Singular wave collapse

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Spontaneous nonlinear wave-field singularities do not vanish in many cases immediately after absorbing the waves that produce them, but exist a longer time and "suck-in" new waves. The energy absorbed in this manner can exceed significantly the energy consumed in formation of the singularities. In particular, it can be finite under conditions when an infinitesimally low energy is required to form the singularities, i.e., the collapse is weak. These recently advanced premises are verified more thoroughly by investigating the dynamics of a wave field in the spaciotemporal vicinity of the point of formation of the singularity. This is done in the context of the nonlinear Schrödinger equation encountered in many physical problems. It is shown that the dynamics of the wave field in this region is self-similar in a wide range of the parameters of the model employed. The existence of singular self-similar solutions for arbitrary values of the parameters is demonstrated. An alternate variant of wave-field dynamics is proposed for conditions when such solutions are unstable.

1. INTRODUCTION

The possibility of an explosive increase of the amplitude of a nonlinear wave field at individual points of space was recognized approximately a quarter-century ago in the development of the theory of self-focusing of electromagnetic radiation in a medium.[1,2] The various phenomena of this kind were later unified into a concept called wave collapse (first applied in Ref. 3 to a predicted explosive self-compression of a cluster of Langmuir waves in a plasma). Preference was given for a long time to the so-called strong collapse, in which the production of a field singularity requires the expenditure of a finite wave energy. The notion of a "weak" collapse, i.e., a situation in which the energy of the waves localized in the region of the developing singularity tends to zero, has encountered a number of difficulties, which were most clearly manifested in Ref. 4. That reference dealt with a three-dimensional Schrödinger equation with cubic nonlinearity, which is encountered in many branches of physics, particularly in the scalar model of subsonic collapse of Langmuir waves. Only weak collapse regimes were known heretofore within the framework of this model, but the authors of Ref. 4 (see also Ref. 5) have found "semiclassical" strong regimes. These, however, turned out to be unstable to small-scale perturbations, so that the problem of wave absorption could not be eliminated. The absorption mechanism was understood later, not as a result of searching for stable strong-collapse regime but by foregoing the usual identification of the process of singularity formation with the process of wave absorption. A tendency towards a more adequate approach can be seen in an article[6] preceding Refs. 4 and 5, in Sec. 3 of which the "funnel effect" is considered. The gist of this effect is the following.

In the scalar model of the supersonic Langmuir collapse (which is known to be strong), the wave field is described by the Schrödinger equation

$$(i\partial/\partial t+\Delta-V)\psi=0.$$

The role of the potential V is played by the perturbation of the plasma density. After the collapse, the potential has for some time the form

$$V(r)=-A/r^2,\quad 1/4<A,\quad A\sim 1,$$

i.e., the "density funnel" is preserved. It extends from the initial size of the caviton $r\sim a_0$ to the size $r\sim a_f$ of the absorption region, which is physically infinitesimally small compared with a_0. According to Ref. 6, the funnel "sucks-in," from the surrounding space, waves that later "fall to the center" and are absorbed; the "wave function" in the region $a_f\ll r\ll a_0$ takes the form

$$\psi=\psi_0 r^{-1/2-i\beta},\quad \beta=(A-1/4)^{1/2}.$$

The wave-damping strength at the center of the funnel, needed to realize the above picture, was not discussed in Ref. 6. Yet it is clear that to absorb a free quasiparticle with wavelength of order a_0 the damping decrement ν must be at least a_0/a_f times larger than the reciprocal of the time of flight of the particle over the caviton (since only a small fraction of the energy is concentrated at each instant of time in the region $r\lesssim a_f$). On the other hand, the stationary-funnel approximation used in Ref. 6 is valid only during the characteristic time of the caviton deepening in the concluding stage of the collapse. The time is estimated to be equal to ν^{-1}, since it is just the damping of the trapped wave which limits the explosive growth of its amplitude. In a time ν^{-1} the distance covered by a free wave of length a_0 does not exceed a_f, so that no noticeable energy, compared with that absorbed in the collapse, can enter the funnel from the outside.

