\eta')\frac{\partial^2F(\mathbf{r},\eta',t)}{\partial r_\alpha\,\partial r_\beta} +\left(\frac{\partial^2}{\partial\eta^2}+3\frac{\partial}{\partial\eta}\right)\int_{-\infty}^{\infty}D(\eta-\eta')F(\mathbf{r},\eta',t)\,d\eta' \tag{28}$$

where we introduced instead of the momentum the variable $\eta=\ln(P/P_0)$, in which case

$$\frac{\partial^2}{\partial\eta^2}+3\frac{\partial}{\partial\eta}=\frac{1}{p^2}\frac{\partial}{\partial p}p^4\frac{\partial}{\partial p}.$$

Averaging over all field harmonics except those from a narrow range Δk we get from the transport Eq. (1)

$$\frac{\partial\tilde{F}}{\partial t}=\int_{-\infty}^{\infty}d\eta'\,\chi_{\alpha\beta}'\frac{\partial^2\tilde{F}(\mathbf{r},\eta',t)}{\partial r_\alpha\,\partial r_\beta}+\left(\frac{\partial^2}{\partial\eta^2}+3\frac{\partial}{\partial\eta}\right) \times\int_{-\infty}^{\infty}D'(\eta-\eta')\tilde{F}\,d\eta'-\delta u_\alpha\frac{\partial}{\partial r_\alpha}\tilde{F}+\frac{1}{3}\frac{\partial}{\partial\eta}\tilde{F}\frac{\partial\delta u_\alpha}{\partial r_\alpha}. \tag{29}$$

It is convenient to Fourier transform with respect to the momentum variable η. Denoting the Fourier variable by s and the Fourier transforms of $\tilde{F}$, $\chi_{\alpha\beta}'$, and D' by $\tilde{F}_s$, $\chi_{\alpha\beta}'(s)$, and $\tilde{D}'(s)$, respectively, we get

$$\frac{\partial F_s}{\partial t}=\tilde{\chi}_{\alpha\beta}'(s)\frac{\partial^2 F_s}{\partial r_\alpha\,\partial r_\beta}-(s^2+3is)\tilde{D}'(s)F_s$$

$$-\delta u_\alpha\frac{\partial}{\partial r_\alpha}F_s-\frac{is}{3}F_s\frac{\partial}{\partial r_\alpha}\delta u_\alpha. \qquad (30)$$

The problem is now reduced to averaging an equation such as (18) and it can be solved by the same method as in Sec. 3. The fluctuating correction to the distribution function δF_s is given by the equation

$$\delta F_s(\mathbf{r},t)=-\int d\mathbf{r}'\int_{-\infty}^{t}dt'G(\mathbf{r}-\mathbf{r}',t-t',s)$$

$$\times\left\{\delta u_\alpha(\mathbf{r}',t')\frac{\partial}{\partial r_\alpha'}F_s(\mathbf{r}',t')\right.$$

$$\left.+\frac{is}{3}F_s(\mathbf{r}',t')\frac{\partial}{\partial r_\alpha'}\delta u_\alpha(\mathbf{r}',t')\right\}, \qquad (31)$$

where the Green function is

$$G(\rho,\tau,s)=(4\pi\tilde{\chi}(s)\tau)^{-3/2}\exp\{-\rho^2/4\tilde{\chi}\tau-(s^2+3is)\tilde{D}\tau\}. \quad (32)$$

Using (31) to average Eq. (30) we get

$$\frac{\partial F_s}{\partial t}=[\tilde{\chi}_{\alpha\beta}'(s)+\delta\tilde{\chi}_{\alpha\beta}(s)]\frac{\partial^2 F_s}{\partial r_\alpha\,\partial r_\beta}-(s^2+3is)[\tilde{D}'+\delta\tilde{D}]F_s,$$

$$(33)$$

i.e., the Fourier transform of Eq. (28) in which

$$\delta\tilde{\chi}_{\alpha\beta}=\int d\rho\int_0^\infty d\tau G(\rho,\tau,s)\left\{\delta K_{\alpha\beta}(\rho,\tau)\right.$$

$$\left.+\rho_\alpha\frac{\partial}{\partial\rho_\gamma}\delta K_{\gamma\beta}(\rho,\tau)+\frac{1}{18}(s^2+3is)\rho_\alpha\rho_\beta\frac{\partial^2}{\partial\rho_\gamma\,\partial\rho_\delta}\delta K_{\gamma\delta}\right\}, \qquad (34)$$

$$\delta\tilde{D}=-\frac{1}{9}\int d\rho\int_0^\infty d\tau G(\rho,\tau,s)\frac{\partial^2\delta K_{\alpha\beta}}{\partial\rho_\alpha\,\partial\rho_\beta}. \qquad (35)$$

