Propagation of ultrashort periodic pulses in nonlinear fiber waveguides

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Propagation of femtosecond periodic pulses along a fiber waveguide is considered. Expressions are obtained for the field intensity and amplitude. The Whitham modulation equations are derived and solved to describe an oscillatory region of the field near the point of wave reversal.

1. INTRODUCTION

One of the recent successes in the physics of fiber waveguides has been the development of a method for generating light pulses of ~ 10–100 fs duration (for a review see, for example, Ref. 1). Although pulses of picosecond duration can be described by the familiar nonlinear Schrödinger equation (NSE), in the femtosecond range we have to allow more rigorously for nonlinear effects. It is shown in Ref. 2 (see also Refs. 1 and 3–5) that for typical values of the physical parameters the most important are the effects that follow from the failure of the hypothesis of a quasisteady nonlinear response. Since this quasisteady hypothesis cannot be employed, the evolution of an envelope of a pulse in a fiber waveguide must be described by a generalized NSE

$$i\tilde{q}_z+\tilde{q}_{tt}+2|\tilde{q}|^2\tilde{q}+2ib(|\tilde{q}|^2\tilde{q})_t=0, \tag{1}$$

where use is made of the dimensionless variables $z = Z/L_d$ representing the coordinate along the fiber waveguide, $t = (T - Z/v_g)/\tau_p$ is the "instantaneous" time, and $\tilde{q} = (\gamma/2)^{1/2} A$; here, L_d is the characteristic length or distance in which a pulse spreads out as a result of dispersion, v_g is the group velocity, τ_p is the characteristic duration of a pulse, γ is a parameter representing the nonlinearity of the fiber waveguide, A is the amplitude of the field envelope, and b is a parameter describing departure from the quasisteady approach (see Refs. 1–5). Inclusion of the last term in Eq. (1) is essential also in the description of such effects as the reversal of a "shock" wave in a nonlinear fiber waveguide and formation of an oscillatory region in the vicinity of the reversal point (for experimental results see, for example, Ref. 6). An analysis of problems of this type requires the knowledge not only of the soliton solutions of Eq. (1), which can be found in Ref. 7, but also of its periodic solutions. Since Eq. (1) is integrable (see Refs. 8 and 7), its periodic solutions can be found by the finite-band integration methods.[9,10] We shall use these methods inthe next section to study single-phase periodic solutions of Eq. (1), which are most important from the point of view of physical applications.

Periodic waves in real physical problems are naturally inhomogeneous. If the degree of inhomogeneity is relatively small, the evolution of a periodic wave can obviously be described by the Whitham averaging method.[11] In Sec. 3 we shall derive the Whitham equations for slow changes in the parameters that determine the periodic solutions, and we shall find the solutions corresponding to oscillations of the envelope in the vicinity of the point of wave reversal.

2. PERIODIC SOLUTIONS

We shall simplify somewhat the analysis using the circumstance that the solutions of Eq. (1) are related by the simple expression (see Ref. 12)

$$\tilde{q}(t, z)=b^{-1/2}\exp(-iz/b^2+it/b)\,u(2z/b-t, z)$$

to the solution of another integrable equation

$$iu_z+u_{tt}-2i(|u|^2u)_t=0. \tag{2}$$

The integrability of this "nonlinear Schrödinger equation with a derivative" (NSED), first established in Ref. 13, follows from the fact that it represents the condition of compatibility of two linear equations which contain a spectral parameter λ and which we shall use in the form proposed in Ref. 8:

$$\frac{\partial\psi_1}{\partial t}=-F(\lambda)\psi_1+G(\lambda)q(t,z)\psi_2,\qquad \frac{\partial\psi_2}{\partial t}=G(\lambda)r(t,z)\psi_1+F(\lambda)\psi_2, \tag{3}$$

and

$$\frac{\partial\psi_1}{\partial z}=A(\lambda,r,q)\psi_1+B(\lambda,r,q)\psi_2,\qquad \frac{\partial\psi_2}{\partial z}=C(\lambda,r,q)\psi_1-A(\lambda,r,q)\psi_2, \tag{4}$$

where in the case of the NSED we can assume that

$$q=u,\ r=u^*,\ F=2\lambda^2,\ G=2\lambda,\ A=-8i\lambda^4-4i|u|^2\lambda^2,\quad B=8\lambda^3u+(2iu_t+4|u|^2u)\lambda,\quad C=8\lambda^3u^*+(-2iu_t^*+4|u|^2u^*)\lambda. \tag{5}$$