A more effective Langmuir-wave absorption mechanism connected with the nonstationary character of the caviton remaining after collapse was observed in Ref. 7. According to that reference, the caviton continues, after absorbing the waves that have formed it, to deepen on account of the inertial motion of the ions, and acquires a large number of new bound states relative to the parameter a_0/a_f. Some energy is captured in each of these states. Absorption of the trapped waves no longer requires so strong a damping ν. By the same token, the upper bounds on the caviton deepening time and on the energy flowing into the caviton during this time are lifted. Under rather lax conditions the energy "sucked-in" by the caviton exceeds the energy of the waves forming the singularity by many times relative to the param-

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eter a_0/a_f. In this sense, the energy consumed in the production of the singularity is physically infinitesimally small, just as in a weak collapse, but this does not preclude in any way a finite absorbed energy.

A similar effect takes place in all likelihood in the subsonic collapse of Langmuir waves, and the scalar model of this collapse, already mentioned above in connection with Ref. 4, is certainly realized. This was demonstrated by numerical computation in Ref. 8, where the "sucking-in" of waves by a spontaneously produced singularity was named "distributed collapse."

The hypothesis of wave "sucking-in" by long-lived singularities as a general mechanism of absorption of energy from fields having stable weak-collapse regimes was advanced in Ref. 9, where the existence of singular self-similar solutions and the need for their study were pointed out.

Singular stationary solution of a two-parameter family of nonlinear equations of the Schrödinger type were obtained shortly thereafter.[10] In the opinion of the authors of that paper, weak collapse in a wide range of the parameters of this model was indeed accompanied by formation of a long-lived singularity of the wave field.

The main purpose of the present paper is an investigation of the (obviously nonstationary) dynamics of a nonlinear wave field in the spatiotemporal vicinity of the point where the singularity is produced. Such an investigation is needed for a more profound elucidation of the very possibility of existence of a somehow prolonged spontaneously produced singularity.

2. QUALITATIVE DISCUSSION OF THE BASIC EQUATIONS

The nonlinear Schrödinger equation

$$(i\partial/\partial t+\Delta+|\psi|^s)\psi=0 \tag{2.1}$$

has been the subject of many studies (see, e.g., the review[11]). We consider below centrosymmetric solutions of (2.1), for which this equation takes the form

$$\left(i\frac{\partial}{\partial t}+\frac{\partial^2}{\partial r^2}+\frac{d-1}{r}\frac{\partial}{\partial r}+|\psi|^s\right)\psi=0 \tag{2.2}$$

(d is the dimensionality of space). The most variegated applications are those of the cubic Schrödinger equation ($s = 2$). It describes, in particular, at $d = 1$, excitations in quasi-one-dimensional molecular structures (see, e.g., the review[12]), at $d = 2$ stationary self-focusing of radiation in a medium[1,2] and waves on the surface of a deep liquid,[13–15] and at $d = 3$ the envelope of a quasimonochromatic wave packet (see, e.g., Ref. 16) and a subsonic Langmuir collapse.[17]

To clarify the general properties of the nonlinear equation (2.2) it is useful to understand the dependence of its solutions on the parameters $s\geqslant 1$ and $d\geqslant 1$. The character of this dependence is determined for the most part by the integrals of motion

$$N=\int_0^\infty dr\, r^{d-1}|\psi|^2, \tag{2.3}$$

$$H=\int_0^\infty dr\, r^{d-1}\left(\left|\frac{\partial\psi}{\partial r}\right|^2-\frac{2}{s+2}|\psi|^{s+2}\right), \tag{2.4}$$

which Eq. (2.2) has on regular wave fields ψ. Usually the "number of quanta" N is proportional to the true energy of the wave field, and the Hamiltonian H is proportional to the dispersive-nonlinear increment to this energy. On a single-scale wave field ψ localized in the region $r\lesssim a$, the following estimate is valid for the Hamiltonian:

$$H\sim N/a^2-N^{s/2+1}/a^{sd/2}. \tag{2.5}$$

It can be easily understood from it that collapse is impossible if $sd<4$, since the condition $a\to 0$ is incompatible with conservation of the Hamiltonian. If $sd = 4$, the terms in the right-hand side of (2.5) cancel out at a certain number $N\sim 1$ of quanta, and strong collapse is allowed. For mutual cancellation of the dispersive and nonlinear contributions to H in the case $sd>4$, it suffices to localize in the region $r\lesssim a\to 0$ an infinitesimally small number of quanta, $N\to 0$, corresponding to weak collapse.

