Spin-density correlator of a one-dimensional Fermi gas with strong interaction

A. V. Zabrodin and A. A. Ovchinnikov

N. N. Semenov Institute of Chemical Physics, Academy of Sciences of the USSR, Moscow

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A method for finding the correlation functions in a one-dimensional system of spin-$\frac{1}{2}$ particles with strong interaction (repulsion) is proposed. The long-wavelength asymptotic form of the spin-density correlator at zero temperature is found. The results obtained are compared with conclusions based on the hypothesis of approximate conformal invariance of the model under consideration at large distances.

1. INTRODUCTION

In many cases, the information of greatest interest and practical importance about a system is contained in its correlation properties. However, even in simple models, it is usually much more difficult to find the correlation functions than to find the thermodynamic quantities.

In Refs. 1–3 qualitative features of the correlation functions of one-dimensional quantum systems at zero temperature (the power-law decrease at large distances, and the continuous dependence of the exponents on the coupling constants) were elucidated by various methods. These exponents, by analogy with the theory of phase transitions, are called the critical indices or anomalous dimensions of the corresponding operators; their determination is one of the problems of the theory. Several exact results for special models are known[4–7]; on the whole, however, the analysis has been carried out on the basis of various hypotheses that are plausible but difficult to verify.

In our papers (Refs. 8 and 9) we have found the critical indices of the "density-density" correlators and the density matrix in a one-dimensional system of spinless particles with strong interaction. For a fairly broad class of pair-interaction potentials a simple relationship between the critical indices and the sound velocity v in the system has been established. For example, the index α of the "density-density" correlator was found to be equal to

$$\alpha=4\pi\rho v^{-1}, \tag{1}$$

where ρ is the equilibrium density of the particles. This relation is exact in the sense that it is obtained as a result of taking into account all orders of perturbation theory in the small parameter g^{-1} (g^{-1} is the interaction constant). In addition, it was noted in Refs. 8 and 9 that the range of applicability of the relation (1) is in fact wider than follows from its derivation, and that it should be valid also for short-range potentials (e.g., δ-function potentials) and for small g.

In Refs. 10 and 11, a new, extremely effective method for determining the spectrum of the anomalous dimensions in exactly solvable one-dimensional models of quantum field theory (or, equivalently, two-dimensional models of statistical physics), based on the hypothesis of conformal invariance in the long-wavelength limit, was proposed. In effect, this is equivalent to the conjecture, first put forward by Efetov and Larkin in an earlier paper,[1] that the asymptotic form of the correlation functions is determined by the long-wavelength gapless excitations. After the construction of the formal apparatus of two-dimensional conformal field theory[12] and the investigation of finite-size effects,[13,14] it became possible to find the spectrum of the anomalous dimensions by calculating corrections to the energy of the system in a finite volume; this is especially convenient in exactly solvable models. An already extensive literature has been devoted to these questions (see, e.g., Refs. 10, 11, 15, 16). Conformal invariance at large distances is an extremely useful hypothesis, and our faith in its validity has been strengthened by the fact that the critical indices obtained on the basis of conformal invariance agree with the results known previously. It is conformal invariance that explains the universality of the relation (1).

However, all of this pertains to systems consisting of spinless particles. As regards the more realistic case of Fermi particles with spin $\frac{1}{2}$, practically nothing has been known about the correlators. The point is that, when the internal degrees of freedom are taken into account, the structure of the many-particle wave function becomes considerably more complicated and the combinatoric difficulties that arise in any direct calculation appear, at first sight, to be insurmountable.

In the present paper we show that the direct method of calculation developed for spinless particles in Refs. 17 and 8 leads, in certain cases, to success for spin-$\frac{1}{2}$ particles as well. We consider only fermions, since in the problem of bosons with spin the ground state is ferromagnetic[18] and the complication of the wave function does not occur. The model used is described in Sec. 2; it is the direct generalization of the model of Ref. 8 to the case of particles with spin. In the same section, we give the general form of the many-particle wave function, and also the definitions of the main types of correlation functions.

In Sec. 3 the ground-state wave function is constructed in explicit form; here we use the results of Ref. 8. Next we give the calculation of the spin-density correlator. Its critical index is found to be related to the velocity of the charge-density waves. The formula (32) for this index is our main result.