The equations obtained are the analogs of Eqs. (24). They enable us to write down transcendental equations determining the Fourier transforms of the kernels of the integral operators $\tilde{\chi}(s)$ and $\tilde{D}(s)$:

$$\tilde{\chi}(s)=\varkappa+\frac{1}{3}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\left[\frac{2T(k,\omega)+S(k,\omega)}{\tilde{\chi}(s)k^2+i\omega+\lambda\tilde{D}(s)}\right.$$

$$\left.-\frac{2S(k,\omega)\tilde{\chi}k^2(1+\lambda/6)}{(\tilde{\chi}k^2+i\omega+\lambda\tilde{D})^2}+\frac{4\lambda S\tilde{\chi}^2k^4}{9(\tilde{\chi}k^2+i\omega+\lambda\tilde{D})^3}\right], \qquad (36)$$

$$\tilde{D}(s)=\frac{1}{9}\int\frac{d\mathbf{k}\,d\omega}{(2\pi)^4}\frac{k^2S(k,\omega)}{\tilde{\chi}(s)k^2+i\omega+\lambda\tilde{D}(s)},\qquad \lambda=s^2+3is.$$

$$(37)$$

In these equations, in contrast to (26) and (27), the variable s is an independent parameter. Their solutions will be functions of that parameter. Moreover, both functions we are looking for, $\tilde{\chi}(s)$ and $\tilde{D}(s)$, occur in the integrand.

5. EVALUATION OF THE RENORMALIZED COEFFICIENTS AND DISCUSSION OF THE RESULTS

In what follows we calculate the diffusion coefficients using Eqs. (26) and (27) which correspond to the case of weak acceleration. We consider two actual realizations of an isotropic and stationary turbulent velocity field.

1. A velocity field with exponentially decreasing correlations, described by a binary correlation function of the form

$$K_{\alpha\beta}(\rho,\tau)={}^1\!/_3\delta_{\alpha\beta}\langle u^2\rangle\exp(-\rho^2/L_0^2-|\tau|/\tau_0). \qquad (38)$$

We can assume that the amplitude $\langle u^2\rangle$, the correlation length L_0, and the correlation time τ_0 are either independent, or connected through well defined relations. We find the renormalized diffusion coefficients in the coordinate and momentum spaces from Eqs. (26), (27), which we can write in the present case in the form

$$D_1=\varepsilon+\frac{1}{6\pi^{1/2}}\int_0^\infty\frac{1+{}^1\!/_3D_1a^2x^2}{(1+D_1a^2x^2)^2}e^{-x^2/4}x^2\,dx, \qquad (39)$$

$$D_2=\frac{1}{54\pi^{1/2}}\int_0^\infty\frac{e^{-x^2/4}x^4\,dx}{1+D_1a^2x^2}. \qquad (40)$$

Here $a=u\tau_0/L_0$, $u=\langle u^2\rangle^{1/2}$, and we have introduced the dimensionless quantities $D_1=\chi/uL_0$, $D_2=DL_0/p^2u$, and $\varepsilon=\varkappa/uL_0$. One easily checks by an analysis of Eqs. (39) and (40) that for fixed a in the limit $\varepsilon\gg 1$ we have the quasilinear asymptotic behavior $D_1\propto\varepsilon$, and $D_2\propto\varepsilon^{-1}$. In the limit $a\to 0$ we have finite limits $D_1\to\frac{1}{3}+\varepsilon$ and $D_2\to\frac{2}{9}$. When $a\gg 1$ we have $D_2\approx 3.6\times 10^{-2}/D_1a^2$. The asymptotic behavior in that limit is as follows:

$$D_1\approx\begin{cases}1/(3\cdot 2^{1/2})a, & 3\cdot 2^{1/2}a\varepsilon<1,\\ \varepsilon, & 3\cdot 2^{1/2}a\varepsilon>1.\end{cases}$$

A decrease in D_2 (i.e., in the rate of particle acceleration) with increasing τ_0 for fixed u and L_0 is found in agreement with the results of the quasilinear theory indicating the importance of the stochasticity of the field for the acceleration effect (see the end of Sec. 2). The behavior of the coefficients D_1 and D_2 for finite values of a and ε, found by computer calculations, is illustrated in Fig. 1.

