Multiscale percolation systems

A. V. Neĭmark

State Scientific-Research and Design Institute of the Chlorine Industry
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A new model of the structure of a polydisperse medium with arbitrary volume density and with fractal interfaces between the phases, called a multiscale percolation system (MPS), is presented. In the construction of the MPS a symbiosis of the fractal and percolation approaches to the construction of models of disordered systems is achieved. Expressions are obtained for the structural characteristics of the MPS (the porosity, the specific surface, the size distribution of the pores, and the fractal dimensionality). The problem of percolation in the MPS is formulated. By means of the percolation renormalization-group transformation an equation is obtained for the calculation of the percolation threshold of a self-similar MPS.

For the study of the physicochemical, electrophysical, and mechanical properties of disordered disperse and porous materials wide use is made of different structural models.[1] Among these are: a) percolation models based on the randomization of lattices or mosaics[2,3]; b) fractal models based on modification of mathematical fractals of the Sierpinski-carpet or Koch-island type[4–6]; c) models of growth and disintegration that mimic the process of formation of a disperse structure as a result of stochastic agglomeration or decomposition.[7–9] However, not one of the known models is adapted to the description of disordered polydisperse media with arbitrary volume density (porosity) and a fractal interphase boundary. The percolation models and the models of growth and disintegration are essentially monodisperse—they consist of primary elements of one characteristic size. Polydispersity is inherent in fractal models, the volume density of which tends in the limit to the extreme values 0 or 1.

In the present article we develop a new model of polydisperse media with fractal properties, called a multiscale percolation system (MPS). In the construction of an MPS a symbiosis of the fractal approach and the percolation approach to the construction of models of disordered systems is realized. Stage-by-stage fractionation of the scale is accompanied by randomization and by the division of elements into classes—in particular, into conductors and insulators. Of the models discussed in the literature previously, the closest to an MPS are the random-fractal models introduced into analysis by Mandelbrot,[4] the model of a discrete multiscale random medium,[10] the model of scales,[11] and the model of a multifractal lattice.[12] A number of papers are devoted to percolation on a Sierpinski carpet.[13,14]

DEFINITION OF A MULTISCALE PERCOLATION SYSTEM

An MPS is obtained as a result of the following iterative procedure (Fig. 1a). As the basis we take a certain mosaic (dimensionality $d = 2$) or stacking ($d = 3$), consisting of blocks of size r_1. These blocks, called first-rank blocks, are divided into three classes: X-, Y-, and Z-blocks. The choice of the type of block is made randomly, with probabilities x_1, y_1, and z_1, respectively ($x_1 + y_1 + z_1 = 1$), as is done in the modeling of a three-component percolation system.

In the first step in the construction of the MPS each first-rank Z-block is divided, with preservation of the topology of the original mosaic, into n_1^d second-rank blocks of size $r_2 = r_1/n_1$, which are decomposed randomly into X-, Y-, and Z-blocks with probabilities x_2, y_2, and z_2, respectively ($x_2 + y_2 + z_2 = 1$). In the second step, each second-rank Z-block is divided analogously into third-rank X-, Y-, and Z-blocks of size $r_3 = r_2/n_2$ with probabilities x_3, y_3, and z_3, respectively, and so on. In each step, the X- and Y-blocks remain in the system, and the Z-blocks are subjected to fractionation into X-, Y-, and Z-blocks of smaller size. The parameters n_i are called fractionation multiplicities. By repeating the procedure of Z-block fractionation $N - 1$ times, we obtain an N-scale percolation system (an N-MPS) consisting of X- and Y-blocks of size $r_1, r_2, r_3, \ldots, r_i, \ldots, r_N$ and Z-blocks of size r_N. Its properties are determined by the $3N$ parameters r_1, n_i ($i = 1,\ldots,N-1$), x_i, y_i ($i = 1,\ldots,N$).

In application to the modeling of a porous medium the X-blocks are associated with pores (voids), and the Y-blocks with particles forming the skeleton of the porous body or dispersed aggregates (clusters) of finite size. By choosing the parameters n_i, x_i, *and* y_i appropriately, it is possible to obtain a polydisperse medium with a given size distribution of the pores and (or) particles. Here, the specific volume v_i^x of the pores of size r_i (X-blocks of rank i) is equal to

$$v_i^x = x_i \prod_{j=1}^{i-1} z_j, \tag{1}$$

the specific volume v_i^y of the skeleton particles of size i (Y-blocks of rank i) is equal to

$$v_i^y = y_i \prod_{j=1}^{i-1} z_j, \tag{2}$$

and the porosity ε_N (the total volume fraction of the X-blocks) is equal to

$$\varepsilon_N = \sum_{i=1}^{N} x_i \prod_{j=1}^{i-1} z_j. \tag{3}$$

If from experiment, e.g., from electron-microscopy data, we know the size distribution of the pores and particles for the sequence of characteristic scales $r_1 \gg r_2 \gg \ldots \gg r_N$, i.e., we know the values of v_i^x and v_i^y, then, in accordance with (1) and (2), we can determine the parameters x_i and y_i of the corresponding N-MPS.

