Universal two-flux spectra of weak acoustic turbulence

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The structural instability of the isotropic Kolmogorov spectrum of weak wave turbulence is discussed. The universal two-flux spectra of weak acoustic turbulence, transporting fluxes of both conserved quantities (energy and momentum), are derived.

The universality hypothesis in the theory of developed turbulence is usually formulated as follows: in the interval of scales intermediate between the source and the sink, the turbulence is isotropic and the energy distribution over the scales depends on a single external parameter—the energy flux in **k**-space.[1,2] In accordance with this hypothesis isotropic one-parameter spectra (usually called Kolmogorov spectra) have been obtained for both hydrodynamic eddy turbulence[1-3] and weak wave turbulence in hydrodynamics, plasma physics, and acoustics.[4] It should be pointed out, however, that in all cases the interaction of both waves and eddies conserves, aside from the energy, the total momentum. Any real source of turbulence is, however, anisotropic and asymmetric, which causes the system to have a nonzero momentum.

Stationary corrections δn_k, transporting a small momentum flux $\mathbf{R}$, to weakly-turbulent Kolmogorov solutions n_k, carrying an energy flux P, were constructed by Kats and Kontorovich.[5] For waves with power-law dispersion $\omega \propto k^\alpha$ these so-called drift corrections have the simple form

$$\delta n_{\mathbf{k}}/n_k \propto (\mathbf{Rk})\,\omega_k/(Pk^2) \propto \cos\theta_k(\omega_k/k) \propto k^{\alpha-1}\cos\theta_k, \qquad (1)$$

determined by the fact that $y = (\mathbf{Rk})\omega_k/(Pk^2)$ is the only dimensionless parameter which can be constructed from the quantities under study.

In the case of decay dispersion laws we have $\alpha > 1$ and the relative contribution of the anisotropic part of the spectrum increases as k increases, i.e., in a direction from the source of waves into the inertial interval. In the nondecay situation an analogous effect arises for Kolmogorov solutions transferring a flux of wave action Q into the long-wavelength region. In this case the relative contribution of the drift correction

$$\delta n_{\mathbf{k}}/n_k \propto (\mathbf{Rk})/(Qk^2) \propto \cos\theta k^{-1} \qquad (2)$$

also increases in the direction into the interior of the inertial interval, i.e., towards small values of k.

In accordance with the results obtained by Kats and Kontorovich the drift corrections cause the collision integral for waves, which is linearized with respect to an isotropic Kolmogorov solution, to vanish. The following question arises: are the solutions 1 and 2 established when the source of waves is anisotropic?

Fal'kovich and Shafarenko[6] and Zakharov and Balk[7] found that the drift corrections are established against the background of spectra with an energy flow for capillary and sound waves and are not established for spectra of gravity waves with an action flux. Starting from the general criterion derived by Zakharov and Balk for the stability of isotropic Kolmogorov spectra the following theorem can be proved: drift solutions can be established only in the case when the momentum flux transported by them in the scale space is directed in the same direction as the flux of the main integral of motion.[8] Indeed, it is easy to verify by direct calculation that in the three cases mentioned above the momentum flux is directed toward large values of k, in the same direction as the energy flux for capillary and sound waves and in a direction opposite to the action flux for gravity waves.

Thus there exist cases when drift corrections are established and their relative contribution to the spectrum increases as the distance away from the source in k-space increases. Therefore the isotropic solution becomes structurally unstable: a small anisotropy of the source leads to a substantially anisotropic stationary distribution of waves in the inertial interval. The existence of structural instability of the isotropic Kolmogorov solution indicates that the simplest universality is absent—one parameter (the energy flux P or the action flux Q) is not sufficient to obtain a stationary distribution. How many parameters *are* necessary?

The answer to this question is probably least obvious for turbulence of sound waves with small positive dispersion $\omega_k \propto k^{1+\varepsilon}$, $\varepsilon \ll 1$. In this case, as shown in Refs. 6 and 7, in the presence of a small anisotropy of the source not only the drift solutions (1) which transport a momentum flux but also the stationary anisotropic corrections constructed by L'vov and Fal'kovich[9], corresponding to angular harmonics of highest order

$$\delta n_k/n_k \propto P_m(\cos\theta)\,k^{\varepsilon m(m+1)/2}, \qquad \varepsilon m^2 < 1 \qquad (3)$$

are established in the inertial interval. Here P_m are Legendre polynomials. For $m = 1$ the expression (3) is identical to the expression (1).

As one can see from the expression (3) the higher the number of the harmonic the more rapidly its contribution to the spectrum increases as k increases. Thus the turbulence of weakly dispersed waves is an example of the structurally most unstable system—for this system the largest number of angular harmonics is excited by a weakly anisotropic source ($m \approx \varepsilon^{-1/2}$). For the remaining systems studied (see Ref. 7) harmonics with numbers greater than unity are not excited. A set of anisotropic stationary solutions of the kinetic equation was also constructed by L'vov and Fal'kovich[9] and Fal'kovich[10] for weakly dispersed waves:

$$n_m(k) = k^{-9/2+\varepsilon m(m+1)/4} P_m^{1/2}(\cos\theta). \qquad (4)$$

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These solutions were constructed within the framework of the differential approximation in angular variables and are valid only in the regions where $P_m(\cos\theta) > 0$. They describe the set of noninteracting jets with constant (with respect to k) angular width of order π/m.

