Macroscopic properties of superfluid atomic hydrogen H↑

E. P. Bashkin and L. A. Kuchuk

Institute of Physical Problems, Academy of Sciences of the USSR

(Submitted 27 June 1989)
Zh. Eksp. Teor. Fiz. **96**, 1714–1727 (May 1989)

The properties of superfluid gaseous H↑ consisting of hydrogen atoms in the states $|a\rangle$ and $|b\rangle$, are analyzed. The presence of two components means that there are two Bose-Einstein condensation points, T_{ca} and T_{cb}. At temperatures in the interval $T_{cb} < T < T_{ca}$ the system is a solution of normal particles of type $|b\rangle$ in a superfluid background of component $|a\rangle$. At $T < T_{cb}$ the properties of H↑ are detemined by the simultaneous presence of two condensates. The energy spectrum of the system and the dynamics of the condensate wave function are derived. A corresponding diagram technique is developed. Thermodynamic functions, the velocities of acoustic vibrations, and the kinetic coefficients of gaseous H↑ are calculated.

1. INTRODUCTION

There has recently been considerable research interest, both theoretical and experimental, in some very diverse properties of spin-polarized atomic hydrogen, H↑. Gaseous H↑ is a unique entity which does not condense into a liquid even at absolute zero, according to the theory. It exhibits a number of macroscopic quantum-mechanical effects, being one of the most striking examples of the so-called quantum gases.[1–3] One of the most important directions in this research field is seeking for superfluidity of gaseous H↑, associated with a Bose condensation of hydrogen atoms as the temperature of the system is lowered. Observing a phase transition at temperatures which are not too low requires fairly high gas densities. Although the gas pressures which have already been achieved at $T \sim 0.1$–0.8 K are extremely high ($N \approx 4.5 \cdot 10^{18}$ cm^{-3}; Refs. 4 and 5), three-particle recombination[6] leaves little hope that it will prove possible to achieve the densities required for realizing superfluidity in this temperature interval. There is accordingly major interest in plans for a variety of magnetic traps and in research on spin-polarized atomic hydrogen at lower temperatures.[7]

To some extent or other, the present study goes in both of these directions. In this paper we examine thermodynamic characteristics, hydrodynamic processes, and kinetic phenomena in spin-polarized atomic hydrogen below the point of the transition to a superfluid state. In addition, we study the effect of the phase transition on the nuclear spin dynamics, in particular, on the spin-wave spectrum. This information may prove extremely important in NMR experiments, for identifying Bose-Einstein condensation in a gas. We work from the circumstance that gaseous H↑ is usually a mixture of atoms which are in two distinct states as a result of a hyperfine interaction:

$$|a\rangle = |\uparrow\not\downarrow\rangle - \eta|\downarrow\not\uparrow\rangle, \quad |b\rangle = |\uparrow\not\uparrow\rangle$$

(η is the hyperfine interaction parameter). As a rule, the density of particles in state $|a\rangle$ is significantly higher than that in state $|b\rangle$; i.e., we have $N_a \gg N_b$ or $x = N_b/N_a \ll 1$. For this reason, the critical temperature of the transition to the superfluid state for component a,

$$T_{ca} \sim \hbar^2 N_a^{2/3}/m$$

(m is the mass of the hydrogen atom), is far higher than the Bose condensate points for the particles in state b,

$$T_{cb} \sim \hbar^2 N_b^{2/3}/m \ll T_{ca}.$$

Here thus exists a temperature interval $T_{cb} < T < T_{ca}$ in which only one component of gaseous H↑ is superfluid. The particles of the other component form a normal (nonsuperfluid) system, which is dissolved in a superfluid gas of hydrogen atoms in state $|a\rangle$. In this situation, we can call up the entire arsenal of the phenomenological theory of superfluid solutions of quantum fluids (Ref. 9, for example) to describe the properties of H↑. On the other hand, since we are dealing with a low-density Bose fluid, i.e., since we are actually dealing with a slightly nonideal Bose gas, we can use the Bogolyubov method[10] for a systematic microscopic calculation of all the phenomenological parameters, including the energy of the ground state.

In the interval $T < T_{cb} < T_{ca}$, gaseous H↑ is a system with two Bose condensates, so a corresponding phenomenological description is required. In particular, low-frequency vibrational processes in a gas of this sort are described by the equations of three-velocity hydrodynamics with two "superfluid" velocities and one "normal" velocity.[11] Again in this case, it is a straightforward matter to work from first principles to find the energy of the ground state, to find the spectrum of elementary excitations, and to express all the macroscopic observable quantities in terms of the scattering length for the s-wave scattering of hydrogen atoms by each other. This length is $a = 0.72$ Å.

2. SPECTRUM OF EXCITATIONS; THERMODYNAMICS

We will treat gaseous H↑ as a two-component system of quantum-mechanically distinct particles of species a and b, i.e., as a mixture of hydrogen atoms in states $|a\rangle$ and $|b\rangle$. The Hamiltonian of this binary system can be written in the standard form

$$\begin{aligned} \hat{H} &= \hat{H}_{aa} + \hat{H}_{bb} + \hat{H}_{ab}, \\ \hat{H}_{ii} &= -\frac{\hbar^2}{2m}\int \hat{\Psi}_i^+(\mathbf{r},t)\,\nabla^2\Psi_i(\mathbf{r},t)\,d\mathbf{r} \\ &\quad + \frac{1}{2}\int \hat{\Psi}_i^+(\mathbf{r},t)\,\hat{\Psi}_i^+(\mathbf{r}',t)\,U_{ii}(\mathbf{r}-\mathbf{r}') \\ &\quad \cdot\hat{\Psi}_i(\mathbf{r}',t)\,\hat{\Psi}_i(\mathbf{r},t)\,d\mathbf{r}\,d\mathbf{r}', \\ \hat{H}_{ab} &= \int \hat{\Psi}_a^+(\mathbf{r},t)\,\hat{\Psi}_b^+(\mathbf{r}',t)\,U_{ab}(\mathbf{r}-\mathbf{r}')\,\hat{\Psi}_b(\mathbf{r}',t)\,\hat{\Psi}_a(\mathbf{r},t)\,d\mathbf{r}\,d\mathbf{r}'. \end{aligned} \tag{2.1}$$

Here $\hat{\Psi}_i^+$ and $\hat{\Psi}_i$ are field operators subject to the standard Bose commutation relations:

0038-5646/89/110968-07$04.00

$$\hat{\Psi}_i(\mathbf{r},t)\,\hat{\Psi}_k^+(\mathbf{r}',t)-\hat{\Psi}_k^+(\mathbf{r}',t)\,\hat{\Psi}_i(\mathbf{r},t)=\delta_{ik}\delta(\mathbf{r}-\mathbf{r}'),$$

$$\hat{\Psi}_i(\mathbf{r},t)\,\hat{\Psi}_k(\mathbf{r},t)-\hat{\Psi}_k(\mathbf{r},t)\,\hat{\Psi}_i(\mathbf{r},t)=0. \qquad (2.2)$$

Gaseous atomic hydrogen is studied at fairly low temperatures, which definitely satisfy the condition $T\ll\hbar^2/mr_0^2$, where r_0 is the range of the interaction between particles. For this reason we are dealing with a system of slow particles, $pr_0/\hbar\ll 1$, for which the interaction is dominated by long-wave s-wave scattering, and which is amenable to an exact microscopic description in the gas approximation $Nr_0^3\ll 1$ (Refs. 2, 10, and 12).