The systems of equations (3) and (4) have two basic solutions $\psi = (\psi_1, \psi_2)$ and $\varphi = (\varphi_1, \varphi_2)$, which satisfy different boundary conditions. In finding the periodic solutions of the NSED it is convenient to go over to systems of equations which are satisfied by the "squares of the basis functions" (see Refs. 14 and 15):

$$f=-\tfrac{1}{2}i(\varphi_1\psi_2+\varphi_2\psi_1),\ g=\varphi_1\psi_1,\ h=-\varphi_2\psi_2. \tag{6}$$

These systems are described by

$$\partial f/\partial t=-iGrg+iGqh,\ \partial f/\partial z=-iCg+iBh,\quad \partial g/\partial t=2iGqf-2Fg,\ \partial g/\partial z=2iBf+2Ag,\quad \partial h/\partial t=-2iGrf+2Fh;\ \partial h/\partial z=-2iCf-2Ah. \tag{7}$$

We can easily demonstrate that the expression

$$f^2-gh=P \tag{8}$$

is independent of t and z, so that P is a function of λ alone. Periodic solutions are distinguished by the following condition: $P = P(\lambda)$, which is a polynomial of λ. Substituting the system (5) corresponding to the NSED into the system (7),

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we can demonstrate that in this particular case the polynomial $P(\lambda)$ can contain only even powers of λ. Nontrivial solutions are obtained if $P(\lambda)$ is of degree $\geqslant 6$, and degrees 6 and 8 correspond to one-band periodic solutions. The knowledge of these solutions is sufficient for the description of such typical physical tasks as, for example, the formation of solitons as a result of a reversal of a shock wave, so that we shall confine our treatment to these solutions. Moreover, in the case of many-band solutions describing the interaction of nonlinear waves, the inverse scattering problem gives us so far insufficiently workable expressions. We shall show later that the solutions corresponding to the sixth degree of the polynomial $P(\lambda)$ are special cases of the solutions of the eighth-degree $P(\lambda)$. We shall therefore assume that the polynomial $P(\lambda)$ is

$$P(\lambda)=\prod_{i=1}^{4}(\lambda^2-\lambda_i^2)=\lambda^8-s_1\lambda^6+s_2\lambda^4-s_3\lambda^2+s_4, \qquad (9)$$

where $\pm\lambda_i$ are zeros of $P(\lambda)$. We can then easily show that the system (7) is satisfied by the expressions

$$f=\lambda^4-f_1\lambda^2+f_2,\ g=u\lambda(\lambda^2-\mu),\ h=u^*\lambda(\lambda^2-\mu^*), \qquad (10)$$

where

$$\partial u/\partial t=-4iu(f_1-\mu),\ \partial u/\partial z=8iu[2f_2-(f_1-\mu)(2f_1+|u|^2)], \qquad (11)$$

and the quantities f_1, f_2, $|u|^2$, and μ are related by the following condition deduced from Eq. (8):

$$(\lambda^4-f_1\lambda^2+f_2)^2-|u|^2\lambda^2(\lambda^2-\mu)(\lambda^2-\mu^*)=P(\lambda). \qquad (12)$$

The variable μ is known as a point of an additional spectrum in the eigenvalue problem (3) with periodic boundary conditions. The dependences of μ on t and z can be obtained from the system (7) if we substitute there $\lambda^2=\mu$ and bear in mind that $f(\mu^{1/2})=(P(\mu^{1/2}))^{1/2}$.

$$\partial\mu/\partial t=\pm4i(P(\mu^{1/2}))^{1/2},\ \partial\mu/\partial z=\pm8i(2f_1+|u|^2)(P(\mu^{1/2}))^{1/2}. \qquad (13)$$