The existence of the solutions (4) and (3) led previously to the assumption that in the region of large k the angular shape of the spectrum can be irregular with a scale of the order of the small angle of interaction [see the relations (7) below]. This would indicate, essentially, the absence of any universality of the spectrum of turbulence in the inertial interval. As we shall now show, for acoustic turbulence with positive dispersion a universal stationary solution of the kinetic equation, depending on two fluxes of conserved quantities (energy and momentum), can be constructed analytically. Of course, this solution will be anisotropic.

The kinetic equation for weakly dispersed waves has the following form[4]:

$$\frac{\partial n_k}{\partial t}=\int kk_1|\mathbf{k}-\mathbf{k}_1|[\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}-\mathbf{k}_1})\times(n_{\mathbf{k}_1}n_{\mathbf{k}-\mathbf{k}_1}-n_{\mathbf{k}}n_{\mathbf{k}_1}-n_{\mathbf{k}}n_{\mathbf{k}-\mathbf{k}_1})-2\delta(\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_1-\mathbf{k}}-\omega_{\mathbf{k}})(n_{\mathbf{k}}n_{\mathbf{k}_1-\mathbf{k}}-n_{\mathbf{k}_1}n_{\mathbf{k}_1-\mathbf{k}}-n_{\mathbf{k}_1}n_{\mathbf{k}})]d\mathbf{k}_1. \tag{5}$$

Assuming that the dispersion law is scale invariant $\omega_k \propto k^{1+\varepsilon}$ and is virtually identical to the acoustic spectrum ($\varepsilon \ll 1$), we integrate (5) over the polar angle and obtain the following equation in the stationary case ($x = k_1/k$):

$$\int_0^1 (1-x^{1+\varepsilon})^{(2-\varepsilon)/(1+\varepsilon)}x^2(n_1n_2-n_1n_k-n_2n_k)\,dx\,d\varphi-2\int_1^\infty (x^{1+\varepsilon}-1)^{(2-\varepsilon)/(1+\varepsilon)}x^2(n_kn_2-n_1n_2-n_1n_k)\,dx\,d\varphi=0. \tag{6}$$

Here $n_i = n[k_i, \theta_{kk_i}(k_1,\varphi)]$ must be taken, in accordance with the δ function in Eq. (5), on the surface given by the relations

$$\cos\theta_{kk_1}=[1+x^2-(1-x^{1+\varepsilon})^{2/(1+\varepsilon)}]/(2x)\approx 1+\varepsilon[x\ln x+(1-x)\ln(1-x)](1-x)/x \tag{7a}$$

in the first integral and

$$\cos\theta_{kk_1}=[(x^{1+\varepsilon}-1)^{2/(1+\varepsilon)}-1-x^2]/(2x)\approx 1-\varepsilon[x\ln x-(x-1)\ln(x-1)](x-1)/x \tag{7b}$$

in the second integral. As regards $n_2 = n(k_2, \theta_{kk_2})$, we have $\theta_{kk_2} = -x\theta_{kk_1}/(1-x)$. As one can see, the angles between the interacting waves are small, with the exception of narrow regions near $x = 0$ and 1 and also in the limit $x \to \infty$. However because the integrals in the kinetic equation converge [see Eq. (9) below] these regions do not contribute to the interaction of the waves. We shall seek the axisymmetric solution, depending on k, P, and $(\mathbf{R}\cdot\mathbf{k})$, of Eq. (6). In accordance with the dimensional relation $Pk \propto R\omega_k$ this solution should have the following form:

$$n_k=P^{1/2}k^{-9/2}f[(\mathbf{R}\cdot\mathbf{k})\omega_k/(Pk^2)]=P^{1/2}k^{-9/2}f(y). \tag{8}$$

Here f is an unknown function of the dimensionless parameter y. The weakness of the dispersion makes it possible to employ the differential approximation in the variable y, since

$$y_1=(\mathbf{R}\mathbf{k}_1)\omega_1/(Pk_1^2)=(k_1/k_a)^\varepsilon\cos(\mathbf{k}_1\mathbf{R})\approx(k/k_a)^\varepsilon[\cos(\mathbf{k}\mathbf{R})+\sin(\mathbf{k}\mathbf{R})\sin\varphi\,\theta_{kk_1}-\cos(\mathbf{k}\mathbf{R})\,\theta_{kk_1}^2/2+\varepsilon\cos(\mathbf{k}\mathbf{R})\ln(k_1/k)],$$