We first consider the case in which the Bose condensation occurs in only one component of the system, i.e., the case with $T_{cb}<T<T_{ca}$, where

$$T_{ci}=\left[\frac{2^{1/2}\pi^2}{\Gamma(^3/_2)\zeta(^3/_2)}\right]^{2/3}\frac{\hbar^2N_i^{2/3}}{m},\quad i=a,b. \qquad (2.3)$$

Here $\zeta(x)$ is the Riemann zeta function and $\Gamma(x)$ is the gamma function. Switching to operators which create and annihilate particles for the a and b components in (2.1), and taking into account the effective short range, $pr_0\ll\hbar$, we write the Hamiltonian in the form

$$\hat{H}=\hat{H}(a)+\hat{H}(b)+u_{ab}\sum_p\hat{a}^+_{p1}\hat{a}_{p2}\hat{b}^+_{p3}\hat{b}_l \qquad (2.4)$$

or

$$\hat{H}(c)=\sum_p\frac{p^2}{2m}\hat{c}_p{}^+\hat{c}_p+\frac{1}{2}u_{cc}\sum_p\hat{c}^+_{p1}\hat{c}_{p2}\hat{c}_{p3}\hat{c}_{p4}.$$

Since the hyperfine effects are small, all the binary-interaction potentials can very accurately be assumed to be identical, regardless of the particular species of the particles involved in the scattering: $u_{aa}=u_{ab}=u_{bb}=u$. If we are not interested in the corrections to the ground-state energy of higher order in the density, proportional to $N^{5/2}$, we can express u in terms of the scattering length a in a very simple way:

$$2g=u=4\pi\hbar^2a/m,\quad a>0. \qquad (2.5)$$

Using next the standard Bogolyubov transformations[10] for the operators $\hat{a}_p{}^+,\hat{a}_p$ and the relations

$$a_0^2+\sum_{p\neq 0}\hat{a}_p{}^+\hat{a}_p=N_a,\quad\langle\hat{b}_p{}^+\hat{b}_p\rangle=\mathscr{N}_b(\mathbf{p}), \qquad (2.6)$$

we find the total energy of the system with the help of (2.1), (2.2):

$$E=gN_a^2+\int\frac{dp}{(2\pi\hbar)^3}\{\varepsilon(\mathbf{p})n_a(\mathbf{p})+\mathscr{E}_b(\mathbf{p})\mathscr{N}_b(\mathbf{p})\}. \qquad (2.7)$$

In (2.1), (2.7), and below we are omitting a term $-\mathbf{M}\cdot\mathbf{H}$ ($\mathbf{M}$ is the magnetization), which stems from the existence of an external magnetic field. The first term in (2.7) determines the ground-state energy, in which we ignore small corrections on the order of $gN_a^2(N_aa^3)^{1/2}$ which are inconsequential because of the low gas density. The contribution of weakly excited states is determined by two independent branches of the spectrum of elementary excitations, $\varepsilon_a(\mathbf{p}),\mathscr{E}_b(\mathbf{p})$, and by the corresponding distribution functions $n_a(\mathbf{p}),\mathscr{N}_b(\mathbf{p})$. The energy spectrum and the distribution function of the impurity quasiparticles of component b, dissolved in the superfluid of atoms in state $|a\rangle$, are

$$\mathscr{E}_b(\mathbf{p})=\delta E/\delta\mathscr{N}_b(\mathbf{p})=2gN_a+p^2/2m,$$
$$\mathscr{N}_b(\mathbf{p})=\left\{\exp\left[\frac{\mathscr{E}_b(\mathbf{p})-\mu_b}{T}\right]-1\right\}^{-1},$$
$$\int\mathscr{N}_b(\mathbf{p})\frac{d\mathbf{p}}{(2\pi\hbar)^3}=N_b, \qquad (2.8)$$

where μ_b is the chemical potential of component b. [In principle, we could also retain the term $4gN_b$ in dispersion relation (2.8); that term stems from the interaction of impurity particles with each other.] The spectrum of excitations associated with the existence of superfluid component a has the typical Bogolyubov form

$$c_a^2=\frac{N_a}{m}\frac{\partial^2}{\partial N_a^2}(gN_a^2)=\frac{2gN_a}{m}. \qquad (2.9)$$

The distribution function $n_a(\mathbf{p})$ is also described by the Bose-Einstein formula with a zero chemical potential.

If the concentration of component b is not too small,

$$1\gg(N_b/N_a)^{2/3}\gg N_a^{1/3}a, \qquad (2.10)$$

the condition $T_{ca}>T>T_{cb}\gg gN_a$ clearly holds, so the basic contribution of Bogolyubov excitations to the thermodynamics of the system is determined by the massive part of the spectrum at large momenta $p\gg mc_a$. The density of the normal component,

$$\rho^{(n)}=-\frac{1}{3}\int\frac{\partial n}{\partial\varepsilon}p^2\frac{d\mathbf{p}}{(2\pi\hbar)^3} \qquad (2.11)$$

is thus given under conditions (2.10) by the sum of the Bogolyubov and impurity components:

$$\rho^{(n)}=\rho_a^{(n)}+\rho_b^{(n)}=mN_a(T/T_{ca})^{3/2}+mN_b. \qquad (2.12)$$

It can be seen from (2.12) that at $T\sim T_{cb}$ the two contributions to the thermodynamics are the same in order of magnitude. Specifically, at $T_{ca}\gg T\gg T_{cb}\gg gN_a$ the impurity component of the heat capacity, for example, is always small:

$$C_V=\frac{5}{2}\frac{\Gamma(^5/_2)\zeta(^5/_2)}{\Gamma(^3/_2)\zeta(^3/_2)}\left(\frac{T}{T_{ca}}\right)^{3/2}N_a+\frac{3}{2}N_b. \qquad (2.13)$$