It is appropriate to note here that at $sd>4$ it is possible to "organize" also a strong collapse, but this calls for a two-scale wave field ψ. If the field ψ varies over length a

$$\lambda\sim|\partial\ln\psi/\partial r|^{-1} \tag{2.6}$$

that is small compared with its localization scale, it is necessary to replace a by λ in the first term of the right-hand side of (2.5). The condition for mutual cancellation of the contribution of the dispersion and of the nonlinearity H determines the connection between λ and a:

$$\lambda\sim N^{-s/4}a^{sd/4}. \tag{2.7}$$

It is clear therefore that the assumption $\lambda\ll a$ is indeed justified for $a\to 0$ and $sd>4$. The character of the temporal decrease of a and λ is simplest to elucidate with the aid of the continuity equation

$$\frac{\partial n}{\partial t}+\frac{\partial j}{\partial r}=0, \qquad n=r^{d-1}|\psi|^2,\quad j=2r^{d-1}|\psi|^2\frac{\partial}{\partial r}\arg\psi. \tag{2.8}$$

If the scales of the spatial variation of the functions n and j are estimated to be equal to a, it follows from (2.8) that

$$\lambda a\sim t_*-t, \tag{2.9}$$

where t_* is the instant of singularity formation. The "quasiclassical" collapse regimes described by the estimates (2.7) and (2.9) are, as already noted, unstable to small-scale perturbations. The most unstable perturbations have a wavelength of order λ and evolve within a time $\tau\sim\lambda^2$ which is short compared with t_*-t.

Another type of strong collapse of two-scale wave fields ψ comprise spherical soliton layers converging to a center. For $d = 3$ and $s = 2$ such solutions date back to Refs. 17–19, and are stable within the framework of Eq. (2.2). In the context of the initial equation (2.1), spherical layers are most likely destroyed by tangential modulation, as is also a planar soliton.[20,21]

Judging from the results of a numerical solution the Cauchy problem for Eq. (2.2) with $d = 3$ and $s = 2$ (see Refs. 17, 22, 23, and 10) and with $d = 1$ and $s = 6$ (see Refs. 24 and 23), a weak collapse is stable, but the observed explosion picture is quite crude and hardly lends itself to a significant examination in the more general (for $d\neq 1$) model (2.1).

3. FORMATION OF SINGULARITY

In all the heretofore known cases a weak collapse obeys, as $t \to t_*$, the self-similar law

$$\psi(r,t)=a^{-2(s^{-1}+i\varkappa)}f(r/a),\quad a=(t_*-t)^{1/2}. \tag{3.1}$$

In the region $r \gg a$ the field cannot change noticeably within a time $t_* - t$ and so to speak "freezes." In the self-similarity region $r \ll a_0$ [where Eq. (3.1) is valid] the "frozen" field takes the form

$$\psi(r,t)\approx\psi(r,t_*)=Cr^{-2(s^{-1}+i\varkappa)}. \tag{3.2}$$

The function $f(\rho)$ in (3.1) satisfies the ordinary differential equation

$$\left[i\left(s^{-1}+i\varkappa+\frac{1}{2}\rho\frac{d}{d\rho}\right)+\frac{d^2}{d\rho^2}+\frac{d-1}{\rho}\frac{d}{d\rho}+|f|^s\right]f=0. \tag{3.3}$$

Its solution as $\rho\to\infty$ has the asymptote

$$f\approx C\rho^{-2(s^{-1}+i\varkappa)}+C_1\rho^{2(s^{-1}+i\varkappa)-d}\exp(-i\rho^2/4). \tag{3.4}$$

The first term of this equation corresponds to the frozen field (3.2). The second term (which decreases as $\rho\to\infty$ faster than the first by virtue of the weak-collapse condition $sd>4$) describes a spherical wave that converges to a center. Such a wave, which produces a singularity even in the linear problem, can occur only under very special initial conditions. In the general case there is no convergent wave, i.e., $C_1=0$. Recognizing that wave fields obtained from one another by adding an arbitrary real number to the phase are identical, we can choose a positive coefficient C in the asymptote (3.4). A subfamily that depends on two real parameters C and $\varkappa$ (s and d are fixed), which decrease as $\rho\to\infty$, is thus selected from the family of the solutions of (3.3).