A particular case of a general MPS model with $y_i \equiv 0$ is the random Sierpinski carpet (a model of a discrete multi-

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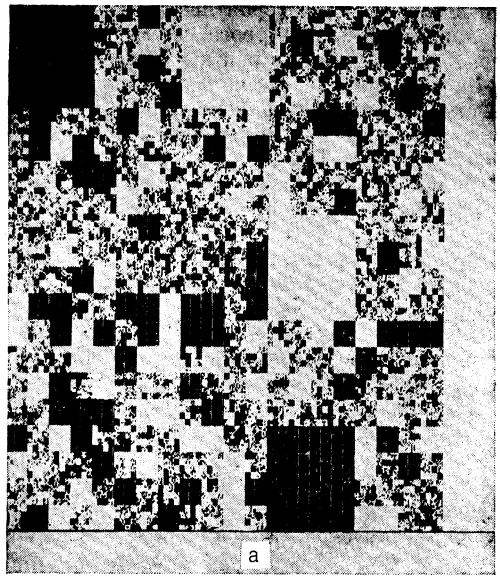

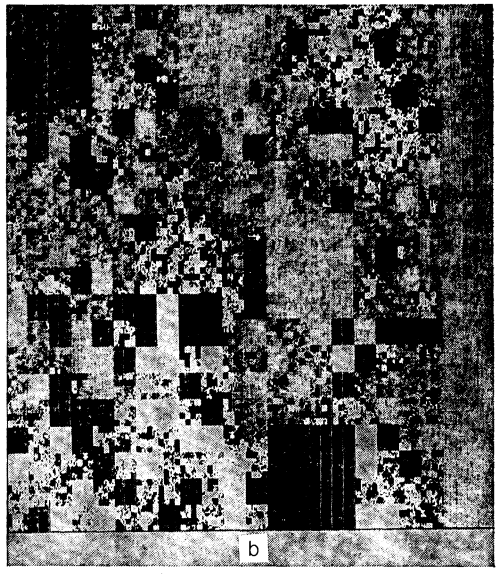

FIG. 1. a—Computer-modeled fragment of an MPS on a square lattice. The MPS fragment, containing 25 first-rank elements, is obtained as a result of three fractionation stages, with parameters $x_1 = 0.125$, $x_2 = 0.167$, $x_3 = 0.25$, $x_4 = 0.6$, $z_1 = 0.75$, $z_2 = 0.666$, $z_3 = 0.5$, $z_4 = 0$, $n_1 = n_2 = n_3 = 4$. The X-blocks are not shaded, and the Y-blocks are shaded black. b—Percolation through X-blocks on the MPS fragment depicted in Fig. a. The infinite cluster in contact with the right boundary of the fragment is shaded grey. Finite clusters of X-blocks are not shaded.

scale random medium[10]), the porosity of which tends to unity as the number of iterations increases. If here we confine ourselves to one iteration, we obtain a two-scale or bidisperse percolation system.[15]

Another particular case of an MPS is the model of scales of Ref. 11, which is obtained, for a constant fractionation multiplicity $n_i \equiv n$, if the division of the elements into X- and Y-blocks is implemented in accordance with the index of the iteration; namely, in the even steps we have $x_{2i} \equiv x$, $y_{2i} \equiv 0$, while in the odd steps we have $x_{2i+1} \equiv 0$, $y_{2i+1} \equiv y$.

SPECIFIC SURFACE OF MANY-SCALE PERCOLATION SYSTEMS

The specific surface of an MPS is defined as the surface ($d = 3$) or perimeter ($d = 2$) of the interface between the X- and Y-blocks per unit volume of the system as a whole. Interfaces between X- and Y-blocks of different ranks make a contribution to this surface. The interfaces between blocks of rank i will be called boundaries of rank i. They appear in the ith stage of the construction of the MPS as a result of fractionation of Z-blocks of rank $i-1$.