Hence it follows that the point μ moves along an elliptic Riemann surface (w, λ) defined by the equation $w^2=P(\lambda^{1/2})$. The usual procedure (for example, integration of the NSE in Refs. 14 and 15) involves integration of the system (13) for μ subject to the initial conditions selected so as to satisfy the equality (12). However, we can readily avoid this additional condition in our one-band case if right from the beginning we consider the motion of μ only along such paths which satisfy always the equality (12). (This method of deriving workable expressions for one-band periodic solutions of the NSE was proposed in Ref. 16.) In fact, equating the coefficients of like powers of λ in Eq. (12), we obtain

$$s_1=2f_1+I,\ s_2=f_1^2+2f_2+I(\mu+\mu^*),$$
$$s_3=2f_1f_2+I\mu\mu^*,\ s_4=f_2^2, \qquad (14)$$

where I denotes the square of the modulus of the field $I=|u|^2$. Hence it follows that the natural parameter for specifying the path of μ is the variable I. Solving the system (14) for μ, we obtain

$$\mu=[4s_2\pm8(s_4)^{1/2}-(I-s_1)^2+i(-R(I))^{1/2}]/8I, \qquad (15)$$

where $R(I)$ is a fourth-degree polynomial in I:

$$R(I)=[4s_2\pm8(s_4)^{1/2}-(I-s_1)^2]^2+64I[\pm(s_4)^{1/2}(I-s_1)-s_3]. \qquad (16)$$

In the above expressions $(s_4)^{1/2}$ is understood to mean $(s_4)^{1/2}=\lambda_1\lambda_2\lambda_3\lambda_4$. The polynomials $R(I)$ will be called "resolvents" of the polynomial $P(\lambda)$, because their zeros are related to the zeros of $P(\lambda)$ by the following simple symmetric expression:[1)] the upper signs in (16) correspond to the zeros

$$I_1=(\lambda_1+\lambda_2+\lambda_3-\lambda_4)^2,\ I_2=(\lambda_1+\lambda_2-\lambda_3+\lambda_4)^2,$$
$$I_3=(\lambda_1-\lambda_2+\lambda_3+\lambda_4)^2,\ I_4=(-\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2, \qquad (17)$$

and the lower signs in Eq. (16) correspond to the zeros of the resolvent

$$I_1=(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2,\ I_2=(\lambda_1+\lambda_2-\lambda_3-\lambda_4)^2,$$
$$I_3=(\lambda_1-\lambda_2+\lambda_3-\lambda_4)^2,\ I_4=(-\lambda_1+\lambda_2+\lambda_3-\lambda_4)^2. \qquad (18)$$

The above transition to the sixth-degree polynomial $P(\lambda)$ is obtained if one of the values λ_i is assumed to vanish.

It follows from Eqs. (13) and (14) that μ depends only on the phase

$$W=t+2s_1z,\ d\mu/dW=\pm4i(P(\mu^{1/2}))^{1/2}=\pm4if(\mu^{1/2}). \qquad (19)$$

The convenience of introduction of the resolvent $R(I)$ becomes particularly obvious if we go over from the equation for μ to the equation for I. Differentiating $P(\mu^{1/2})=f^2(\mu^{1/2})$ with respect to I, we obtain

$$\frac{dP(\mu^{1/2})}{d\mu}\frac{d\mu}{dI}=2f(\mu^{1/2})\left[2\mu\frac{d\mu}{dI}+\frac{\mu}{2}+\frac{I-s_1}{2}\frac{d\mu}{dI}\right],$$

which yields the expression for the derivative

$$\frac{dI}{d\mu}=\frac{i(-R(I))^{1/2}}{4f(\mu^{1/2})}. \qquad (20)$$

Multiplying Eqs. (19) and (20), we find that the intensity I satisfies

$$\frac{dI}{dW}=(-R(I))^{1/2}. \qquad (21)$$

This equation is readily solved using elliptic functions. If I is known, then the system (11) readily yields $u(t, z)$. Using Eq. (14), we find then that

$$u(t, z)=\exp(16i(s_4)^{1/2}z)\tilde{u}(W), \qquad (22)$$

where $\tilde{u}(W)$ is obtained from the equation

$$d\tilde{u}/dW=4i\left(-\frac{s_1}{2}+\frac{I}{2}+\mu\right)\tilde{u}.$$

Substituting here Eq. (15) and using Eq. (21), we find that

$$\frac{d\ln\tilde{u}}{dW}={}^1/_2\frac{d\ln I}{dW}+4i\left[-\frac{s_1}{4}+\frac{3}{8}I+\frac{(I_1I_2I_3I_4)^{1/2}}{8}\frac{1}{I}\right], \qquad (23)$$

where use is also made of the identity

$$(4s_2\pm8(s_4)^{1/2}-s_1^2)^2=I_1I_2I_3I_4,$$

which follows from Eq. (16).