At $T\gtrsim T_{cb}$ the situation changes, and the two components become comparable in magnitude:

$$C_V=\frac{5}{2}\frac{\Gamma(^5/_2)\zeta(^5/_2)}{\Gamma(^3/_2)\zeta(^3/_2)}\left[N_a\left(\frac{T}{T_{ca}}\right)^{3/2}+N_b\left(\frac{T}{T_{cb}}\right)^{3/2}\right]$$
$$-3\left[\frac{\Gamma(^3/_2)\zeta(^3/_2)}{2\pi}\right]^2N_b\left(1-\frac{T_{cb}^{3/2}}{T^{3/2}}\right)\left(5\frac{T^{3/2}}{T_{cb}^{3/2}}+1\right). \qquad (2.14)$$

Under these conditions the chemical potential of component b vanishes as a power law:

$$\mu_b=-\frac{2\pi^2\hbar^6}{m^3}\frac{N_b}{T^2}\left(\frac{T^{3/2}}{T_{cb}^{3/2}}-1\right)^2. \qquad (2.15)$$

If the concentration of particles in state $|b\rangle$ is sufficiently low,

$$1\gg N_a^{1/3}a\gg(N_b/N_a)^{2/3}, \qquad (2.16)$$

the relation between the contributions of the a and b components to the thermodynamics of the system also depends on the temperature. At $T_{cb}<gN_a<T<T_{ca}$ the density of the normal component is again given by (2.12), but in this temperature region the relation $\rho_b{}^{(n)}\ll\rho_a{}^{(n)}$ always holds. If, on the other hand, we have $T_{cb}<T<gN_a<T_{ca}$, the contribu-

tion of component *a* to the thermodynamics is determined by the phonon part of spectrum (2.9), so the total density of the normal component is given by

$$\rho^{(n)}=2\pi^2T^4/45\hbar^3c_a{}^5+mN_b. \tag{2.17}$$

If the temperature is not too low, $T_{cb} < T^* < T < gN_a$, where

$$T^*=gN_a\left[\left(\frac{N_b}{N_a}\right)^{2/3}\frac{1}{N^{1/3}a}\right]^{3/8}, \tag{2.18}$$

the phonon component in (2.17) will be significantly greater than the impurity component. As the temperature is lowered, $T_{cb} < T < T^*$, we have, on the contrary, $mN_b \gg \rho_a^{(n)}$.

Corresponding conclusions can be reached for the heat capacity (and for other thermodynamic quantities). In the case $T_{cb} < gN_a < T < T_{ca}$, for example, the value of C_V is again given by (2.13), with $C_V{}^{(a)} \gg N_b$. If the temperature is sufficiently low, $T_{cb} < T < gN_a < T_{ca}$, the heat capacity of the gas is determined by the sum of the phonon component

$$C_V^{(a)}={}^{16}/{}_{15}\pi^5(T/2\pi\hbar c_a)^3 \tag{2.19}$$

and the impurity component $C_V{}^b$, which is given by the corresponding component in (2.13) or (2.14), depending on the relation between T and T_{cb}.

3. ENERGY SPECTRUM OF TWO-CONDENSATE SYSTEM

We now consider the case $T < T_{cb} < T_{ca}$, in which gaseous H↑ has two Bose condensates simultaneously. The presence of two condensates means that there exist two macroscopic wave functions that describe the behavior of the *a* and *b* particles in the superfluid H↑. The space-time dependence of the Heisenberg operators Ψ is determined by the usual Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_i(\mathbf{r},t)=[\Psi_i(\mathbf{r},t)\hat{H}'],\quad i=a,b. \tag{3.1}$$

The Hamiltonian in (3.1) is $\hat{H}' = \hat{H} - \Sigma_i\mu_iN_i$, where $\hat{H}$ is given by (2.1).

Substituting (2.1) into (3.1), and using commutation relations (2.2), we find

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_i(\mathbf{r},t)=\left(-\frac{\hbar^2}{2m}\nabla^2-\mu_i\right)\hat{\Psi}_i(\mathbf{r},t)$$
$$+\hat{\Psi}_i(\mathbf{r},t)\int\hat{\Psi}_i{}^+(\mathbf{r}',t)U_{ii}(\mathbf{r}-\mathbf{r}')$$
$$\times\hat{\Psi}_i(\mathbf{r}',t)d\mathbf{r}'+\hat{\Psi}_i(\mathbf{r},t)\int\hat{\Psi}_k{}^+(\mathbf{r}',t)U_{ik}(\mathbf{r}-\mathbf{r}')\hat{\Psi}_k(\mathbf{r}',t)d\mathbf{r}',$$
$$i,\ k=a,\ b,\ k\neq i. \tag{3.2}$$

By definition, the Heisenberg operator $\hat{\Psi}$ in a system with a Bose condensate can written as the sum of a "quasiclassical" wave function $\Xi(\mathbf{r},t)$ and of an above-condensate term $\hat{\Psi}'(\mathbf{r},t)$:

$$\hat{\Psi}(\mathbf{r},t)=\Xi(\mathbf{r},t)+\hat{\Psi}'(\mathbf{r},t). \tag{3.3}$$

Near absolute zero, nearly all the particles of a slightly nonideal gas are in the condensate, so the above-condensate part $\hat{\Psi}'$ can be ignored in the leading approximation. Substituting $\hat{\Psi}$ (which is equal to Ξ) into Schrödinger equation (3.2), we find the following result for slightly nonuniform distributions, for which the length scale of the variation in the function $\Xi(\mathbf{r},t)$ is significantly greater than the range of the potential $U_{ik}(\mathbf{r})$:

$$i\hbar\frac{\partial}{\partial t}\Xi_i=\left(-\frac{\hbar^2}{2m}\nabla^2-\mu_i\right)\Xi_i+\Xi_i|\Xi_i|^2u_{ii}+\Xi_i|\Xi_k|^2u_{ik},$$
$$i,\ k=a,\ b,\quad i\neq k. \tag{3.4}$$

In (3.4), the chemical potential μ_i of each component is determined by the obvious relation

$$\mu_i=N_iu_{ii}+N_ku_{ik}. \tag{3.5}$$

In (3.5) we have omitted terms $\pm\beta H$ (β is the magnetic moment of the particle), which are unimportant for the analysis below, since all the terms in Hamiltonian $\hat{H}'$ which depend on the magnetic field cancel out.

We find the spectrum of excitations of the system by examining small oscillations of the condensate wave functions about their expectation values $N_a{}^{1/2}$, $N_b{}^{1/2}$ (Ref. 10). We write slight perturbations of the functions Ξ_i in the form

$$\Xi_a=N_a^{1/2}+A\exp[i(\mathbf{kr}-\omega t)]+B^*\exp[-i(\mathbf{kr}-\omega t)],$$
$$\Xi_b=N_b^{1/2}+C\exp[i(\mathbf{kr}-\omega t)]+D^*\exp[-i(\mathbf{kr}-\omega t)], \tag{3.6}$$

where A, B^*, C, and D^* are small complex amplitudes.