The specific surface $\tilde{s}_1$ of all the first-rank boundaries (independently of the type of blocks separated by them) is equal to the ratio of half the surface (perimeter) of a first-rank block to its volume:

$$\tilde{s}_1 = \alpha d r_1^{-1}. \tag{4}$$

Here α is the shape factor of the block; for a cubic mosaic and for a square mosaic, $\alpha = 1$; for a triangular mosaic, $\alpha = \sqrt{3}$. The specific surface $\tilde{s}_i$ of all the boundaries of rank i is proportional to the volume fraction v_{i-1}^z of Z-blocks of rank $i-1$:

$$\tilde{s}_i = \alpha d (n_{i-1} - 1) r_{i-1}^{-1} v_{i-1}^z, \quad v_{i-1}^z = \prod_{j=1}^{i-1} z_j, \quad i = 2, \ldots, N. \tag{5}$$

The fraction of the surface of the rank-i boundaries that belongs to interfaces between X- and Y-blocks is equal to the probability p_i that, at an arbitrarily chosen point of a rank-i boundary, an X-block lies on one side of the boundary and a Y-block on the other. In an N-MPS this probability is equal to

$$p_i^N = 2\{x_i + z_i[x_{i+1} + z_{i+1}(x_{i+2} + \ldots + z_{N-1}x_N)\ldots]\} \times \{y_i + z_i[y_{i+1} + z_{i+1}(y_{i+2} + \ldots + z_{N-1}y_N)]\}. \tag{6}$$

Here the first factor in the curly brackets is the probability that, on a particular side of the point under consideration on a rank-i boundary, there is either an X-block of rank i (probability x_i), or an X-block of rank $i+1$ (probability $z_i x_{i+1}$), or an X-block of rank $i+2$ (probability $z_i z_{i+1} x_{i+2}$), etc., up to the X-block of rank N (probability $z_i z_{i+1} z_{i+2} \ldots z_{N-1} x_N$). The second factor is the analogous probability of realization of a Y-block.

Using the probabilities p_i^N, we can write the expression for the specific surface of the N-MPS in the following form:

$$s_N = \sum_{i=0}^{N} \tilde{s}_i p_i^N = 2\alpha d r_1^{-1} \left(\sum_{j=1}^{N} x_j \prod_{k=1}^{j-1} z_k\right)\left(\sum_{j=1}^{N} y_j \prod_{k=1}^{j-1} z_k\right)$$
$$+ 2\alpha d \sum_{i=2}^{N} (n_{i-1} - 1) r_{i-1}^{-1} \left(\sum_{j=1}^{N} x_j \prod_{k=1}^{j-1} z_k\right)\left(\sum_{j=1}^{N} y_j \prod_{k=1}^{j-1} z_k\right). \tag{7}$$

The specific surface (7) of the MPS does not always tend to a finite quantity as $N \to \infty$. Under certain conditions the quantity s_N increases without limit as the number of iterations increases. The surface of the MPS can then have a fractal character (see below).

SELF-SIMILAR MANY-SCALE PERCOLATION SYSTEMS

In the case when the parameters do not depend on the index of the iteration, i.e., the case of constant probabilities $x_i \equiv x$, $y_i \equiv y$, $z_i \equiv z \equiv 1 - x - y$ and constant fractionation multiplicity $n_i \equiv n$, the local structures of the MPS at each step of the construction turn out to be similar. As the number of steps tends to infinity a Z-block of rank i can be ob-

tained from a Z-block of rank j as a result of a change of all linear scales by a factor of n^{i-j}. In this case all the specific characteristics of the Z-blocks, per unit volume, are independent of their size.

For a self-similar MPS (SMPS) the following power law holds for the decrease of the size of the blocks:

$$r_i=r_1/n^{i-1}. \tag{8}$$

The specific volumes of X- and Y-blocks of rank i are equal to

$$v_i^x=xz^{i-1}, \quad v_i^y=yz^{i-1}. \tag{9}$$

The porosity of a self-similar N-MPS is equal to

$$\varepsilon_N=x\sum_{i=1}^{N}z^{i-1}=x\frac{1-z^N}{1-z}=\frac{x}{x+y}(1-z^N). \tag{10}$$