The parameters characterizing the solutions need not satisfy any additional requirements apart from the obvious condition that I be real and positive. It follows from Eqs. (17) and (18) that this condition is satisfied, for example, if all the quantities λ_i are real, so that all the intensities I_i are real and greater than zero; if two zeros of λ_i are complex-conjugate and the other two are real, then the two values of I_i are real and positive; if λ_i split into two complex-conjugate pairs, then again the two values of I_i are real and positive. We shall first give the appropriate expressions for I.

If all λ_i are real, we shall renumber them to satisfy the inequalities $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant 0$, so that in both cases of Eqs. (17) and (18) we have $I_1 \geqslant I_2 \geqslant I_3 \geqslant I_4 \geqslant 0$. The intensity I can vary within the limits $I_1 \geqslant I \geqslant I_2$ and $I_3 \geqslant I \geqslant I_4$, where the resolvent $R(I) = \Pi(I - I_i)$ is negative. Integration of Eq. (21) is carried out using standard expressions (see, for example, Ref. 17) and yields

$$I=\frac{I_1(I_2-I_4)+I_4(I_1-I_2)\,\mathrm{sn}^2(2\theta,k)}{I_2-I_4+(I_1-I_2)\,\mathrm{sn}^2(2\theta,k)}, \quad I_1\geqslant I\geqslant I_2 \qquad (24)$$

or

$$I=\frac{I_4(I_1-I_3)+I_1(I_3-I_4)\,\mathrm{sn}^2(2\theta,k)}{I_1-I_3+(I_3-I_4)\,\mathrm{sn}^2(2\theta,k)}, \quad I_3\geqslant I\geqslant I_4, \qquad (25)$$

where

$$\begin{aligned}\theta&=[(I_1-I_3)(I_2-I_4)]^{1/2}(W-W_0)/4,\\ W&=t+2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)z,\\ k^2&=\frac{(I_1-I_2)(I_3-I_4)}{(I_1-I_3)(I_2-I_4)}=\frac{(\lambda_1^2-\lambda_2^2)(\lambda_3^2-\lambda_4^2)}{(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)}.\end{aligned} \qquad (26)$$

where W_0 is the value of the phase $z=0$ when $t=0$. If, however,

$$\lambda_1=\alpha+i\beta,\ \lambda_2=\alpha-i\beta,\ \lambda_3=\gamma-i\delta,\ \lambda_4=\gamma+i\delta, \qquad (27)$$

the equations in the system (18) give the following values for the zeros of the resolvent

$$\begin{aligned}I_1&=4(\alpha+\gamma)^2,\ I_2=4(\alpha-\gamma)^2,\\ I_3&=-4(\beta-\delta)^2,\quad I_4=-4(\beta+\delta)^2.\end{aligned} \qquad (28)$$

The intensity I can oscillate only in the interval $I_1 \geqslant I \geqslant I_2$ and is described by Eq. (24). One possible case, when two zeros of λ_i are complex-conjugate and the other two are real, leads to more complex expressions, but we shall not use them and not write them down.