Substituting (3.6) into equations (3.4) and (3.5), and linearizing them with respect to small deviations, we arrive at the characteristic system

$$\begin{aligned}A\hbar\omega&=(p^2/2m)A-\mu_aA+u_{aa}N_a(B+2A)\\&\quad+u_{ab}[(N_aN_b)^{1/2}(C+D)+N_bA],\\-B\hbar\omega&=(p^2/2m)B-\mu_aB+u_{aa}N_a(A+2B)\\&\quad+u_{ab}[(N_aN_b)^{1/2}(C+D)+N_bB],\\C\hbar\omega&=(p^2/2m)C-\mu_bC+u_{bb}N_b(D+2C)\\&\quad+u_{ab}[(N_aN_b)^{1/2}(A+B)+N_aC],\\-D\hbar\omega&=(p^2/2m)D-\mu_bD+\\&\quad+u_{bb}N_b(C+2D)+u_{ab}[(N_aN_b)^{1/2}(A+B)+N_aD].\end{aligned} \tag{3.7}$$

The condition under which system (3.7) is consistent determines the unknown dispersion relation:

$$\begin{aligned}&[(p^2/2m)^2+u_{aa}N_ap^2/m-(\hbar\omega)^2]\\&\times[(p^2/2m)^2+u_{bb}N_bp^2/m-(\hbar\omega)^2]\\&\quad-4u_{ab}N_aN_b(p^2/2m)^2=0.\end{aligned} \tag{3.8}$$

As we mentioned earlier, in the case of gaseous hydrogen $H\uparrow$ all the potentials u_{ik} are quite accurately equal: $u_{aa} = u_{ab} = u_{bb} = 2g$. Solving Eq. (3.8) at the same accuracy level, we find a dispersion relation for small oscillations of the condensate wave functions. This result determines the spectrum of both branches of elementary excitations in the given two-component system:

$$\begin{aligned}\hbar\omega_1&=\varepsilon_1(\mathbf{p})=[2gNp^2/m+(p^2/2m)^2]^{1/2},\\\hbar\omega_2&=\varepsilon_2(\mathbf{p})=p^2/2m,\end{aligned} \tag{3.9}$$

where $N = N_a + N_b$. As before, the spectrum $\varepsilon_1(\mathbf{p})$ is of Bogolyubov form, and its phonon part is characterized by the same sound velocity, $c_a{}^2 = 2gN/m \approx 2gN_a/m$, at this accuracy level. In the spectrum $\varepsilon_2(\mathbf{p})$, in contrast, the corresponding sound velocity is zero. This result stems from the circumstance that the interaction constants are the same for all the particles of the system, and it is a macroscopic reflection of the appearance of an additional symmetry, under the substitution[1)] $a \leftrightarrow b$, in the initial Hamiltonian.

This assertion also agrees completely, in the appropriate limit, with the results derived by Mineev[13] through a direct diagonalization of the Hamiltonian. The fact that the velocity of acoustic vibrations vanishes in the spectrum $\varepsilon^2(\mathbf{p})$ corresponds to the boundary of absolute thermodynamic stability of the mixture with respect to stratification. Incorporating in the total energy the hyperfine interaction, which is responsible for the different values of the constants u_{ik}, and also corrections of higher order in the gas density leads to a nonzero sound velocity in the spectrum $\varepsilon_2(\mathbf{p})$ in the stability region of the systems.

In our approximation, (3.9), the second phase transition to a superfluid state at $T = T_{cb}$ formally does not alter the spectrum of excitations of the system, so we could say that we are dealing with a single, doubly degenerate condensate. On the other hand, since the chemical potential μ_b of the impurity excitations is zero at $T \leqslant T_{cb}$ (in contrast with the situation at temperatures $T > T_{cb}$, where we have $\mu_b < 0$), the presence of a second critical temperature T_{cb} has a strong effect on the thermodynamics of the system. At temperatures which are not too low, $gN_a \ll T < T_{cb}$, for example, the contributions to the thermodynamics from the quasiparticles of the two species are equal:

$$\rho^{(n)} = mN_a\left(\frac{T}{T_{ca}}\right)^{3/2} + mN_b\left(\frac{T}{T_{cb}}\right)^{3/2} = \frac{2^{1/2}}{\pi^2}\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)\frac{m^{5/2}T^{3/2}}{\hbar^3},$$

$$C_V = \frac{5}{2^{1/2}\pi^2}\Gamma\left(\frac{5}{2}\right)\zeta\left(\frac{5}{2}\right)\frac{m^{3/2}T^{3/2}}{\hbar^3}. \tag{3.10}$$

If, on the other hand, the gas temperature is sufficiently low, $T \ll \min(gN_a, T_{cb})$, the density of the normal component and the heat capacity of the superfluid gas will be given by the obvious equations

$$\rho_a^{(n)} + \rho_b^{(n)} = {}^1/_2\rho^{(n)} + 2\pi^2T^4/45\hbar^3c_a^5,$$

$$C_V^{(a)} + C_V^{(b)} = {}^1/_2C_V + {}^{16}/_{15}\pi^5(T/2\pi\hbar c_a)^3, \tag{3.11}$$

where $\rho^{(n)}$ and C_V are given by (3.10). The first term in (3.11) is significantly larger than the second.

4. ENERGY EIGENFUNCTIONS

Above we derived equations which describe the behavior of the condensate wave function. The basic assertions can also be formulated in terms of the above-condensate operators $\hat{\Psi}'(\mathbf{r},t)$. For this purpose, it is most convenient to use a diagram technique for slightly nonideal Bose systems.[10] A theory of solutions of two Bose fluids has been derived by Nepomnyashchiĭ.[14]

In the case at hand, it is convenient to slightly alter the description of the system in terms of Green's functions; it is also convenient to immediately make use of the small value of the gas parameter in order to derive the final results. We define the normal and anomalous Green's functions for the two components by the customary relations

$$iG_i(X_1, X_2) = \langle \mathcal{N}_i, \mathcal{N}_k | T\hat{\Psi}_i'^{+}(X_1), \hat{\Psi}_i'(X_2) | \mathcal{N}_i, \mathcal{N}_k \rangle,$$
$$iF_i(X_1, X_2) = \langle \mathcal{N}_i+2, \mathcal{N}_k | T\hat{\Psi}_i'^{+}(X_1), \hat{\Psi}_i'^{+}(X_2) | \mathcal{N}_i, \mathcal{N}_k \rangle,$$
$$i.\ k = a,\ b,\ i \neq k. \tag{4.1}$$