Any solution of (3.3) is automatically regular for all finite values $\rho>0$. [The singularity discussed in Ref. 4 actually does not satisfy Eq. (3.3), as can be easily verified by using the continuity equation (2.8).] The constraints on the parameters C and $\varkappa$ are imposed exclusively by the condition that the function $f(\rho)$ be regular at $\rho=0$. The general solution, infinitely close to the regular one, has for $\rho\to0$ the asymptote

$$f\approx A+A_1\begin{cases}\rho^{2-d}, & d\neq2,\\ \ln\rho, & d=2.\end{cases} \tag{3.5}$$

To reach the regular asymptote as ρ decreases it is necessary to make the complex quantity A_1 equal to zero, i.e., to satisfy two real relations. This condition selects an enumerable set of values of the parameters C and $\varkappa$. Analysis shows that only the first of the corresponding set of self-similar solutions (arranged in increasing order of $\varkappa$) is stable. On the set of regular stable weak-collapsing regimes, the parameters C and $\varkappa$ are uniquely connected with d and s in the entire $ds>4$ region. In a number of limiting cases this connection can be described analytically. The most interesting is the limit $ds-4\to0$, since the dynamics of the pre-critical weak collapse has the characteristic features not only of the case $ds>4$, but also of the critical case[1] $ds=4$. As $ds-4\to0$, the following relation turns out to be very useful:

$$\rho^{d-1}|f(\rho)|^2\left[\rho+4\frac{\partial}{\partial\rho}\arg f(\rho)\right]=(d-4s^{-1})\int_0^\rho d\rho_1\,\rho_1^{d-1}|f(\rho_1)|^2. \tag{3.6}$$

It is obtained by multiplying (3.3) by $\rho^{d-1}f^*$ and integrating the imaginary part of the result from zero to ρ. According to (3.6), as $d-4s^{-1}\to0$ we have in the region $\rho\ll\rho_*$ [where the amplitude $|f(\rho)|$ is not too small] the relation

$$\arg f(\rho)\approx-\rho^2/8+\text{const}. \tag{3.7}$$

With the aid of the substitution

$$f(\rho)=\bar f(\rho)\exp(-i\rho^2/8), \tag{3.8}$$

which eliminates the oscillations of $f(\rho)$ in the region $\rho\ll\rho_*$, Eq. (3.3) is reduced to the form

$$\left[\frac{d^2}{d\rho^2}+\frac{d-1}{\rho}\frac{d}{d\rho}+|\bar f|^s+\frac{\rho^2}{16}-\varkappa-\frac{i}{4}(d-4s^{-1})\right]\bar f=0. \tag{3.9}$$

It becomes clear next that $\varkappa\to\infty$ as $d-4s^{-1}\to0$. Taking this into account, we can define ρ_* by the equation

$$\rho_*=4\varkappa^{1/2}. \tag{3.10}$$

In the region $\rho\leqslant\varkappa^{-1/2}$ the solution of (3.9) is

$$\bar f(\rho)\approx\varkappa^{1/s}R(\varkappa^{1/2}\rho), \tag{3.11}$$

where $R(r)$ is the stationary state of Eq. (2.2) with $ds=4$, a state having a single "binding energy"

$$\left(-1+\frac{d^2}{dr^2}+\frac{d-1}{r}\frac{d}{dr}+|R|^s\right)R=0. \tag{3.12}$$

The function $R(r)$ for $s=d=2$ was calculated in the known Ref. 31. The number of quanta in the soliton (3.11) is independent of $\varkappa$ and is equal to

$$N_c=\int_0^\infty dr\,r^{d-1}|R(r)|^2 \tag{3.13}$$

(according to Ref. 31, $N_c=1.86$ at $s=d=2$).