i.e., $|y_1 - y| \ll y$. We expand the functions $f(y_1)$ and $f(y_2)$ in (6) up to second order in ε; next, separating the second integral into two identical terms, we make the substitution $x \to x^{-1}$, in one term and the substitution $x \to (1-x)^{-1}$ in the other term (the Zakharov transformation). After this, since $n_k \propto k^{-9/2}$ is an exact solution of Eq. (6), we obtain the following equation to first order in ε:

$$\left[\left(\frac{df}{dy}\right)^2+f\frac{d^2f}{dy^2}\right]\sin^2(\mathbf{k}\mathbf{R})\left(\frac{k}{k_a}\right)^\varepsilon\times\int_0^1 x^2(1-x)^2[x\ln x+(1-x)\ln(1-x)]\times[x^{-9/2}(1-x)^{-9/2}-(1-x)^{-9/2}-x^{-9/2}]\,dx=0. \tag{9}$$

The equation for $f(y)$ appears as a factor in front of a converging integral. The solution is found trivially:

$$f(y)=\begin{cases}(ay+b)^{1/2}, & y>-b/a,\\ 0, & y<-b/a.\end{cases} \tag{10}$$

According to Eqs. (8) and (10) n_k should vanish on some surface in k-space. Indeed as $y \to -b/a$ the derivatives of the function $f(y)$ grow rapidly and the conditions for the differential approximation to be applicable no longer hold in a narrow neighborhood of the surface (with $y + b/a \lesssim \varepsilon^{1/2}$). The solution of the complete equation (6) should lead to a smooth, but rapidly decaying on the scale of the characteristic interaction angle (i.e., $\varepsilon^{1/2}$), function $f(y)$. The function $f(y)$ should converge to zero as $y \to \mathrm{M}\infty$. The constants of integration a and b can be incorporated into the definition of the fluxes R and P. In so doing the constant a should be regarded as positive, since the substitution $a \to -a$ simply means that the coordinate system undergoes the rotation $\theta \to \pi \to \theta$. At the same time the two opposite signs of b give two different families of solutions:

$$n_k=k^{-9/2}[(R\omega_k\cos\theta)/k+P]^{1/2}, \tag{11a}$$

$$n_k=k^{-9/2}[(R\omega_k\cos\theta)/k-P]^{1/2}. \tag{11b}$$

The first of the solutions, Eq. (11a), corresponds to a spectrum that becomes narrower as k increases. In particular, it should describe a stationary distribution, engendered by a weakly anisotropic source with a weak momentum flux ($R\omega_k \ll Pk$). In this case, expanding (11a) in powers of $R\omega_k/(Pk)$, for small k we obtain an isotropic Kolmogorov solution in zeroth order, the drift correction (1) in first order, and higher order harmonics (3), whose relative contribution increases as k increases, in subsequent orders. For large values of k practically all waves are concentrated in the right hand hemisphere. The solution (11b) describes an expanding spectrum. Its width $\Delta\theta(k)$ increases with k according to the law

$$R\omega_k \cos \Delta\theta(k) = Pk.$$

If at the boundary of the inertial interval (at $k = k_0$) we have $R\omega(k_0) \approx Pk_0$, then the starting width of the spectrum $\Delta\theta(k^0)$ can be very small. The quantity $\Delta\theta(k_0)$ is bounded from below only by the angle of interaction $\varepsilon^{1/2}$, because for this width the differential approximation, on the basis of which the solutions (11) were found, becomes meaningless. Thus one would think that the solution (11b) must be engendered by narrow sources, whose width satisfies $0 < \Delta\theta < \pi/2$. They could also include extremely narrow sources with $\Delta\theta$ of the order of or even less than the angle of interaction $\varepsilon^{1/2}$. In particular, a source function that is strongly irregular as a function of the angle can engender a solution which in the intermediate asymptotic range contains a collection of narrow, but expanding jets (11b) which for large values of k merge into one smooth distribution of the type (11). It is significant that in the limit of large k and $-\pi/2 < \theta < \pi/2$ both solutions (11a) and (11b) become identical. The spectrum is determined solely by the momentum flux and consists of one wide stream, whose angular shape does not depend on the form of the boundary conditions:

$$n_k \to R^{1/2} k^{-9/2+\varepsilon/2} (\cos\theta)^{1/2}.$$

In order to determine which stationary solution is established in the inertial interval under the action of a concrete source—the multistream solution (4) or the universal solution (11)—it is necessary to solve a transient problem, which falls outside the scope of this work.

It appears, however, that the existence of the solutions (11) indicates that in the inertial interval the spectrum can have a universal form. The fact that the solutions (11) admit a wide set of boundary conditions for small k (in the region of the source)—ranging from an isotropic collection up to extremely narrow with scale $\varepsilon^{1/2}$—evidently means that the stationary spectrum engendered by an arbitrary source in the region of large k has the universal form (11). Thus the existence of structural instability of the isotropic Kolmogorov solution by no means indicates absence of universality. It simply means that the spectrum in the inertial interval should, generally speaking, depend on the fluxes of all integrals of motion.

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