As the number of steps increases without bound ($n\to\infty$), with the corresponding decrease to zero of the scale of the blocks that are constructed, the porosity of an SMPS tends to a finite quantity

$$\varepsilon=x/(x+y). \tag{11}$$

The specific surface of an SMPS depends on the porosity and fractionation multiplicity. Taking into account that, for an SMPS,

$$\tilde{s}_i=\alpha d(n-1)r_1^{-1}n^{i-2}z^{i-1}, \tag{12}$$

$$p_{i+1}^N=2xy\left(\sum_{j=0}^{N-i}z^j\right)^2=2xy(1-z)^{-2}(1-z^{N-i+1})^2, \tag{13}$$

we obtain, according to (7),

$$s_N=2\alpha dr_1^{-1}xy(1-z)^{-2}\Big\{(1-z^{N+1})^2+(n-1)n^{-1}\times\Big[(nz)^{N+1}\frac{n(n+1)(1-z)^2}{(nz+1)(n-1)(n-z)}-\frac{nz}{nz-1}+\frac{2nz^{n+1}}{n-1}-\frac{nz^{2n+1}}{n/z-1}\Big]\Big\}. \tag{14}$$

Equation (14) is valid under the condition $nz\neq 1$. In the exceptional case when the fractionation multiplicity is given by $n=1/z$, the specific surface of the SMPS is equal to

$$s_N=2\alpha dr_1^{-1}\frac{xy}{(1-z)^2}\Big\{(1-z^{N+1})^2+(1-z)\Big[N-\frac{z(2+z)}{1-z^2}+\frac{2z^{N+1}}{1-z}-\frac{z^{2N+2}}{1-z^2}\Big]\Big\}. \tag{15}$$

The value $n=1/z$ is the critical value of the fractionation multiplicity. For $n<1/z$, the specific surface of the SMPS tends, with unlimited increase of the number of iterations ($N\to\infty$), to a finite quantity:

$$s_N|_{nz<1}\xrightarrow[N\to\infty]{}2\alpha dr_1^{-1}xy(1-z)^{-2}\left(1+\frac{n-1}{n}\frac{nz}{1-nz}\right)=2\alpha dr_1^{-1}xy(1-z)^{-1}(1-nz)^{-1}. \tag{16}$$

If the fractionation multiplicity satisfies $n>1/z$, as $N\to\infty$ the specific surface of the SMPS grows without limit in accordance with an exponential law:

$$s_N|_{nz>1}\xrightarrow[N\to\infty]{}\frac{2\alpha dr_1^{-1}xy(n^2-1)}{(nz+1)(n-1)(n-z)}[(nz)^{N+1}+o((nz)^N)]. \tag{17}$$

For the critical fractionation multiplicity $n=1/z$ the value of s_N increases linearly with N:

$$s_N|_{nz=1}\xrightarrow[N\to\infty]{}2\alpha dr_1^{-1}xy(1-z)^{-1}[N+o(N)]. \tag{18}$$

FRACTAL DIMENSIONALITY OF THE SURFACE OF A SELF-SIMILAR MANY-SCALE PERCOLATION SYSTEM

The exponential law (17) for the growth of the specific surface is evidence of the fractal properties of an SMPS. The situation is entirely analogous to the classical example of the determination of the length of a coast-line.[4] The experimentally determined magnitude of the surface of an MPS depends on the size of the measuring instrument used. The smaller the size r of the measuring instrument, the greater is the measured surface area $s(r)$, since as the size of the measuring instrument decreases finer details of the surface relief can be identified. The boundary of a block of rank i cannot be identified by means of a measuring instrument whose size r exceeds the linear dimensions r_i of the block. With a measuring instrument of size r_i it is possible to determine the surface area $s(r_i)$ of the interface between blocks whose rank is not greater than i. In an N-scale system the surface area $s_N(r_i)$, determined with the aid of a measuring instrument of size r_i, is equal to the surface area of the i-scale system from which, as a result of the following $N-i$ fractionation stages, the N-scale system under investigation was obtained, i.e.,

$$s_N(r_i)=s_i, \quad i\leqslant N. \tag{19}$$

The actual surface area s_N of the N-scale system can be determined with the aid of a measuring instrument of a size smaller than or equal to the size r_N of the smallest blocks; in particular, $s_N(r_N)=s_N$.

The fractal dimensionality d_{fs} of the surface of any multiscale system is defined as

$$d_{fs}=-\lim_{r\to 0}\frac{\ln[s(r)/r^{d-1}]}{\ln r}=d-1-\lim_{r\to 0}\frac{\ln s(r)}{\ln r}. \tag{20}$$

This definition is fully consistent with the classical definition of fractal dimensionality.[4] Here it is assumed that the range of scales in the system under consideration makes it possible to perform measurements of the surface with a sufficiently large number of measuring instruments of different scales. The limit in the definition (20) is to be understood in the sense of tending to some smallest length scale of nonuniformity of the system.