In the region $\rho\gg\varkappa^{-1/2}$ one can neglect the nonlinear term in Eq. (3.9). The linear equation can already be solved analytically. The solution decreases exponentially in the below-barrier region $\varkappa^{-1/2}\ll\rho<\rho_*$, is "drawn" in standard fashion through the vicinity of the stopping point $\rho=\rho_*$ (see, e.g., Ref. 32), and reaches at $\rho\gg\rho_*$ the semiclassical asymptote

$$\bar f(\rho)=C\rho^{-2(s^{-1}+i\varkappa)}\exp(i\rho^2/8), \tag{3.14}$$

corresponding to the frozen field (3.2). In the region $\rho\gg\rho_*$, $C^2\ln(\rho/\rho_*)\ll1$, where the asymptote (3.14) is valid, the integral in the right-hand side of (3.6) is practically independent of ρ and is equal to N_c, while from (3.6) it follows that

$$C^2=(d-4s^{-1})N_c. \tag{3.15}$$

The known equation for the below-barrier coefficient for the passage of a particle from the region $\rho\lesssim\varkappa^{-1/2}$ into the region $\rho>\rho_*$ makes it possible to establish a connection between $\varkappa$ and C:

$$\varkappa=-\frac{1}{\pi}\ln C\approx-\frac{1}{2\pi}\ln\left(1-\frac{4}{ds}\right). \tag{3.16}$$

If the difference $d-4s^{-1}$ is increased, Eq. (3.15) remains valid somewhat longer than (3.16). For example, in the case $d=3$, $s=2$, when the true values of C^2 and $\varkappa$ are

$$C^2\approx2.02,\ \varkappa\approx0.54,$$

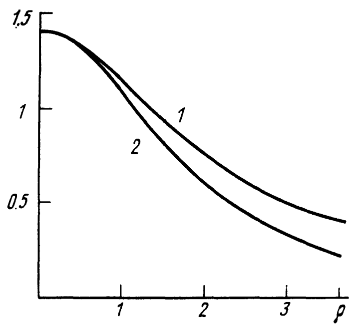

FIG. 1. Plots of self-similar solution $|f(\rho)|$ (curve 1) and of the soliton (3.11) (curve 2) with $\varkappa$ chosen such that the soliton amplitude coincided with its true value at $\rho = 0$.

the asymptotic equations (3.15) and (3.16) yield

$$C^2 \approx 1.86, \quad \varkappa \approx 0.18.$$

In Fig. 1 are compared the true self-similar solution $|f|$ and the soliton (3.11) (with a true value $|f(0)|$) at $d = 3$ and $s = 2$. With so noticeable a departure from the region where the asymptotic relations are valid, the correspondence can be regarded as good.

4. EARLY STAGE OF SINGULARITY

At the instant $t = t_*$ when the singularity is formed, the wave field is frozen in the entire self-similarity region $r \ll a_0$:

$$\psi(r, t_*) = Cr^{-2(s^{-1}+i\varkappa)}. \tag{4.1}$$

The field (4.1) corresponds to a negative "quantum flux," i.e., directed towards the singularity,

$$j(r, t_*) = -2\varkappa C^2 r^{d-2-4s^{-1}}. \tag{4.2}$$

Within the time $t - t_*$ of the onset of the singularity the field manages to change in the region

$$r \leqslant b = (t - t_*)^{1/2}, \tag{4.3}$$

remaining frozen on the level (4.1) at $r \gg b$. In the region $r \ll b$, where the field is capable of changing in a time $\delta t \sim r^2$ short compared with $t - t_*$, it is natural to expect a quasistationary state to set in, the quantum flux to the singularity being independent of r. The dynamics of the wave field should have then a self-similar character:

$$\psi(r, t) = b^{-2(s^{-1}+i\varkappa)} g(r/b). \tag{4.4}$$

The function $g(\rho)$ satisfies the ordinary differential equation

$$\left[-i\left(s^{-1} + i\varkappa + \frac{\rho}{2}\frac{d}{d\rho}\right) + \frac{d^2}{d\rho^2} + \frac{d-1}{\rho}\frac{d}{d\rho} + |g|^s\right] g = 0 \tag{4.5}$$