Explicit expressions for the periodic NSED solutions can be derived conveniently using the Weierstrass elliptic functions. We shall therefore introduce zeros of the corresponding Weierstrass cubic polynomial (see Ref. 19):

$$\begin{aligned}e_1&=[2(I_1-I_3)(I_2-I_4)-(I_1-I_2)(I_3-I_4)]/12,\\ e_2&=[2(I_1-I_2)(I_3-I_4)-(I_1-I_3)(I_2-I_4)]/12,\\ e_3&=-[(I_1-I_2)(I_3-I_4)+(I_1-I_3)(I_2-I_4)]/12.\end{aligned} \qquad (29)$$

The familiar relationship between the Jacobi elliptic sine and the Weierstrass $\wp$ function can be used to transform Eqs. (24) and (25) to the unified form

$$I=I_0\frac{\wp(W-W_0)-\wp(\rho)}{\wp(W-W_0)-\wp(\varkappa)}, \qquad (30)$$

where I_0 is the initial intensity when $W = W_0$. We shall simplify the expressions by assuming that $W_0 = 0$. Then, Eq. (24) corresponds to $I_0 = I_1$ and the parameters $\varkappa$ and ρ are defined by

$$\begin{aligned}\wp(\varkappa)&=e_3-(I_1-I_2)(I_1-I_3)/4,\\ \wp(\rho)&=e_3-I_4(I_1-I_2)(I_1-I_3)/4I_1,\end{aligned} \qquad (31)$$

whereas in Eq. (25) we have $I_0 = I_4$ and $\varkappa$ and ρ are given by

$$\begin{aligned}\wp(\varkappa)&=e_3-(I_2-I_4)(I_3-I_4)/4,\\ \wp(\rho)&=e_3-I_1(I_2-I_4)(I_3-I_4)/4I_4.\end{aligned} \qquad (32)$$

Substituting Eq. (30) into Eq. (23), and integrating this equation employing standard expressions from the theory of elliptic functions, we obtain the following expression for the periodic solution of the NSED (2):

$$\begin{aligned}u(t,z)=\exp\{&i(-s_1+3I_0/2+(I_1I_2I_3I_4)^{1/4}/8I_0)W\\ &+(3\zeta(\varkappa)-\zeta(\rho))W\\ &+16i(s_4)^{1/2}z\}\,(I_0)^{1/2}\frac{\sigma(\varkappa)\sigma(W+\rho)\sigma(W-\varkappa)}{\sigma(\rho)\sigma^2(W+\varkappa)},\end{aligned} \qquad (33)$$

where ζ and σ are the Weierstrass functions.

It is of interest to investigate the soliton limit of these solutions when $I_2 = I_3$, so that the modulus of the elliptic functions of Eq. (26) is $k = 1$ and we have

$$\begin{aligned}e_1&=e_2=a=(I_1-I_2)(I_2-I_4)/12,\\ e_3&=-2a=-(I_1-I_2)(I_2-I_4)/6.\end{aligned} \qquad (34)$$

Introducing the notation

$$2\theta=(3a)^{1/2}W,\ \cos^2(\Gamma/2)=(I_2-I_4)/(I_1-I_4), \qquad (35)$$

we find that the solution (33) with $I_0 = I_1$ and with the parameters $\varkappa$ and ρ given by Eq. (31) is readily reduced to

$$\begin{aligned}u(t,z)=&\tfrac{1}{2}\exp\{i(-s_1+3I_2/2)W-(-I_1I_4)^{1/4}W/2+16i(s_4)^{1/2}z\}\\ &\times\frac{\mathrm{ch}(2\theta+i\Gamma/2)}{\mathrm{ch}(2\theta-i\Gamma/2)}\left[I_1^{1/2}+I_4^{1/2}+(I_1^{1/2}-I_4^{1/2})\frac{\mathrm{ch}(2\theta+i\Gamma/2)}{\mathrm{ch}(2\theta-i\Gamma/2)}\right].\end{aligned} \qquad (36)$$

We shall consider two characteristic special cases of this solution.

1. We shall first assume that $\lambda_1 = \lambda_4 = \alpha + i\beta$ and $\sigma_2 = \lambda_3 = \alpha - i\beta$, so that

$$I_1=16\alpha^2,\ I_2=I_3=0,\ I_4=-16\beta^2,\ \cos^2(\Gamma/2)=\beta^2/(\alpha^2+\beta^2).$$

The above expression suggests the parametrization

$$\alpha=\Delta\sin(\Gamma/2),\ \beta=\Delta\cos(\Gamma/2).$$

Substitution of these two parametric expressions into Eq. (36) yields the soliton solution

$$u(t,z)=4\Delta\sin\Gamma\frac{e^{2i\Phi}}{e^{2\theta}+e^{-2\theta+i\Gamma}}\frac{e^{4\theta}+e^{-i\Gamma}}{e^{4\theta}+e^{i\Gamma}}, \qquad (37)$$

where

$$\begin{aligned}\Phi&=2t\Delta^2\cos\Gamma-8z\Delta^4\cos2\Gamma,\\ \theta&=2t\Delta^2\sin\Gamma-8z\Delta^4\sin2\Gamma.\end{aligned}$$

The above differs only in notation from the soliton solution given in Ref. 13.