2. A velocity field with a power-law spectrum and a Lorentz shape for the frequency dependence. We specify this field in terms of a spectral function $S(k,\omega)=T(k,\omega)$ with the frequency dependence (11), where the parameters are described by the equations

$$S(k)=\frac{\langle u^2\rangle}{3}\frac{\Gamma(\nu/2+1)}{\pi^{3/2}\Gamma(\nu/2-{}^1\!/_2)}\frac{k_0^{\nu-1}}{(k^2+k_0^2)^{\nu/2+1}},$$

$$\Gamma_k/2=\langle u^2\rangle^{1/2}k_0^{(\nu-1)/2}k^{(3-\nu)/2},\qquad \omega_0=\langle u^2\rangle^{1/2}k. \qquad (41)$$

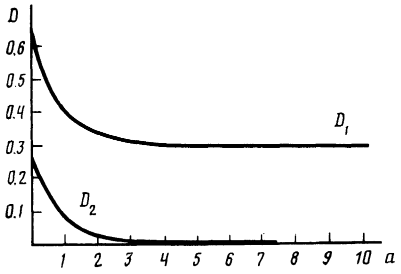

FIG. 1. Renormalized kinetic coefficients D_1 and D_2 as functions of the correlation time $a=\tau_0u/L_0$ for $\varepsilon=0.3$.

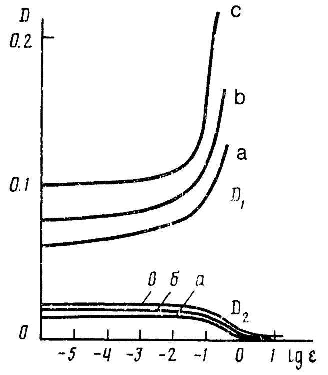

FIG. 2. Renormalized kinetic coefficients D_1 and D_2 as functions of lgε for the case of a power-law fluctuation spectrum for various values of the spectral index ν(a—$\nu=\frac{3}{2}$, b—$\nu=\frac{5}{3}$, c—$\nu=2$).

The results of calculating D_1 and D_2 as functions of ε for spectra with exponents $\nu = 2, \frac{5}{3}$, and $\frac{3}{2}$ are shown in Fig. 2. The kinetic coefficients D_1 and D_2 have finite limits as $\varepsilon \to 0$ which give the values of the turbulent transport coefficients. The way the kinetic coefficients depend on the spectral exponent reduces to an increase of D_1 and D_2 with increasing ν. This increase indicates for the chosen way of normalizing the spectrum (41) the predominant contribution of the large-scale fluctuations. We must bear in mind that the spectral index ν determines in (41) not only the energy distribution over various scales, but also the way the resonance width Γ_k depends on the scale.

We note that the results of the calculations give in the case of both spectra considered $D_1 > D_2$ which justifies the use of the differential form of the operators with respect to the momentum variable. If a calculation of the renormalized kinetic coefficients through Eqs. (26) and (27) leads to $D_1 \sim D_2$, we must for such fluctuation spectra use the renormalization scheme expounded in Sec. 4, which is based upon Eqs. (36) and (37), for in that case we have strong particle acceleration over a length L_0.

In conclusion we discuss a comparison of the results of the calculation of the renormalized kinetic coefficients with the results of a computer experiment. Such a comparison is particularly important since the method for calculating the renormalized coefficients proposed above is approximate. The main inaccuracy of the method lies, apparently, in the fact that we consider diffusive propagation of the particles at scales of order L_0 whereas such a consideration is valid only for distances $R \gg L_0$.

In a paper by Drummond *et al.*[7] the results were given of a computer simulation of the spatial particle transport by incompressible single-scale hydrodynamic flow with a Gaussian distribution of the realizations of the velocity amplitudes. The spectrum corresponding to the velocity field realized in the computer experiment of Ref. 7 has the form

$$\frac{T(k,\omega)}{(2\pi)^4} = \frac{u_0^2}{4(2\pi)^{5/2}k_0^2\omega_0}\,\delta(k-k_0)\exp(-\omega^2/2\omega_0^2). \tag{42}$$

The spectrum (42) has a single scale so that formally the approach developed in Sec. 3 as a basis for a method to calculate kinetic coefficients cannot be used. Nonetheless, one can verify that by reasoning as in Sec. 3 we find for a small section $\Delta\omega$ of the frequency spectrum the same Eqs. (26) and (27). We can thus use Eq. (26) to calculate the diffusion coefficients for the spectrum (42). For a wide range of the parameters ω_0 and k_0 the results of the calculation agree well with the data of the computer experiment of Ref. 7, the maximum difference being of the order of 10%, while for most points it does not exceed 3–6%. This confirms the reasonable nature of the approximations and the possibility to use the proposed method to describe the transport and acceleration of particles by large-scale turbulence. Perturbation-theory calculations proposed in Ref. 4 gave somewhat better agreement with the computer experiment, their accuracy being a few percent. However, these calculations are appreciably more complicated and, what is especially important, become even more complicated when one tries to include into the discussion the effects of particle energy changes, which was the main problem of our paper.

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