In the case in which, as the size of the measuring instrument decreases, the value of the measured surface area $s(r)$ tends to a finite limit, the fractal dimensionality of the surface of the multiscale system is equal to the dimensionality of the surface (or perimeter) of one block: $d_{fs}=d-1$. In the case of a power dependence $s(r)\propto r^{-\Delta}$, where $0<\Delta<1$, the fractal dimensionality of the surface of the multiscale system exceeds the dimensionality of the surface of one block by an amount Δ, which we shall call the fractal-dimensionality effect:

$$d_{fs}=d-1+\Delta. \tag{21}$$

Of interest is the case in which, as r decreases, $s(r)$ is observed to increase without limit more slowly than by a power law, e.g., logarithmically: $s(r) \propto -\ln r$. In this case, despite the unbounded increase of the surface of the multiscale system, its fractal dimensionality (20) is equal to the dimensionality of the surface of one block. Such surfaces will be called subfractal.

We determine the fractal dimensionality of the surface of an MPS using the block sizes r_i as a sequence of measuring rules. Taking into account the relation (19), we have

$$d_{fs}=d-1-\lim_{i\to\infty}[\ln s_{i+1}/\ln r_i]. \tag{22}$$

For an SMPS the value of the fractal dimensionality of the surface is determined by the relative magnitudes of the fractionation multiplicity and the fraction of Z-blocks. If we take into account that $r_i = r_1 n^{-i+1}$, it follows from the asymptotic forms (16)–(18) that

$$\ln s_N(r_i) \xrightarrow[i\to\infty]{} \begin{cases} \text{const}, & n<1/z, \\ \ln(-\ln r), & n=1/z, \\ -\ln r \ln(nz)/\ln n, & n>1/z, \end{cases} \tag{23}$$

and, correspondingly,

$$d_{fs}=\begin{cases} d-1, & n\leqslant 1/z, \\ d-1+\ln(nz)/\ln n, & n>1/z. \end{cases} \tag{24}$$

Thus, for the critical fractionation multiplicity ($n = 1/z$) the surface of the SMPS is subfractal, while for a fractionation multiplicity greater than the critical value ($n > 1/z$) the surface of the SMPS possesses fractal properties.

By varying the parameters of the SMPS it is possible to obtain a model of a many-scale system with arbitrary porosity $\varepsilon = x/(x+y)$ and surface fractal dimensionality d_{fs}, which can lie in the range from $d-1$ for $z\to 1/n$ to d for $z\to 1$.

We note that the fractal dimensionality of the surface of an SMPS does not depend on its porosity and is equal to the fractal dimensionality of the Sierpinski mosaic (carpet) obtained in the limit $y = 0$ and $\varepsilon = 1$.

POROMETRIC FRACTAL DIMENSIONALITY

One of the most common methods of determining the fractal dimensionality of real materials with a polydisperse pore-size distribution is to analyze the integrated pore-size distribution functions $V(r)$, obtained, e.g., by the methods of mercury porometry or capillary condensation.[1,5] The quantity $V(r)$ is the total volume of pores of size r and larger. Then, if in a certain range of sizes V has a power dependence on r, one associates the parameter d_p in the relation

$$-dV(r)/dr \propto r^{2-d_p} \tag{25}$$

with the fractal dimensionality of the material under investigation (the subscript p indicates that d_p is determined from the porometric curve).

We shall apply this method to an SMPS. In this case, the integrated pore-size distribution function is equal to

$$V(r_i)=\sum_{j=i}^{\infty} v(r_j)=\frac{x}{x+y}(1-z^i)=\frac{x}{x+y}\left[1-z\left(\frac{r_i}{r_1}\right)^{-\ln z/\ln n}\right]. \tag{26}$$

Hence,

$$-\frac{dV}{dr}=\frac{x}{x+y}z\left(-\frac{\ln z}{\ln n}\right)\frac{1}{r_1}\left(\frac{r_i}{r_1}\right)^{-\ln(nz)/\ln n}, \tag{27}$$

$$d_p=2+\ln(nz)/\ln n. \tag{28}$$

We see that the porometric fractal dimensionality d_p of an SMPS for $n\geqslant 1/z$ coincides with the surface fractal dimensionality d_{fs}, while for $n < 1/z$ the quantity d_p has no physical meaning. This example demonstrates that one must take a critical view of the quantity d_p determined from experimental data on the basis of the relation (25). A value of d_p in the range from 2 to 3 is evidence of fractal properties of the surface of the material ($d_{fs} = d_p$) and tells us nothing about the fractal properties of the volume of the pores or the volume of the particles. For $d_p < 2$, only additional experiments, independent of porometry, can confirm the presence of fractal properties of the material under investigation. Since a value $d_p < 2$ is characteristic of many colloid systems of the aerogel type,[1,16] this remark is important.