and has the same asymptote as f for $\rho \to 0$:

$$g(\rho) = C\rho^{-2(s^{-1}+i\varkappa)}. \tag{4.6}$$

Equation (4.5) differs from (3.3) in the sign of the first term. As a result, the field (4.4) is automatically singular at the point $\rho = 0$, and the corresponding quantum flux

$$j(r, t) = 2b^{d-2-4s^{-1}} J(\rho), \qquad J(\rho) = \rho^{d-1} |g|^2 \frac{\partial}{\partial\rho} \arg g \tag{4.7}$$

does not vanish as $\rho \to 0$:

$$J(0) < 0. \tag{4.8}$$

The actual form of the asymptote $g(\rho)$ as $\rho \to 0$ depends on the parameters d and s. In the region $d \geqslant 2 + 2s^{-1}$ the principal terms of this asymptote can be calculated using the equations obtained in Ref. 10 for stationary singular solutions, since the first term of (4.5), which is due to nonstationarity, is small as $\rho \to 0$, viz.,

$$g(\rho) \approx \left[\frac{-J(0)}{\rho^{d-1}}\right]^{2/(s+4)} \exp\left\{i \frac{s+4}{sd-2s-4}\left[\frac{-J(0)}{\rho^{sd-2s-4}}\right]^{1/(s+4)}\right\} \tag{4.9}$$

for $d > 2 + 4s^{-1}$,

$$g(\rho) \approx A\rho^{-2s^{-1}+iJ(0)/|A|^2}, \quad |A|^2[|A|^s - 4s^{-2}]^{1/2} = -J(0) \tag{4.10}$$

for $d = 2 + 4s^{-1}$,

$$g(\rho) \approx A\rho^{-2s^{-1}}\left[1 + \frac{iJ(0)}{|A|^2}\frac{\rho^{2+4s^{-1}-d}}{2+4s^{-1}-d}\right], \qquad |A|^s = 2s^{-1}(d-2-2s^{-1}) \tag{4.11}$$

for $2 + 2s^{-1} < d < 2 + 4s^{-1}$, and

$$g(\rho) = \rho^{-2/s}\left(\frac{s^2}{2}\ln\frac{1}{\rho}\right)^{-1/s} A\left[1 + \frac{iJ(0)}{2/s}\left(\frac{s^2\rho}{2}\ln\frac{1}{\rho}\right)^{2/s}\right], \qquad |A| = 1, \tag{4.12}$$

for $d = 2 + 2s^{-1}$. This last equation was obtained earlier[8] for the particular case $s = 2$ and $d = 3$. In the context of the present article, the semiclassical character of the asymptote (4.9) can be qualitatively attributed to the increase of the quantum flux into the singularity with time at $d > 2 + 4s^{-1}$ [see (4.7)].

Singular self-similar solutions exist also in the region $d < 2 + 2s^{-1}$. In this case, in the calculations of the principal terms of the asymptotes as $\rho \to 0$, we can neglect not only the first but also the nonlinear term of (4.5):

$$g \approx \begin{cases} A/\rho^{d-2} + iJ(0)/(d-2)A^*, & d > 2, \\ A\ln(1/\rho) + iJ(0)/A^* & d = 2, \\ A + iJ(0)\rho^{2-d}/(2-d)A^*, & d < 2. \end{cases} \tag{4.13}$$

Plots of the functions $|g(\rho)|$ and $d \arg g(\rho)/d\ln\rho$ for different values of the parameters d and s are shown in Fig. 2. Figure 3 demonstrates the realization of a singular self-similar solution $g(\rho)$ for the time-dependent problem at $d \geqslant 2 + 2s^{-1}$. Similar calculations for $4s^{-1} < d < 2 + 2s^{-1}$ at-