Very recently, a paper by Izergin, Korepin, and Reshetikhin[19] has appeared in which the conformal approach to the calculation of critical indices is extended to multicomponent integrable models solvable by means of a hierarchy of Bethe substitutions. Fermions with spin $\frac{1}{2}$ and a δ-function interaction of the form $V(x-y)=g\delta(x-y)$ belong to this class of models.

As is well known, excitations above the ground state in this model are gapless, but there are branches of excitations with different sound velocities.[20] Therefore, the hypothesis of conformal invariance at large distances is somewhat less

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well-founded than in the one-component case. In fact, for the hypothesis to be valid it is necessary that, in the long-wavelength limit, the system effectively decompose into two noninteracting subsystems with their own sound velocities, and this is not obvious in advance.

In view of this, it is very interesting to compare our results with the predictions obtained within the conformal approach. Section 4 is devoted to this. Unfortunately, such a comparison is possible, as yet, only in the limit $g\to\infty$ in the model of electrons with a δ-function interaction. Nevertheless, it turns out to be extremely instructive, and leads to the conclusion that the results of the two indicated approaches, at least in the limit $g\to\infty$, agree with each other.

Section 5 contains a discussion of the results obtained, and some concluding remarks. Necessary information on the exact solution of the model of free fermions with δ-function interaction is collected in the Appendix.

2. DESCRIPTION OF THE MODEL AND BASIC PROPERTIES OF THE WAVE FUNCTION

We shall consider a one-dimensional system of spin-$\frac{1}{2}$ Fermi particles, the Hamiltonian of which has the form

$$H[\psi^+,\psi]=\sum_{\beta=\pm1}\int_0^L dx\,\frac{\partial\psi_\beta^+}{\partial x}\frac{\partial\psi_\beta}{\partial x}+\frac{1}{2}g\sum_{\beta,\beta'=\pm1}\int_0^L dx\,dx'\,\rho_\beta(x)V(x-x')\rho_{\beta'}(x'),\tag{2}$$

where

$$\rho_\beta(x)=\psi_\beta^+(x)\psi_\beta(x)\tag{3}$$

is the operator of the density of particles with a fixed spin projection; ψ_β^+ and ψ_β are Fermi operators, satisfying the standard single-time anticommutation relations; L is the length of the system; $g>0$ is the interaction constant; $V(x)$ is a certain potential. We assume the external magnetic field to be equal to zero.

We introduce the spin-density operator $\sigma(x)$:

$$\sigma(x)=\rho_1(x)-\rho_{-1}(x).\tag{4}$$

The following correlation functions are of greatest interest: the one-particle density matrix

$$\delta_{\beta\beta'}G(R)=\langle\psi_\beta^+(R)\psi_{\beta'}(0)\rangle\tag{5}$$

and the pair spin-density correlator

$$S(R)=\langle\sigma(R)\sigma(0)\rangle.\tag{6}$$

The function $S(R)$ is expressed in terms of the auxiliary density correlators

$$W_{\beta,\beta'}(R)=\langle\rho_\beta(R)\rho_{\beta'}(0)\rangle\tag{7}$$

as follows:

$$S(R)=2[W_{1,1}(R)-W_{1,-1}(R)].\tag{8}$$

The angular brackets denote averaging over the physical vacuum at zero temperature or Gibbs averaging at nonzero temperature. We shall study only the case of zero temperature.

Since in the system (2) the number of particles is conserved, we can go over to the first-quantization formalism and rewrite the Hamiltonian in a sector with $2N$ particles:

$$\hat H=-\sum_{i=-N}^{N}{}'\frac{\partial^2}{\partial x_i^2}+g\sum_{i<j}{}'V(x_i-x_j).\tag{9}$$

With respect to the spin variables, (9) is the unit matrix. We shall find it convenient to label the particles from $-N$ to -1 and from 1 to N; the prime on the summation symbol means that $i\neq0$. We impose periodic boundary conditions, i.e., place the particles on a circle of circumference L. Here, strictly speaking, it is necessary to assume that the potential $V(x)$ is periodic with period L, but in the thermodynamic limit

$$L\to\infty,\quad N\to\infty,\quad L/2N=a=\rho^{-1}\tag{10}$$

this does not play a role.