2. Let us now assume that all values of λ_i are real and that

$$\lambda_1=(\alpha+\beta)/2,\ \lambda_2=\lambda_3=\beta/2,\ \lambda_4=-(\alpha-\beta)/2, \qquad (38)$$

so that

$$I_1=4\beta^2,\ I_2=I_3=\alpha^2,\ I_4=0,\ \cos^2(\Gamma/2)=\alpha^2/(4\beta^2).$$

Substitution of these expressions in Eq. (36) yields

$$u(t,z)=\alpha e^{i\Phi}\frac{\mathrm{ch}\,2\theta\,\mathrm{ch}(2\theta+i\Gamma/2)}{\mathrm{ch}^2(2\theta-i\Gamma/2)}, \tag{39}$$

where

$$\Phi=(\alpha^2+\beta^2)t+[(\alpha^2+\beta^2)^2-4\beta^4]z,$$
$$\theta=\alpha(4\beta^2-\alpha^2)^{1/2}[t+(\alpha^2+2\beta^2)z]/4. \tag{40}$$

This is a "bright" soliton appears against a background of a constant pedestal, as can be seen particularly readily from the expression for the square of the modulus of the field

$$I=|u(t,z)|^2=4\alpha^2\beta^2/[\alpha^2+(4\beta^2-\alpha^2)\mathrm{th}^2\,2\theta], \tag{41}$$

which naturally is identical with the corresponding limit of Eq. (24).

The soliton limit of the solution (33) with $I_0=I_4$ can be considered in a fully analogous manner. We shall give only the final result for the case of the distribution of zeros in accordance with Eq. (38):

$$u(t,z)=i\alpha e^{i\Phi}\frac{\mathrm{sh}\,2\theta\,\mathrm{ch}(2\theta-i\Gamma/2)}{\mathrm{ch}^2(2\theta+i\Gamma/2)}, \tag{42}$$

where $\sin^2(\Gamma/2)=\alpha^2/(4\beta^2)$, whereas Φ and θ are given by Eq. (40). The square of the modulus of the field is [see also Eq. (25)]

$$I=|u(t,z)|^2=4\alpha^2\beta^2/[\alpha^2+(4\beta^2-\alpha^2)\mathrm{cth}^2\,2\theta]. \tag{43}$$

Obviously, Eqs. (42) and (43) describe a "dark" soliton against a constant pedestal. Solutions of this type have already been obtained by numerical integration of the NSED (Ref. 4).

An important qualitative circumstance revealed by our analysis and distinguishing the solutions of the NSED from the solutions of the NSE is the existence of two solutions described by Eqs. (24) and (25) for the same set of zeros λ_i of the polynomial $P(\lambda)$. It follows that at an inhomogeneous solution point where $I_2=I_3$ we can match a solution of the type given by Eq. (25) characterized by smaller oscillations of the amplitude to a solution of the Eq. (24) type corresponding to large oscillations of the amplitude. For a qualitative description of the inhomogeneous solutions we have to derive the Whitham equations governing the evolution of the parameters λ_i.