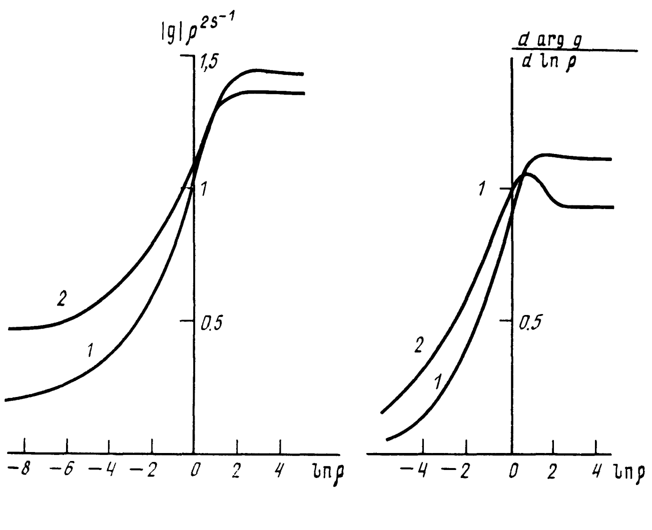

FIG. 2. Plots of the functions $|g(\rho)|\rho^{2s^{-1}}$ and $d\arg g/d\ln\rho$ at $d=3$ and $s=2$ (curves 1) and $s=2.5$ (curves 2).

test to the instability of the corresponding solutions $g(\rho)$. This instability is due to the weakness of the nonlinear effects on the asymptotes (4.13) and to the obvious vanishing of the sigularity following the smallest stirring of a linear wave field focused into a point. Once the focus vanishes, the linear wave is reflected as usual from the center and becomes divergent. Its subsequent evolution can be tracked analytically for $d-4s^{-1}\ll 1$, when the constant C in the frozen asymptote of the field $\psi(r,t)$ is small [see (3.15)] and the linear approximation is applicable in the entire self-similarity region [and not only if $r\ll b(t)$]. The character of the evolution remains qualitatively the same (i.e., the field singularity at the point $r=0$, $t=t_*$ turns out to be isolated) and for other values of the parameters d and s from the region $4s^{-1}<d<2+2s^{-1}$. Since a single collapse hardly changes the number N of the quanta but leads to the appearance of small-scale field fluctuations, it is natural to expect a gradual displacement of the spectral density of the waves into the small-scale region. Such a displacement should continue in all likelihood until the field drops below the modulation-instability threshold, i.e., until the nonlinearity is suppressed by dispersion (which in fact determines the subsequent evolution). It should be noted that this does not contradict the theorem stating that collapse is inevitable at $H<0$ (see Ref. 33 as well as the review[11]), since the Hamiltonian H, in contrast to the number N of the quanta, changes noticeably at each singularity and is positive in the final (weakly nonlinear) state of the field.

5. PRINCIPAL RESULTS

We have investigated the dynamics of a wave field described by the nonlinear Schrödinger equation (2.2) with two parameters $d\geqslant 1$ and $s\geqslant 1$, in the spatiotemporal vicinity of the point $r=0$, $t=t_*$ where a singularity sets in. We have shown that in the case $d\geqslant 2+2s^{-1}$ there exist and are ob-