We note that for

$$V(x)=\delta(x)\tag{11}$$

the model (9) can be solved exactly.[21,22] For

$$V(x)=x^{-2}\tag{12}$$

there is also an exact solution, but of an entirely different type; this is the so-called Sutherland model.[7]

We shall comment on the potentials for which our method is valid. As follows from Refs. 8 and 9, the potential $V(x)$ should fall off sufficiently rapidly with distance (faster than x^{-1}), but should also not have too short a range. Thus, the potential (11) will not do, while (12) is in the range of applicability. To formulate particular exact conditions for the potential is rather difficult, but their origin is perfectly clear: The potential should be such that the particles in the ground state should form a "Wigner crystal" [with lattice constant a; see (10)], which serves as the zeroth approximation for the perturbation theory in g^{-1} developed in Ref. 8. This implies that the complete wave function should have a rather sharp maximum at that point of configuration space at which the particles form a regular lattice.

We now recall, following Ref. 18, the general structure of the wave function of the ground state of the system (9). In Ref. 18, a proof was given of the theorem that, in the absence of a magnetic field, the ground state has zero total spin. Therefore, we shall assume immediately that the spin part of the wave function corresponds to zero projection of the spin. Let s_j be the spin variables, taking the values $+1$ and -1. The complete wave function should be antisymmetric under the simultaneous interchange of the coordinates and spins of any two particles. Therefore, it can be sought in the form

$$\Psi\left\{\begin{pmatrix}x_{-N}\\s_{-N}\end{pmatrix},\dots,\begin{pmatrix}x_{-1}\\s_{-1}\end{pmatrix},\begin{pmatrix}x_1\\s_1\end{pmatrix},\dots,\begin{pmatrix}x_N\\s_N\end{pmatrix}\right\}=\sum_P(-1)^{\eta(P)}\Phi(x_{P(-N)},\dots,x_{P(-1)}|x_{P(1)},\dots,x_{P(N)})\times\prod_{k=1}^{N}\delta_{-1,s_{P(-k)}}\delta_{+1,s_{P(k)}},\tag{13}$$

where $\eta(P)$ is the parity of the permutation P and the sum is taken over all permutations of the $2N$ particles. The coordinate part $\Phi(x_1,\dots,x_N|y_1,\dots,y_N)$, which depends on two

sets of variables, should be antisymmetric in the $\{x_j\}$ and the $\{y_j\}$ separately. The equal numbers of Kronecker symbols $\delta_{+1,s}$ and $\delta_{-1,s}$ in (13) ensures that the projection of the total spin is equal to zero. The condition that Ψ corresponds to zero total spin (the Fock condition) has the form

$$\left(1-\sum_{i=1}^{N}\hat{P}_{x_N,y_i}\right)\Phi(\{x_j\}\,|\,\{y_j\})=0, \tag{14}$$

where $\hat{P}_{x,y}$ is the operator whose action on the function Φ corresponds to interchange of the arguments x and y.

The correlator (7) can be expressed in terms of Ψ as follows:

$$W_{s,s'}(R)=\sum_{s_{-N+1}\ldots s_{N-1}}\int_0^L\left(\prod_{k=1}^{N-1}dx_k\,dx_{-k}\right)\times\left|\Psi\left\{\begin{pmatrix}0\\s'\end{pmatrix},\begin{pmatrix}R\\s\end{pmatrix},\begin{pmatrix}x_{-N+1}\\s_{-N+1}\end{pmatrix},\ldots,\begin{pmatrix}x_{N-1}\\s_{N-1}\end{pmatrix}\right\}\right|^2. \tag{15}$$

Next we shall consider the correlator $W_{1,-1}(R)$; the calculation of $W_{1,1}(R)$ is completely analogous. Substituting (13) into (15), we obtain, after summation over the spin variables,

$$W_{1,-1}(R)=\int_0^L\prod_{k=1}^{N-1}dx_k\,dx_{-k}\,|\,\Phi(0,x_{-N+1},\ldots,x_{-1}\,|\,x_1,\ldots,x_{N-1},R)\,|^2. \tag{16}$$

In going from (15) to (16) we have omitted a numerical factor of no importance for us. Since we shall not pursue the normalization, henceforth, in certain expressions [e.g., (24), (30), etc.], the equalities are to be understood as fulfilled to within a constant. For us, the only thing of importance is that this constant is the same for $W_{1,-1}$ and $W_{1,1}$, and this is easily verified. The relation (16) is the starting point for the subsequent calculations.