PERCOLATION PROPERTIES OF A MANY-SCALE PERCOLATION SYSTEM

We formulate the problem of percolation in an MPS (Fig. 1b). Let the X-blocks be conductors, and the Y-blocks insulators. Two X-blocks will be regarded as in contact or linked if they have a common boundary of dimensionality $d-1$ (a common face for $d = 3$, or a common edge for $d = 2$). All the X-blocks are separated into clusters, consisting of X-blocks of different ranks in contact. If the fraction of X-blocks is sufficiently large, besides finite clusters there exists in the MPS an infinite cluster (IC) of X-blocks, which determines the electrophysical and mechanical properties of the MPS. It is of interest to determine the conditions under which a percolation transition (the formation of an IC) is observed in the MPS, and to study the properties of the MPS in the neighborhood of the percolation-transition point. This problem can be solved by successively applying the percolation renormalization-group transformation (PRGT) (Ref. 17) to the Z-blocks of ranks $N-1$, $N-2$, etc., up to those of rank 1. The conditions under which it is possible to apply the PRGT should be stipulated separately at each stage.

We shall consider N-MPS's in which the Z-blocks of rank N are conductors or insulators with probabilities π_N and $1-\pi_N$, respectively. Here, the fraction of conductors among the blocks of rank N amounts to

$$p_N=x_N+z_N\pi_N. \tag{29}$$

The subsequent analysis has meaning when the value of p_N is so close to the percolation threshold p_c of the initial mosaic that the correlation length $\xi_N = |p_N - p_c|^{-\nu}$ (ν is the critical index) of the percolation system of the conductors and insulators in a Z-block of rank $N-1$ is much greater than its size—the fractionation multiplicity n_{N-1}. When the strong inequality $\xi_N \gg n_{N-1}$ is fulfilled, we can apply to the Z-blocks of rank $N-1$ the percolation renormalization-group transformation associated with increase of the microscopic scale of the percolation system by a factor of n_{N-1}. Here, the probability π_{N-1} that a Z-block of rank $N-1$ is an effective conductor is equal to

$$\pi_{N-1}=p_{c,n_{N-1}}+k_{n_{N-1}}(p_N-p_{c,n_{N-1}}). \tag{30}$$

In this relation $p_{c,n}$ is the fixed point of the PRGT associated with increase of the microscopic scale by a factor of n; as $n\to\infty$ we have $p_{c,n}\to p_c$ and $k_n\to n^{1/\nu}$. The value of π is interpreted as the probability that there exists in a block of n^d elements an overlapping cluster of conductors that ensures that the block is an effective conductor.[17] Accordingly, we shall assume that, in its macroscopic properties, an N-MPS is equivalent to an $(N-1)$-MPS in which the Z-blocks of rank $N-1$ have been replaced by conductors and insulators with probabilities π_{N-1} and $1-\pi_{N-1}$, respectively. After this transformation, the fraction of conductors among the blocks of rank $N-1$ amounts to

$$p_{N-1}=x_{N-1}+z_{N-1}\pi_{N-1}. \tag{31}$$

When the strong inequality

$$\xi_{N-1}=|p_{N-1}-p_c|^{-\nu}\gg n_{N-2}$$

is fulfilled, the PRGT can be applied again. By applying the PRGT to the Z-blocks of rank $N-2$, we obtain the equivalent $(N-2)$-MPS, in which the Z-blocks of rank $N-2$ are replaced by conductors and insulators with probabilities π_{N-2} and $1-\pi_{N-2}$, respectively, where

$$\pi_{N-2}=p_{c,n_{N-2}}+k_{n_{N-2}}(p_{N-2}-p_{c,n_{N-2}}), \tag{32}$$

and the fraction of conductors among the blocks of rank $N-2$ amounts to

$$p_{N-2}=x_{N-2}+z_{N-2}\pi_{N-2}. \tag{33}$$

If the parameters of the system are such that, after successive application of the PRGT, it turns out to be possible to make the next application of the PRGT to the Z-blocks of lowest rank, then, as a result of $(N-1)$-fold application of the PRGT, we obtain a one-scale percolation system of conductors and insulators which is equivalent in its macroscopic properties to the initial N-MPS and in which the fraction p_1 of conductors is determined by the chain of recursion relations