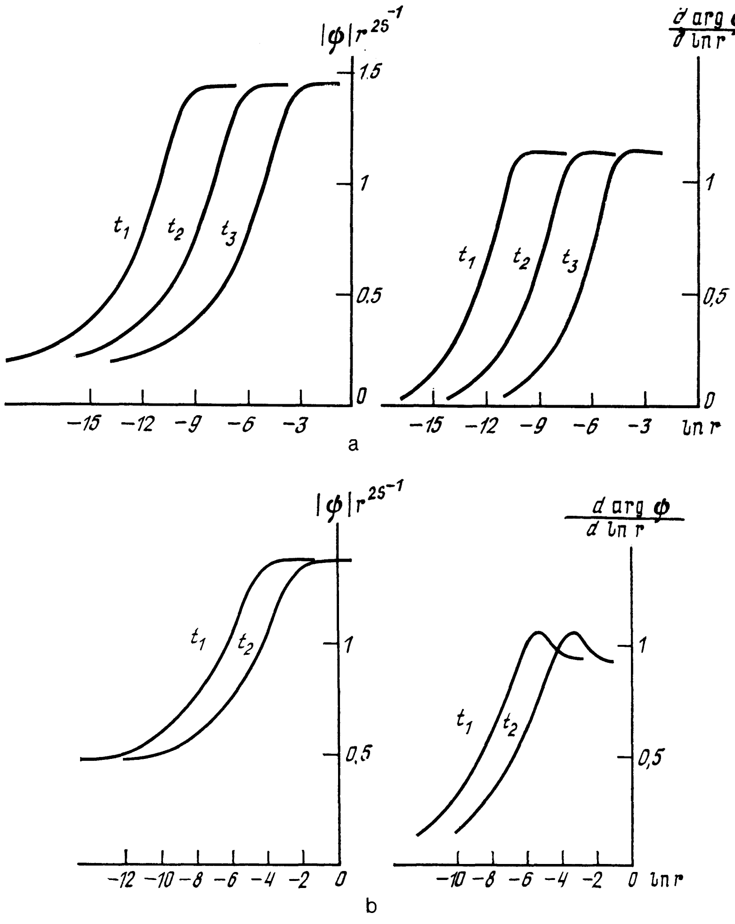

FIG. 3. Temporal evolution of the wave field after the onset of the singularity ($t_*=0$) at $d=3$: a) $s=2$, $t_1=6.7\cdot10^{-8}$, $t_2=2.6\cdot10^{-5}$, $t_3=1.05\cdot10^{-2}$; b) $s=2.5$, $t_1=6.7\cdot10^{-4}$, $t_2=4.8\cdot10^{-3}$.

tained singular self-similar solutions with nonzero quantum flux proportional to $(t-t_*)^{d/2-1-2/s}$ into the singularity. In the case $4s^{-1}<d<2+2s^{-1}$, when the singularity is isolated in space-time, a hypothesis has been advanced that the wave field relaxes gradually (in view of the scale fragmentation due to the collapse) to a state in which the nonlinearity is suppressed by dispersion and collapse is impossible.

APPENDIX 1

The numerical solution of the Cauchy problem for Eq. (2.2) was carried out in terms of the independent variables $x=\ln r$ and t. This made it possible to approach, to any required degree, the singularity $(r=0)$ without changing the mesh of the spatial net. The functions $r^{2/s}|\psi|$ and $\partial \arg \psi/\partial x$ were calculated. In the initial state (having a field maximum at the point $r=0$ and a negative Hamiltonian) these functions decreased exponentially as $x\to-\infty$. When t was increased, plateaus with heights of the order of unity began to be "pull out" from them to the left along x. For $t\to t_*$ the plateaus extended to $-\infty$, meaning formation of a singularity (which was investigated earlier by another method and was therefore not shown in the figures of the present article). At any instant $t<t_*$ a numerical calculation was required only for distances, not very large compared with unity, from the point $x=(1/2)\ln(t_*-t)$. To the right of this region [i.e., for $r\gg(t_*-t)^{1/2}$] the field ψ froze, and to the left [i.e., for $r\ll(t_*-t)^{1/2}$] it was practically independent of x.

After the onset of the singularity the plateaus of the functions $r^{2/s}|\psi|$ and $\partial \arg \psi/\partial x$ began to "sag" to the left. The "sagging" in the vicinity of the point x began at $t-t_*\sim e^{2x}$. A quasistationary state with a quantum flux, independent of x, to the left was established next in this vicinity (at $d\geqslant 2+2s^{-1}$). The numerical calculations were performed in a finite vicinity of the point $x=\frac{1}{2}\ln(t-t_*)$. This vicinity could be shifted, when convenient, to the right so that the field on its right-hand boundary remained frozen. On the left boundary of the computation region the time derivative of the field was calculated by using a smooth extrapolation. The extrapolation order and the computation-region width were chosen such that their increase did not alter the results.

[1)]It was already noted above that in the critical case $(s=d=2)$ the collapse is strong. Many attempts were made to elucidate its dynamics in many papers (see Refs. 22, 23, 25–29, and the citations therein), but not successfully. The remaining difficulties could be overcome only most recently.[30]

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