Thus, the problem reduces to finding the function Φ. It is clear that it is sufficient to know it in the sector Ω of configuration space determined by the conditions

$$x_{-N}<x_{-N+1}<\ldots<x_{-1};\quad x_1<x_2<\ldots<x_N,$$

and it can be continued into all the other sectors in an antisymmetric manner. Therefore, for the Schrödinger equation with the Hamiltonian (9) it is necessary to find the lowest-energy solution that vanishes on the boundary of the sector Ω. These boundary conditions are less stringent than for spinless fermions: In the spinless case (or in the ferromagnetic state) we require the solution to vanish on the boundary of the sector Δ specified by the conditions

$$x_{-N}<x_{-N+1}<\ldots<x_{-1}<x_1<x_2<\ldots<x_N$$

and lying within the sector Ω. Therefore, the energy of a state with zero spin will, generally speaking, be lower than that of a state with nonzero spin.

3. CALCULATION OF THE SPIN-DENSITY CORRELATOR

In the case of large g and R the main contribution to the integral (16) should be made by small neighborhoods of the sharp maxima of the function Φ that correspond to assignments of the particles to the sites of a regular lattice.[9] It is obvious that there exist $(2N)!$ such maxima, corresponding to the various permutations of the $2N$ particles. The decisive simplification for large g is that, to within terms that are exponentially small in g, the coordinate wave function in the sector Ω can be expressed in terms of a simpler wave function in the sector Δ. In fact, for strong repulsion, the boundary conditions in Δ and in the enveloping sector Ω are automatically consistent with each other.

To be more precise, let P be some permutation of the $2N$ particles. We denote by Δ_P the sector in which

$$x_{P(-N)}<x_{P(-N+1)}<\ldots<x_{P(-1)}<x_{P(1)}<x_{P(2)}<\ldots<x_{P(N)}$$

(so that $\Delta=\Delta_I$, where I is the identical permutation). Let $\Phi_0(x_{-N},\ldots,x_N)$ be the lowest-energy eigenfunction of the Hamiltonian (9) in the sector Δ. We recall that if g is large, and we are interested only in the neighborhood of a maximum of the wave function, the boundary conditions in Δ have practically no effect on the shape of the maximum. More precisely, for $g=\infty$ the ground state is degenerate in the spin; for large but finite g the degeneracy is lifted and the energy begins to depend on the boundary conditions, but the splitting of the levels is exponentially small in g (Ref. 20). The sector Ω contains many sectors of the type Δ_P. In the region of intersection of Ω and Δ_P we can represent $\Phi(\{x_{-j}\}|\{x_j\})$ in the form

$$\Phi(\{x_{-j}\}\,|\,\{x_{+j}\})=F(P)\Phi_0(x_{P(-N)},\ldots,x_{P(N)}). \tag{17}$$

The function Φ_0 does not depend on the sector Δ_P (only the order of its arguments changes). The dependence on Δ_P is contained in the numerical factor $F(P)$. We note that in the spinless case the representation (17) is, of course, also valid: $F(P)=(-1)^{\eta(P)}$ (see Ref. 9). But, in a more general situation, nothing prevents $F(P)$ from having different moduli in different sectors.

What determines the form of the factor $F(P)$? One of the constraints is the Fock condition (14), but this is not sufficient to fix $F(P)$. The explicit form of $F(P)$ can be determined by noting that the degeneracy of the ground state for $g\to\infty$ is lifted by an effective Heisenberg Hamiltonian

$$\hat{h}=J\sum_{k=-N}^{-2}\mathbf{S}_k\mathbf{S}_{k+1}+J\mathbf{S}_{-1}\mathbf{S}_1+J\sum_{k=1}^{N-1}\mathbf{S}_k\mathbf{S}_{k+1}+J\mathbf{S}_N\mathbf{S}_{-N}, \tag{18}$$

where the $\mathbf{S}_k$ are spin operators (σ-matrices) and $J>0$ is the exchange integral, which is exponentially small in g. Let k_1, k_2, ..., k_N be the labels of the variables x_1, x_2, ..., x_N after the permutation P [i.e., $k_j=P(j)$], and let P_- and P_+ be the permutations of the sets $\{x_{-N},\ldots,x_{-1}\}$ and $\{x_1,\ldots,x_N\}$, induced by the permutation P. Then