In a binary system with two Bose condensates, it is necessary to introduce some additional Green's functions, which correspond to the transitions of the particles of species a and b into different condensates and vise versa:

$$iS_{ik}(X_1, X_2) = \langle \mathcal{N}_i, \mathcal{N}_k | T\hat{\Psi}_i'(X_1), \hat{\Psi}_k'(X_2) | \mathcal{N}_i+1, \mathcal{N}_k+1 \rangle,$$
$$iS_{ik}^{+}(X_1, X_2) = \langle \mathcal{N}_i+1, \mathcal{N}_k+1 | T\Psi_i'^{+}(X_1), \hat{\Psi}_k'^{+}(X_2) | \mathcal{N}_i, \mathcal{N}_k \rangle,$$
$$iQ_{ik}(X_1, X_2) = \langle \mathcal{N}_i+1, \mathcal{N}_k | T\hat{\Psi}_i'^{+}(X_1), \hat{\Psi}_k'(X_2) | \mathcal{N}_i, \mathcal{N}_k+1 \rangle,$$
$$iQ_{ik}(X_1, X_2) = \langle \mathcal{N}_i, \mathcal{N}_k+1 | T\hat{\Psi}_i'(X_1), \hat{\Psi}_k'^{+}(X_2) | \mathcal{N}_i+1, \mathcal{N}_k \rangle, \tag{4.2}$$

where $\mathcal{N}_i$ is the number density of above-condensate particles of species i, and X_i is a space-time 4-vector. Many of the eight functions in (4.2) are not independent and can be expressed in terms of each other. The interchanges $X_1 \leftrightarrow X_2$ and $a \leftrightarrow b$, combined with a time ordering in (4.2), lead to the equality $S_{ab}(X_1, X_2) = S_{ba}(X_2, X_2)$, which has the following meaning in the momentum representations:

$$S_{ab}(P) = S_{ba}(-P). \tag{4.3}$$

Using a property of the Heisenberg operator $\hat{\Psi}'$ for an immobile fluid,

$$\hat{\Psi}^{+}(X) = \hat{\tilde{\Psi}}(-X),$$

we also find

$$S_{ab}^{+}(P) = S_{ba}(-P), \quad S_{ab}(P) = S_{ab}^{+}(P),$$
$$Q_{ab}^{+}(P) = Q_{ab}(P). \tag{4.4}$$

The basic graphical elements of the diagram technique for describing all of the Green's functions introduced above are the following energy eigenfunctions (Fig. 1). For example, the difference between functions $G(P) - G_i^{(0)}(P)$, where $G_i^{(0)}(P)$ is the Green's function of an ideal Bose gas, is determined by a combination of different chains of the type in Fig. 2. Pruning the outermost link from all the chains in Fig. 2, and going through the standard procedure, we find an equation which determines the exact Green's function $G_i(P)$:

$$G_a(P) - G_a^{(0)}(P) = G_a^{(0)}(P)[\Sigma_{11}^{(aa)}(P)G_a(P) + \Sigma_{20}^{(aa)}(P)F_a(P)$$
$$+ \Sigma_{11}{}^{ba}(P)Q_{ab}(P) + \Sigma_{20}^{(ba)}(P)S_{ab}(P)],$$

$$G_b(P) - G_b^{(0)}(P) = G_b^{(0)}(P)[\Sigma_{11}^{(bb)}(P)G_b(P)$$
$$+ \Sigma_{20}^{(bb)}(P)F_b(P) + \Sigma_{11}^{(ab)}(P)Q_{ab}(P)$$
$$+ \Sigma_{20}^{(ab)}(P)S_{ab}(P)]. \tag{4.5}$$

In deriving (4.5) we made use of relations (4.4).

The equations for the other Green's functions are derived in a completely corresponding way:

$$F_a(P) = G_a^{(0)}(-P)[\Sigma_{11}^{(aa)}(P)F_a(P)$$
$$+ \Sigma_{20}^{(ab)}(P)Q_{ab}(P) + \Sigma_{11}^{(ab)}(P)S_{ab}(P) + \Sigma_{02}^{(aa)}(P)G_a(P)],$$
$$F_b(P) = G_b^{(0)}(-P)[\Sigma_{11}^{(bb)}(P)F_b(P)$$
$$+ \Sigma_{20}^{(bb)}(P)G_b(P) + \Sigma_{11}^{(ba)}(P)S_{ab}(P) + \Sigma_{02}^{(ab)}(P)Q_{ab}(P)],$$
$$Q_{ab}(P) = G_b^{(0)}(P)[\Sigma_{11}^{(bb)}(P)Q_{ab}(P)$$
$$+ \Sigma_{02}^{(bb)}(P)S_{ab}(P) + \Sigma_{11}^{(ab)}(P)G_a(P) + \Sigma_{02}^{(ab)}(P)F_a(P)],$$
$$S_{ab}(P) = G_a^{(0)}(-P)[\Sigma_{02}^{(ab)}(P)G_b(P)$$
$$+ \Sigma_{11}{}^{ab}(P)F_b(P) + \Sigma_{02}^{(ab)}(P)Q_{ab}(P) + \Sigma_{11}^{(aa)}(P)S_{ab}(P)]. \tag{4.6}$$

In deriving (4.6) we made use of the circumstance that in an ideal gas at $T = 0$ all the anomalous functions $F_a^{(0)}$, $F_b^{(0)}$, $S_{ab}^{(0)}$, $Q_{ab}^{(0)}$ are zero.

Taking the small values of the gas parameters $N_i a^3 \ll 1$ into account, we find that the energy eigenfunctions are determined primarily by diagrams of the type in Fig. 3. In the leading approximation, and under the assumption that all the potentials u_{ik} are again equal, we finally find

$$\begin{aligned}
G_a(P)=G_a^{(0)}(P)\,[1&+(2N_a+N_b)\,G_a(P)\,u \\
&+N_a u F_a(P)+u\,(N_aN_b)^{1/2}Q_{ab}(P) \\
&+u\,(N_aN_b)^{1/2}S_{ab}(P)\,],
\end{aligned}$$

$$\begin{aligned}
G_b(P)=G_b^{(0)}(P)\,\{1&+(2N_b+N_a)\,G_b(P)\,u+N_b u F_b(P) \\
&+(N_aN_b)^{1/2}u\,[Q_{ab}(P)+S_{ab}(P)\,]\},
\end{aligned}$$

$$\begin{aligned}
F_a(P)=G_a^{(0)}(-P)\,\{&(2N_a+N_b)\,uF_a(P)+(N_aN_b)^{1/2}u \\
&\times[Q_{ab}(P)+S_{ab}(P)\,]+uN_aG_a(P)\},
\end{aligned}$$

$$\begin{aligned}
F_b(P)=G_b^{(0)}(-P)\,\{&(2N_b+N_a)\,uF_b(P) \\
&+(N_aN_b)^{1/2}u\,[S_{ab}(P)+Q_{ab}(P)\,]+uN_bG_b(P)\},
\end{aligned}$$

$$\begin{aligned}
Q_{ab}(P)=G_b^{(0)}(P)\,\{&u\,(2N_b+N_a)\,Q_{ab}(P)+uN_bS_{ab}(P) \\
&+(N_aN_b)^{1/2}u\,[G_a(P)+F_a(P)\,]\},
\end{aligned}$$

$$\begin{aligned}
S_{ab}(P)=G_a^{(0)}(-P)\,\{&(N_aN_b)^{1/2}u\,[G_b(P)+F_b(P)\,]+uN_aQ_{ab}(P) \\
&+(2N_a+N_b)\,uS_{ab}(P)\}. \qquad (4.7)
\end{aligned}$$

Equation (4.7) determines all the Green's functions of gaseous H↑ with two Bose-Einstein condensates. In particular, after some straightforward manipulations we easily verify that the functions $G_i(P)$ have poles at points determined by conditions (3.9).