$$p_i=x_i+z_i\pi_i, \quad i=N,\ldots,1, \tag{34}$$

$$\pi_i=p_{c,n_i}+k_{n_i}(p_{i,1}-p_{c,n_i}), \quad i=N-1,\ldots,1, \tag{35}$$

which have meaning when the strong inequalities

$$\xi_i=|p_i-p_c|^{-\nu}\gg n_{i-1}, \quad i=N,\ldots,2, \tag{36}$$

are fulfilled. Here, π_i and p_i are the fractions of effective conductors among the Z-blocks of rank i and among all the blocks of rank i, respectively. The following equality serves as the condition for a percolation transition in the system under investigation:

$$p_1=p_c. \tag{37}$$

PERCOLATION THRESHOLD OF A SELF-SIMILAR MANY-SCALE PERCOLATION SYSTEM

In the case of an N-SMPS characterized by the parameters x, z, n, and π_N, the system of equations (34), (35) can be solved explicitly. In the case of an SMPS, it follows from the system of equations (34), (35) that, as a result of applying the PRGT to the Z-blocks of rank i in going from the $(i+1)$-SMPS to the i-SMPS, the following recursion relations hold:

$$p_i-p_{c,n}=x-x_{c,n}+zk_n(p_{i+1}-p_{c,n}), \quad i=N-1,\ldots,1, \tag{38}$$

where

$$x_{c,n}=(1-z)p_{c,n}.$$

Hence, taking (29) into account, we have

$$p_i-p_{c,n}=(x-x_{c,n})+zk_n[(x-x_{c,n})+zk_n[(x-x_{c,n})+zk_n[(x-x_{c,n})+\ldots+zk_n[(x-x_{c,n})+z(\pi_N-p_{c,n})]\ldots]]]. \tag{39}$$

and, consequently, the fraction p_i of effective conductors among the blocks of rank i turns out to be equal to

$$p_i=p_{c,n}+\frac{(zk_n)^{N-i+1}-1}{zk_n-1}(x-x_{c,n})+z(zk_n)^{N-i}(\pi_N-p_{c,n}), \quad zk_n\neq 1,$$
$$p_i=p_{c,n}+(N-i+1)(x-x_{c,n})+z(\pi_N-p_{c,n}). \quad zk_n=1. \tag{40}$$

The value zk_n is specific for the given case. According to (37), the percolation threshold x_c of the N-SMPS, defined as the critical fraction of X-blocks for fixed values of z, N, and π_N, is equal to

$$x_c=x_{c,n}+\frac{zk_n-1}{(zk_n)^N-1}(p_c-p_{c,n})-\frac{(zk_n)^{N-1}(zk_n-1)}{(zk_n)^N-1}z(\pi_N-p_{c,n}), \quad zk_n\neq 1,$$
$$x_c=x_{c,n}+[(p_c-p_{c,n})-z(\pi_N-p_{c,n})]/N, \quad zk_n=1. \tag{41}$$

We shall analyze the correctness of the results (40) and (41). The conditions for applicability of the PRGT are the strong inequalities (36), which, in the case of an SMPS, take the form

$$\left|(p_{c,n}-p_c)+\frac{(zk_n)^{N-i+1}-1}{zk_n-1}(x-x_{c,n})+z(\pi_N-p_{c,n})(zk_n)^{N-i}\right|\ll n^{-1/\nu}, \quad zk_n\neq 1,$$
$$|(p_{c,n}-p_c)+(N-i+1)(x-x_{c,n})+z(\pi_N-p_{c,n})|\ll n^{-1/\nu}, \quad zk_n=1. \tag{42}$$

These conditions will be fulfilled if the strong inequality

$$|(p_{c,n}-p_c)+(x-x_{c,n})+z(\pi_N-p_{c,n})|=|x+z\pi_N-p_c|\ll n^{-1/\nu} \tag{43a}$$

holds; this inequality makes possible the initial application of the PRGT to the Z-blocks of rank $N-1$. For $zk_n<1$, the condition (43a) is also sufficient for the next multiple application of the PRGT to the Z-blocks of lowest rank. For $zk_n>1$, the condition (43a) should be supplemented by the strong inequalities

$$|x-x_{c,n}|\ll N^{-1}n^{-1/\nu}, \quad zk_n=1, \tag{43b}$$

$$|(x-x_{c,n})+(z-k_n^{-1})(\pi_N-p_{c,n})|\ll(zk_n)^{-(N-2)}n^{-1/\nu}, \quad zk_n>1. \tag{43c}$$

It follows from the relations (43) that the possibility of application of the PRGT is determined by the value of the parameter zk_n. The less stringent condition (43a), which is independent of N, holds for $zk_n<1$. For $zk_n>1$, the greater

is N, the closer should the values of x and π_N be to the quantities $x_{c,n}$ and $p_{c,n}$ corresponding to the fixed point of the PRGT.