3. WHITHAM EQUATIONS AND THEIR SELF-SIMILAR SOLUTION

The inverse scattering method has led to the discovery that it is in fact the most effective method for obtaining the Whitham modulation equations directly in the diagonal Riemann form.[18] This made it possible to derive Whitham equations not only for the Korteweg–de Vries equations [11,18] but also for the sine-Gordon equation,[20] the nonlinear Schrödinger equation,[21] and equations describing the self-induced transparency.[22] Integrable equations follow from the compatibility conditions

$$\partial A/\partial t-G(qC-rB)=0,$$
$$G\partial q/\partial z-\partial B/\partial t-2FB-2GqA=0, \tag{44}$$
$$G\partial r/\partial z-\partial C/\partial t+2FC+2GrA=0$$

of the linear systems (3) and (4), and the corresponding Whitham equations can be obtained by the following simple procedure. Using Eqs. (44) and (7), we can easily show that

$$\frac{\partial}{\partial z}\left(\frac{Gq}{g}\right)-\frac{\partial}{\partial t}\left(\frac{B}{g}\right)=0,\quad \frac{\partial}{\partial z}\left(\frac{Gr}{h}\right)-\frac{\partial}{\partial t}\left(\frac{C}{h}\right)=0, \tag{45}$$

which are generating functions of an infinite sequence of conservation laws. Their averaging gives rise to a generating function for the Whitham equations. Substituting Eqs. (5) and (10) in the first expression in Eq. (45) and averaging this first expression over the wave period, we obtain

$$\frac{\partial}{\partial z}\left[(P(\lambda))^{1/2}\left\langle\frac{1}{\lambda^2-\mu}\right\rangle\right]-\frac{\partial}{\partial t}\left[(P(\lambda))^{1/2}\left(4+2s_1\left\langle\frac{1}{\lambda^2-\mu}\right\rangle\right)\right]=0, \tag{46}$$

where we use the normalization $f^2-gh=1$ (see Ref. 18) and average in accordance with the rule

$$\left\langle\frac{1}{\lambda^2-\mu}\right\rangle=\frac{1}{T}\int\frac{dW}{\lambda^2-\mu}=\frac{1}{4T}\int\frac{d\mu}{(\lambda^2-\mu)(-P(\mu^{1/2}))^{1/2}},$$
$$T=\int dW={}^1/_4\int\frac{d\mu}{(-P(\mu^{1/2}))^{1/2}}=\frac{K(k)}{[(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_4^2)]^{1/2}}. \tag{47}$$

where $K(k)$ is a complete elliptic integral of the first kind, whereas k is defined by Eq. (26). Integration in Eq. (47) is along the plots of Eq. (15). We can easily show that these plots surround the cuts between λ_1^2 and λ_2^2 or between λ_3^2 and λ_4^2. In view of their topological equivalence, the result of integration is the same in both cases. Equation (46) has singularities of λ^2 at $\lambda^2=\lambda_i^2$ and the condition of vanishing of the coefficients in front of the singular terms, which result from differentiation of $[P(\lambda)]^{1/2}$ with respect to z and t, yields the following equation for λ_i:

$$\frac{\partial\lambda_i}{\partial z}+\frac{1}{v_i}\frac{\partial\lambda_i}{\partial t}=0, \tag{48}$$

where

$$1/v_i=-2s_1-4\Big/\left\langle\frac{1}{\lambda_i^2-\mu}\right\rangle.$$

The average values in the above expression can be calculated using the self-evident formula

$$\left\langle\frac{1}{\lambda_i^2-\mu}\right\rangle=-\frac{2}{T}\frac{dT}{d\lambda_i^2}=-2\frac{d\ln T}{d\lambda_i^2},$$

so that the group velocities v_i are described by

$$\frac{1}{v_1}=-2\sum\lambda_i^2+\frac{4(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_4^2)K(k)}{(\lambda_2^2-\lambda_4^2)E(k)-(\lambda_1^2-\lambda_4^2)K(k)},$$
$$\frac{1}{v_2}=-2\sum\lambda_i^2-\frac{4(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)K(k)}{(\lambda_1^2-\lambda_3^2)E(k)-(\lambda_2^2-\lambda_3^2)K(k)},$$
$$\frac{1}{v_3}=-2\sum\lambda_i^2+\frac{4(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_4^2)K(k)}{(\lambda_2^2-\lambda_4^2)E(k)-(\lambda_2^2-\lambda_3^2)K(k)},$$
$$\frac{1}{v_4}=-2\sum\lambda_i^2-\frac{4(\lambda_1^2-\lambda_4^2)(\lambda_3^2-\lambda_4^2)K(k)}{(\lambda_1^2-\lambda_3^2)E(k)-(\lambda_1^2-\lambda_4^2)K(k)}. \tag{49}$$

Equations (48) and (49) constitute the required system of Whitham modulation equations for the parameters λ_i governing the periodic solution of the NSED.