$$F(P)=(-1)^{\eta(P_+)+\eta(P_-)}\chi_0(k_{P_+(1)},k_{P_+(2)},\ldots,k_{P_+(N)}), \tag{19}$$

where $\chi_0(k_1,\ldots,k_N)$ is the wave function of the ground state of the antiferromagnetic Heisenberg chain of $2N$ links [(18)] in the sector $k_1<k_2<\ldots<k_N$. Here, the k_j are the coordinates of the inverted spins in the Bethe method.[23] We

note that since for $P_+ = I$ we have $k_1 < k_2 < \ldots < k_N$, we have $k_{P,(1)} < k_{P,(2)} < \ldots k_{P,(N)}$. It can be verified that the function (17) with the factor (19) satisfies the Fock condition.

For the model of fermions with δ-function interaction the above form of the fact $F(P)$ for $g \to \infty$ can be obtained by direct calculation on the basis of the known exact solution. The corresponding calculations are given in the Appendix.

We note that the excitations above the antiferromagnetic vacuum in the model (18) are gapless; the velocity of the spin waves is proportional to J, and so is exponentially small in g for $g \to \infty$, while the velocity v of the charge waves behaves as $g^{1/2}$ and, to within exponentially small corrections, is equal to the sound velocity in the corresponding spinless system. Below, we shall make use of this circumstance.

We now study the function Φ_0 in (17). Obviously, it should coincide with the wave function of the ground state of spinless particles in the sector Δ, for which the following representation was obtained in our paper Ref. 8. In place of the coordinates x_n in Δ we introduce the "phonon" variables φ_n (Ref. 17):

$$\varphi_n = x_n - na, \qquad n \geqslant 1,$$
$$\varphi_n = x_n - (n+1)a, \qquad n \leqslant -1,$$

which, even in the zeroth approximation, take account of the presence of the Wigner crystal. Their Fourier components have the form

$$\varphi_q = (2N)^{-1/2} \sum_{n=0}^{2N-1} \varphi_n e^{iqn}, \qquad \varphi_q^+ = \varphi_{-q}, \tag{20}$$

where q is the dimensionless "momentum" which takes the discrete values $2\pi ma/L$, with m an integer. Then $|\Phi_0|^2$ as a function of the variables φ_q can be written in the form[8]

$$|\Phi_0|^2 = \int_{-\infty}^{\infty} \prod_q dJ_q \exp\left\{ -\frac{1}{2} \sum_q J_q D(q) J_{-q} + \sum_{k=3}^{\infty} \sum_{q_i} D^{(k)}(q_1 \ldots q_k) \delta\left(\sum_{i=1}^{k} q_i \right) \prod_{j=1}^{k} J_{q_j} + i \sum_q J_q \varphi_q \right\}. \tag{21}$$

The summation here is performed over the Brillouin zone from $-\pi$ to π, the J_q are auxiliary integration variables, and $D(q)$ and $D^{(k)}(q_1, \ldots, q_k)$ are certain functions that can be expressed in terms of the exact Green's functions of the phonon system. For us, only their behavior at small momenta is important[9]:

$$D(q) = a(vq)^{-1}, \qquad q \to 0, \tag{22}$$

$$D^{(k)}(q_1 \ldots q_k) \to \text{const}, \qquad q_j \to 0. \tag{23}$$

Here, v is the velocity of propagation of the charge excitations. We shall assume that Φ_0 is continued from Δ into all the other sectors in a symmetric manner.

We divide the integration range $0 < x_j < L$ in (16) into nonintersecting sectors $\Delta^{(n)}$. In the sector $\Delta^{(n)}$, exactly n integration variables lie in the interval from 0 to R. Each sector $\Delta^{(n)}$ is further divided into subsectors $\Delta_P^{(n)} = \Delta^{(n)} \cap \Delta_P$. Collecting (17), (19), and (21) together, we can represent (16) in the form

$$W_{1,-1}(R) = \sum_n \sum_P \chi_0^2(k_{P_+(1)}, \ldots, k_{P_+(N)}) \times \int_{\Delta_P^{(n)}} \prod_{i=1}^{N-1} dx_i \, dx_{-i} |\Phi_0(\{x_{P(j)}\})|^2, \tag{24}$$

where the sum is taken over permutations P satisfying the conditions

$$P(-N) \neq 1, \qquad P(N) = n+1 = k_N,$$
$$P(j) \neq 1, \qquad j = 1, \ldots, N, \tag{25}$$

and we assume $x_{-N} = 0$ and $x_N = R$.