At the percolation threshold $x = x_c$ (41) the following strong inequalities serve as the condition for the correctness of the $(N-1)$-fold application of the PRGT:

$$\begin{aligned} &|(p_{c,\,n}-p_c)+k_n^{-1}(\pi_N-p_{c,\,n})| \ll n^{-1/\nu}, \quad zk_n \geqslant 1, \\ &|(p_{c,n}-p_c)+k_n^{-1}(\pi_N-p_{c,\,n})| \ll (zk_n)^{-1}n^{-1/\nu}, \; zk_n<1. \end{aligned} \tag{44}$$

Since, for the PRGT with moderate values of n, the conditions

$$|p_{c,\,n}-p_c| \ll n^{-1/\nu}, \quad k_n \infty n^{1/\nu},$$

are always fulfilled, for (44) to be fulfilled it is sufficient that π_N be close to the fixed point of the PRGT:

$$|\pi_N - p_{c,\,n}| \ll 1. \tag{45}$$

The condition (45) is the condition for the correctness of the expression (41) obtained by the PRGT method for the percolation threshold of an SMPS. It should be noted that the condition (45) is the least stringent and does not depend on the geometrical parameters of the SMPS. In fact, at the percolation threshold, with $x = x_c$ given by (41), application of the PRGT is always correct [when the condition (45) is fulfilled]. However, for $zk_n > 1$ a slight deviation from the percolation threshold leads to emergence beyond the region of applicability of the PRGT. Therefore, the deviation of x from x_c that is admissible for application of the PRGT decreases exponentially as a function of N. The condition for applicability of the PRGT for $zk_n > 1$ can be represented in the form

$$|x-x_c| \ll (zk_n)^{-(N-2)}n^{-1/\nu}, \quad zk_n>1 \tag{46a}$$

For $zk_n < 1$ the analogous condition does not depend on N:

$$|x-x_c| \ll n^{-1/\nu}, \quad zk_n<1. \tag{46b}$$

This fundamental difference is due to the fact that for $zk_n \geqslant 1$ the application of the PRGT to the Z-blocks of rank i and the corresponding transformation of the $(i+1)$-SMPS to the i-SMPS leads to an increase of the deviation of the fraction p_i of conductors among the blocks of the smallest size from the fixed point $p_{c,n}$ of the PRGT [see the relations (38)].

In conclusion, we write out the asymptotic forms of Eq. (41) for the percolation threshold of an N-SMPS. As $N \to \infty$ we have

$$\lim_{N\to\infty} x_c = \begin{cases} (1-z)\,p_{c,n} + (1-zk_n)(p_c - p_{c,n}), & zk_n < 1, \\ (1-z)\,p_{c,n}, & zk_n = 1, \\ (1-z)\,p_{c,n} - (z-k_n^{-1})(\pi_N - p_{c,n}), & zk_n > 1. \end{cases} \tag{47}$$

When the fractionation multiplicity $n \to \infty$ and, correspondingly, $k_n \to n^{1/\nu}$ and $p_{c,n} \to p_c$, we have

$$x_c = (1-z)\,p_c - \frac{(zn^{1/\nu})^{N-1}(zn^{1/\nu}-1)}{zn^{1/\nu}-1}\,z(\pi_N - p_c). \tag{48}$$

Equation (48) is correct for a sufficiently large fractionation multiplicity n and arbitrary $zn^{1/\nu} \neq 1$.

For $n \to \infty$ the case $\pi_N = 1$, when all the Z-blocks of the smallest size in the N-SMPS are conductors, is of interest. In this case, the following strong inequality serves as the condition for the existence of an IC:

$$\xi(x+z) = (x+z-p_c)^{-\nu} \ll n, \tag{49}$$

which is the opposite of the condition (36) for application of the PRGT. If the condition (49) is fulfilled, each Z-block turns out to be an effective conductor. Consequently, the percolation threshold of the N-SMPS satisfies

$$x_c \to p_c - z \quad \text{as} \quad n \to \infty, \; \pi_N = 1. \tag{50}$$

Equation (48), obtained in the limit opposite from (49), formally satisfies this asymptotic form. We note that under these conditions, for $x = 0$, when the SMPS model goes over into the random-fractal model of Mandelbrot with a volume density of conductors that tends to zero, an IC exists provided that $z > p_c$ is satisfied. The value $z = p_c$ serves as the percolation threshold with respect to the Z-blocks of the smallest size in the limit $n \to \infty$.

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