We shall apply these equations by considering the problem of evolution of the oscillatory region after "reversal" of a wave.[23,9] We shall assume that at the instant of reversal the wave profile is close to a "step," and that the region of transition from a higher intensity $I_1 = I_a$ to a lower intensity $I_4 = I_c$ is very narrow, so that its initial value can be ignored after a sufficiently long time when the asymptotic Whitham method becomes valid and we can use then the self-similar solutions of the system of equations (48). We shall assume that the intensity at the point of reversal, where the derivative becomes infinite, is I_b. Then, the oscillatory region splits into two parts and in one of them there is a transition of the average value $\langle I \rangle$ from I_a to I_b, whereas in the other part the transition is from I_b to I_c with the aid of solutions of the (24) and (25) type, respectively, where the parameters λ_i depend on the self-similar variable $\tau = z/t$. This dependence is given by the system of equations (48) which in the self-similar case become

$$(v_i-\tau)\, d\lambda_i/d\tau=0. \qquad (50)$$

The intensities found above set the limiting values of the parameters λ_i when use is made of the relationships

$$I_a=(\lambda_a+\lambda_b)^2,\quad I_b=(\lambda_a+\lambda_c)^2,\quad I_c=(\lambda_b+\lambda_c)^2,$$

so that in the region $I_a \geqslant \langle I \rangle \geqslant I_b$ only the parameter λ_3 changes from λ_c at the point z_a characterized by $\langle I \rangle = I_a$ to the value λ_b at the point z_b with $\langle I \rangle I_b$, whereas in the range $I_b \geqslant \langle I \rangle \geqslant I_c$ there is only a change in the parameter λ_2 from λ_b at the point z_b characterized by $\langle I \rangle = I_b$ to λ_c at the point z_c with $\langle I \rangle = I_c$. Therefore, the solution of the system (50) is

$$\lambda_1=\text{const}=\lambda_a,\quad \lambda_4=\text{const}=\lambda_c,$$

$$z/t=v_3,\quad k^2=\frac{(\lambda_a^2-\lambda_b^2)(\lambda_3^2-\lambda_c^2)}{(\lambda_a^2-\lambda_3^2)(\lambda_b^2-\lambda_c^2)},\quad z_a \geqslant z \geqslant z_b,$$

$$z/t=v_2,\quad k^2=\frac{(\lambda_a^2-\lambda_2^2)(\lambda_b^2-\lambda_c^2)}{(\lambda_a^2-\lambda_b^2)(\lambda_2^2-\lambda_c^2)},\quad z_b \geqslant z \geqslant z_c.$$

The above expressions describe implicitly the dependence of λ_i on z/t; after substitution of λ_i into Eqs. (17), (24), and (25) we obtain the values of the intensity I in the oscillatory region. The corresponding substitutions of the parameters in Eq. (33) give the expressions for the field amplitude in the oscillatory region. We shall also give the velocities at the boundary points z_a, z_b, and z_c which are readily obtained from Eqs. (51) and (49):

$$\frac{1}{v_a}=-8\lambda_c^2+\frac{2(\lambda_a^2-\lambda_b^2)^2}{2\lambda_c^2-\lambda_a^2-\lambda_b^2},$$

$$\frac{1}{v_b}=-2(\lambda_a^2+2\lambda_b^2+\lambda_c^2),$$

$$\frac{1}{v_c}=-8\lambda_a^2+\frac{2(\lambda_b^2-\lambda_c^2)^2}{2\lambda_a^2-\lambda_b^2-\lambda_c^2}.$$

The solution obtained provides a quantitative explanation of the characteristic pattern of oscillations of the field in the oscillatory region. In the vicinity of the point z_b these oscillations have a large amplitude which decreases away from z_b in either direction, so that the oscillations disappear completely at the boundary points z_a and z_c.

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[1)] In the case of the nonlinear Schrödinger equation the polynomial $P(\lambda)$ is of the fourth degree and the similarly derived polynomial $R(I)$ is its cubic resolvent (see Ref. 16).

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