5. ACOUSTIC MODES

Superfluids are known to allow a greater than normal diversity of low-frequency hydrodynamic vibrations, because of the additional degrees of freedom which stem from the presence of a superfluid motion. A characteristic additional mode in superfluids is second sound, which is a vibration of the density in the system of elementary excitations. The spectrum of all acoustic branches of a superfluid solution is determined from the solution of the linearized hydrodynamic equations,[9,11] which are quite well known, so we will not reproduce them here. The dispersion relation for the vibrations is expressed in terms of exclusively the thermodynamic characteristics of the solution in this case.

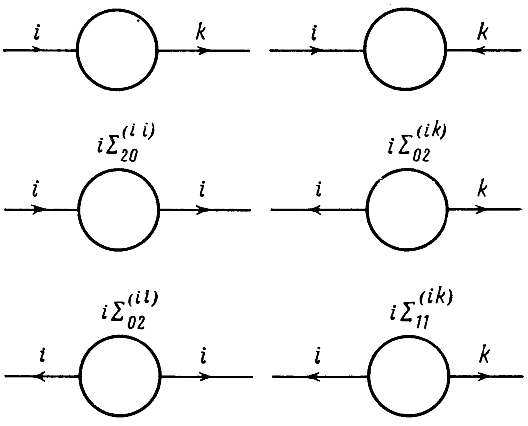

FIG. 1.

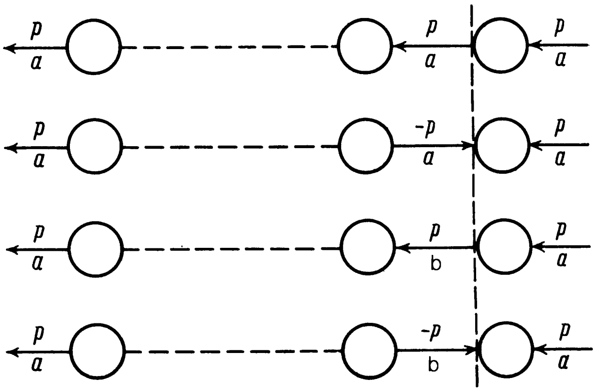

FIG. 2.

Let us apply the results of the solution of the hydrodynamic equations of superfluid solutions to gaseous H↑ at $T_{cb} < T < T_{ca}$. The vibrations of the first type (first sound) are basically vibrations of the gas density $\rho = m(N_a + N_b)$ which propagate at a velocity $u_1^2 = \partial P/\partial\rho$ (P is the pressure). In the leading approximation in the concentration N_b, this velocity is naturally the same as the velocity c_Q^2 from (2.9):

$$u_1^2=c_a^2=2gN_a/m. \qquad (5.1)$$

The velocity of second sound in superfluid atomic hydrogen is given by (cf. Refs. 9 and 15)

$$u_2^2=\left(\frac{\rho}{\rho^{(n)}}-1\right)\left[\bar\sigma^2\left(\frac{\partial T}{\partial\sigma}\right)_{P,c}+\frac{c^2}{m}\frac{\partial}{\partial c}(\mu_b-\mu_a)\right], \qquad (5.2)$$

where $S = \sigma\rho$ is the mixing entropy, and where we have introduced

$$\bar\sigma=\sigma-c\partial\sigma/\partial c, \quad c=N_b/(N_a+N_b). \qquad (5.3)$$

It can be seen from the results of the preceding part of this paper that at temperatures which are not greatly different from T_{ca}, more precisely for $\max(T_{cb}, gN_a) \ll T \ll T_{ca}$, we can ignore the impurity b component in the thermodynamics of the gas, and from (3.2) we find the following result with the help of (2.12) and (2.13) (cf. Ref. 16):

$$u_2^2=\frac{10}{9}\frac{\Gamma(5/2)\zeta(5/2)}{\Gamma(3/2)\zeta(3/2)}\frac{T}{m}\left(1-\frac{T^{3/2}}{T_{ca}^{3/2}}\right). \qquad (5.4)$$

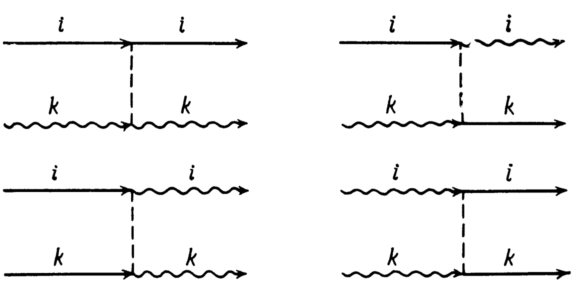

FIG. 3.

The observation of temperature waves (5.4) at $T<T_{ca}$ might serve as a reliable method for identifying the transition of gaseous H↑ to a superfluid state.

Under the condition (2.16), and in the case $T_{cb} \ll T < gN_a$, expression (5.2) for the velocity of second sound takes a very simple and physically lucid form:

$$u_2^2 = \begin{cases} c_a^2/3, & T_{cb} \ll T^* < T < gN_a, \\ 5T/3m, & T_{cb} \ll T < T^* < gN_a, \end{cases} \tag{5.5}$$

where T^* is given by (2.18). Under the same conditions at $T \approx T_{cb}$, the quantity u_2^2 assumes the value

$$u_2^2 = \frac{10}{9} \frac{\Gamma(^5/_2)\zeta(^5/_2)}{\Gamma(^3/_2)\zeta(^3/_2)} \frac{T_{cb}}{m}, \tag{5.6}$$

which is slightly smaller than $5T_{cb}/3m$, which is the value we would find by extrapolating (5.5) to $T = T_{cb}$. To avoid any possible misunderstanding, we recall that simple expressions (5.1)–(5.6) for the spectra of hydrodynamic vibrations are valid ony in the case $u_2 \ll u_1$. For this reason we cannot (for example) use those expressions to calculate u_1 and u_2 at $T \sim T_{cb}$ under conditions (2.11). In this case, we should use the general solution of the original dispersion relation, which we will not reproduce here because of its length.

Since the density of gaseous H↑ is always low under actual conditions, there may arise a situation in which the mean free path l for the excitations of some species or other becomes greater than the size of the experimental cell, d. In such a case the spectra of acoustic vibrations may be quite different from the results of previous calculations.[17] In the simple case in which the condition $l \gg d$ holds for all quasiparticles simultaneously, second sound cannot propagate at all in the system, and the vibrations of the first type are modified into so-called fourth sound, whose velocity is

$$u_4^2 = \frac{mN_a - \rho_a^{(n)}}{m^2} \frac{\partial \mu_a}{\partial N_a} \approx \left(1 - \frac{\rho_a^{(n)}}{mN_a}\right) c_a^2. \tag{5.7}$$

At temperatures below the point of the second phase transition, $T < T_{cb}$, the spectrum of acoustic vibrations is determined by the equations of three-velocity hydrodynamics with two velocities of superfluid motion and one of normal motion. In the case of superfluid two-component H↑ these equations simplify substantially. The reason is that since there is no renormalization of the effective mass in this approximation there is no mutual entrainment of the superfluid components, which is exceedingly important in the case of ^{3}He–^{4}He solutions of quantum fluids.[11] Below we present some simple expressions for the propagation velocities of acoustic modes which can be derived from the results of Refs. 11 and 18 in those regions of the temperature and the concentration in which the conditions $u_1 \gg u_2 \gg u_3$ hold.

The propagation velocity for vibrations of the first type remains essentially unchanged: $u_1^2 = 2gN_a/m$. The dispersion relation for the quanta of second sound is set by the velocity of the corresponding sound wave:

$$u_2^2 = \left[\frac{mN_b - \rho_b^{(n)}}{\rho} + \frac{\rho_b^{(n)}\rho_a^{(n)}}{\rho^{(n)}\rho}\right] \frac{1}{\rho} \frac{\partial^2 E}{\partial c^2}. \tag{5.8}$$

In the limit $T \to 0$ the velocity of second sound given by (5.8) is zero, in total agreement with the absence of a phonon region on the second branch of the spectrum of elementary excitations, $\hbar\omega_2(\mathbf{p})$, from (3.9). The propagation velocity of acoustic vibrations of the third type, basically temperature waves, is given by

$$u_3^2 = \frac{TS^2}{C_V} \frac{mN_b - \rho_b^{(n)}}{\rho^{(n)}(mN_b - \rho_b^{(n)}) + \rho_b^{(n)2}}. \tag{5.9}$$

In the limit $T \to T_{cb}$, and under the conditions $gN_b \ll gN_a \ll T \ll T_{cb}$, the velocity of the third sound vanishes in accordance with

$$u_3^2 = \frac{20}{9} \frac{T_{cb}}{m} \frac{\Gamma(^5/_2)\zeta(^5/_2)}{\Gamma(^3/_2)\zeta(^3/_2)} \left(1 - \frac{T^{3/2}}{T_{cb}^{3/2}}\right). \tag{5.10}$$

The limiting value of u_3^2 as $T \to 0$ turns out to be $c_b^2/3$.

We wish to stress that our analysis is valid only for hydrodynamic, not excessively low, frequencies, $\tau_s^{-1} \ll \omega \ll \tau^{-1}$, where τ is the ordinary relaxation time for the relaxation to an equilibrium in terms of the energy any momenta in the system of particles, and τ_s is the time scale of the restructuring of the spin states (over times longer than τ_s, the quantities N_a and N_b cannot be assumed to remain constant).

6. KINETIC COEFFICIENTS

The transport properties of superfluid atomic hydrogen H↑ are determined by the interaction of all the elementary excitations which are part of the normal component of the solution. The cross sections for the mutual scattering of quasiparticles in this case can be determined very simply, so it is possible to rapidly derive expressions for the kinetic coefficients in some approximation or other. Below we present some results for the temperature regions which are of primary interest from our standpoint.

In the interval $T_{cb} \ll T \lesssim T_{ca}$ the contribution of normal impurities, i.e., of the particles of component b, to such transport phenomena as heat conduction and viscosity is vanishingly small. In this case a quantitative description of these effects reduces to examining the interaction of exclusively the atoms in state $|a\rangle$ with each other. For example, we find the following expression for the thermal conductivity in the τ approximation (which, although an approximation, gives the correct temperature dependence):

$$\varkappa = \frac{2\tau}{3T^2} \int n(n+1) \left(\frac{p^2}{2m}\right)^2 \left(\frac{p^2}{2m} - \frac{ST}{\rho^{(n)}}\right) \frac{d\mathbf{p}}{(2\pi\hbar)^3}, \tag{6.1}$$

where the relaxation time τ is given by

$$\tau^{-1} = \tilde{N}_a \langle \sigma v \rangle = {}^4/_3 m a^2 T^2/\hbar^3, \qquad \tilde{N}_a = \int n_a(p) \frac{d\mathbf{p}}{(2\pi\hbar)^3}. \tag{6.2}$$

Calculations based on (6.1) and the results derived previously for the thermodynamic quantities lead to the expression

$$\varkappa = AT^{1/2}/a^2 m^{1/2}, \tag{6.3}$$

where the constant A is given by the following expression in our approximation:

$$A = 2^{-3/2}\pi^{-3}[{}^7/_2\Gamma(^7/_2)\Gamma(^3/_2)\zeta(^7/_2)\zeta(^3/_2) - \Gamma(^5/_2)\zeta(^5/_2)]/\Gamma(^3/_2)\zeta(^3/_2). \tag{6.4}$$

In our model, the numerical coefficient A in (6.4) is noticeably different from the result given in Refs. 16 and 19.

While the thermal conductivity of the gas at $T \lesssim T_{ca}$ is determined primarily by the particles of component a, the diffusion coefficient D of the hydrogen atoms in state $|b\rangle$ (we are actually dealing with a nuclear spin diffusion) is fundamentally a consequence of the presence of component

b in the system. At high temperatures $T_{cb} \ll T \lesssim T_{ca}$, at which the scattering of particles in state b by excitations corresponding to the "massive" part of the energy spectrum of component a plays a leading role in the diffusion process, calculations for the diffusion coefficient[15]

$$D=\frac{T}{m\tilde{N}_a}\left\langle\frac{1}{\sigma_t v}\right\rangle, \quad \sigma_t=\int (1-\cos\theta)\, d\sigma \tag{6.5}$$

lead to the result

$$D=B\hbar^3/(ma)^2T, \quad B=\pi^{1/2}/2\Gamma(^3/_2)\zeta(^3/_2). \tag{6.6}$$

Under conditions (2.16) and for $T_{cb} < T^* < T < gN_a$, where T^* is given by (2.18), the primary dissipation mechanism is the scattering of the particles of species b by the first-sound phonons. The magnitude of the corresponding scattering cross section completely determines all the transport properties of the system in this temperature region. Since the effective mass of the impurity quasiparticles is the same as m in this case, the Hamiltonian of the interaction of hydrogen atoms in state $|b\rangle$ with the phonon field contains only a single term:

$$\hat{H}_{i-ph}=\frac{\partial\varepsilon_b}{\partial N_a}\delta\hat{N}_a(\mathbf{r},t)=2g\delta\hat{N}_a(\mathbf{r},t). \tag{6.7}$$

To illustrate the situation, we calculate the cross section for inelastic scattering of impurity b particles which are moving in superfluid component a at low temperatures. The probability for a transition of an impurity quasiparticle from an initial state $|i\rangle$ with momentum $\mathbf{p}$ to a final state $\langle f|$ with momentum $\mathbf{p}'$ is given by the well-known expression from quantum mechanics:

$$dw=\frac{1}{\hbar^2}\left|\int_{-\infty}^{+\infty}\langle f|\hat{H}_{i-ph}|i\rangle\exp(-i\omega t)\right|\frac{d\mathbf{p}'}{(2\pi\hbar)^3}, \tag{6.8}$$

where $\hbar\omega=(p^2-p'^2)/2m$. Substituting (6.7) into (6.8), taking an average over the density fluctuations δN_a, and normalizing the inital wave function to a unit flux density, we find the differential scattering cross section (per particle of component a):

$$d\sigma=\left(\frac{2g}{\hbar}\right)^2\frac{m}{pN_a}S(\omega,\mathbf{q})\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3}, \qquad \hbar\mathbf{q}=\mathbf{p}-\mathbf{p}', \tag{6.9}$$

where $S(\omega,\mathbf{q})$ is the dynamic form factor of component a, i.e., the space-time Fourier transform of the correlation function $\langle\delta N_a(\mathbf{r}_1,t_1)\times\delta N_a(\mathbf{r}_2,t)\rangle$. In the phonon temperature region we have the following expression for the function $S(\omega,\mathbf{q})$ of an arbitrary superfluid Bose liquid[13]:

$$S(\omega,\mathbf{q})=N_a\frac{\pi\hbar q}{mu_1}\frac{[\delta(\omega-qu_1)+\exp(-\hbar qu_1/T)\delta(\omega+qu_1)]}{1-\exp(-\hbar qu_1/T)}. \tag{6.10}$$

If we have $T<gN_a$ in this case, then we also have $\hbar qu_1 \gg T$, and expression (6.10) simplifies considerably. From (6.9) and (6.10) we then find the following expression for small-angle scattering:

$$d\sigma=16\pi a^2\frac{P}{mu_1}\sin\left(\frac{\chi}{2}\right)\frac{do'}{4\pi}. \tag{6.11}$$

Cross section (6.11) can be used to calculate the contribution of inelastic scattering by phonons to all the transport characteristics of the system.

Another interesting direction would be study collective spin waves in gaseous H↑. These waxes exist in the system over broad regions of the temperature and the density.[20–22]

We wish to thank A. F. Andreev and L. P. Pitaevskiĭ for useful comments.

1) This circumstance was graciously pointed out to us by A. Rukenshteĭn.

---

[1] I. F. Silvera and J. T. M. Walraven, Scientific American **246**(1), 66 (1982).

[2] T. J. Greytak and D. Kleppner, in *New Trends in Atomic Physics* (*Les Houches Session XXXVIII*) (ed. Grynberg and Stora), North-Holland, Amsterdam, 1984, p. 1129.

[3] E. P. Bashkin, Usp. Fiz. Nauk **148**, 433 (1986) [Sov. Phys. Usp. **29**, 238 (1986)].

[4] R. Sprik, J. T. M.Walraven, and I. F. Silvera, Phys. Rev. Lett. **51**, 479, 942 (1983).

[5] H. F. Hess, D. A. Bell, G. P. Kochonski, *et al.*, Phys. Rev. Lett. **52**, 1520 (1984).

[6] Yu. Kagan, I. A. Vartan'yants, and G. V. Shlyapnikov, Zh. Eksp. Teor. Fiz. **81**, 1113 (1981) [Sov. Phys. JETP **54**, 590 (1981)].

[7] N. Masuhara, J. M. Doyle, J. C. Sandberg, *et al.*, Phys. Rev. Lett. **61**, 935 (1988).

[8] R. van Roijen, J. J. Berkhout, S. Jaakkola, and J. T. M.Walraven, Phys. Rev. Lett. **61**, 931 (1988).

[9] I. M. Khalatnikov, *Theory of Superfluidity* [in Russian], Nauka, Moscow, 1978, Part II.

[10] E. M. Lifshitz and L. P. Pitaevskiĭ, *Statistical Physics*, Part II, Pergamon, Oxford, 1980.

[11] A. F. Andreev and E. P. Bashkin, Zh. Eksp. Teor. Fiz. **69**, 319 (1975) [Sov. Phys. JETP **42**, 164 (1975)].

[12] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[13] V. P. Mineev, Zh. Eksp. Teor. Fiz. **67**, 263 (1974) [Sov. Phys. JETP **40**, 132 (1974)].

[14] Yu. A. Nepomnyashchiĭ, Zh. Eksp. Teor. Fiz. **70**, 1070 (1976) [Sov. Phys. JETP **43**, 559 (1976)].

[15] B. N. Esel'son, V. N. Grigor'ev, V. G. Ivanstsov, *et al.*, *Solutions of Quantum Liquids* [in Russian], Nauka, Moscow, 1973.

[16] C. Gay and A. Griffin, J. Low Temp. Phys. **58**, 479 (1985).

[17] E. P. Bashkin, Zh. Eksp. Teor. Fiz. **73**, 1878 (1977) [*sic*].

[18] I. M. Khalatnikov, Pis'ma Zh. Eksp. Teor. Fiz. **17**, 534 (1973) [JETP Lett. **17**, 386 (1973)].

[19] T. R. Kirkpatrick and J. R. Dorfman, Phys. Rev. A **28**, 2576 (1983).

[20] E. P. Bashkin, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 11 (1981) [JETP Lett. **33**, 8 (1981)].

[21] C. Lhuillier and F. Laloë, J. Phys. (Paris) **43**, 197, 225, 833 (1982).

[22] B. R. Johnson, J. S. Denker, N. Bigelow, *et al.*, Phys. Rev. Lett. **52**, 1508 (1984); **53**, 302.

Translated by Dave Parsons