The last integral in (24) is determined entirely by the sector $\Delta^{(n)}$, and, by virtue of the symmetry of Φ_0, does not depend on P; we shall denote it by $Q(n)$. We note now that the remaining sum over P [we denote it by $H_{1,-1}(n)$] is, under the conditions (25), none other than the correlator (shifted by a constant) of the z-components of the spin $\mathbf{S}_j$ in the model (18):

$$H_{1,-1}(n) = \langle (1/2 + S_1^{(z)})(1/2 - S_{n+1}^{(z)}) \rangle = 1/4 - \langle S_1^{(z)} S_{n+1}^{(z)} \rangle \equiv 1/4 - A(n). \tag{26}$$

We note here that the analogous calculation of $W_{1,1}(R)$ gives

$$H_{1,1}(n) = 1/4 + A(n). \tag{27}$$

As shown in Refs. 8 and 9, in sums of the type (25) the value of the critical index is influenced only by terms with large (of the order of R/a) labels n. Therefore, we need only the asymptotic form of $A(n)$ for $n \gg 1$, which is known from Refs. 2, 6, and 10:

$$A(n) \to (-1)^n/n + \text{const}/n^2 + \ldots. \tag{28}$$

From Ref. 8 we have [it is necessary to make use of the relations (22)–(24)]

$$Q(n) \to \exp[-\pi v(na-R)^2/4a \ln(\pi n)]. \tag{29}$$

Taking the thermodynamic limit, from (24) we find

$$W_{1,\pm 1}(R) = \sum_n [1/4 \pm A(n)] Q(n) = \rho^2/4 + C_1(\rho R)^{-\alpha} \cos(2\pi\rho R) \pm C_2(\rho R)^{-1-\alpha/4} \cos(\pi\rho R) \pm C_3(\rho R)^{-1-9\alpha/4} \times \cos(3\pi\rho R) + \ldots, \quad R/a \gg 1, \tag{30}$$

where the C_j are certain constants and α is given by Eq. (1). In deriving (30) we used the Poisson summation formula and retained only a few leading terms.

Thus, with the aid of (8) we obtain the leading term of the asymptotic form of the pair spin-density correlator:

$$S(R) \propto (\rho R)^{-\mu} \cos(\pi\rho R), \tag{31}$$

in which the critical index μ is expressed in terms of the equilibrium density and the velocity of the charge excitations:

$$\mu = 1 + \pi\rho v^{-1} = 1 + \alpha/4. \tag{32}$$

We note that since, for large g, the ratio ρ/v is small, the index μ is close to unity. In this situation, it makes sense to retain also the next-leading terms in the expression for $S(R)$ [see (30)]:

$$S(R) \propto \sum_{m=0}^{\infty} C_m \cos[(2m+1)\pi\rho R](\rho R)^{-1-\alpha(m+1/2)^2}, \quad (33)$$

and the exponents differ from μ by an amount of order α.

The result obtained has the following simple meaning. The correlator $S(R)$ can be obtained by averaging the spin correlator $A(n)$ in the Heisenberg model over the oscillations of the spins about the equilibrium position. In fact, let $q(R)$ be the operator of the number of particles on the segment from 0 to R. Then the average over the oscillations gives $S(R) \propto \langle A(q(R))\rangle$, which agrees with Eq. (30), since $S(R) \propto \Sigma_n A(n)Q(n)$, and $Q(n)$ is the probability of finding n particles on the segment from 0 to R.

4. THE SPIN-DENSITY CORRELATOR AND CONFORMAL-INVARIANCE HYPOTHESIS

In this section we compare the results of our approach with the predictions of the conformal theory for a model to which both methods are applicable. Unfortunately, up to now only one such model is known—the Fermi gas with the δ-function interaction (11) with $g = \infty$.

Here it is necessary to give some explanation, since this potential can in no way be called long-range, and would appear to lie outside the region of applicability of our method. Indeed, the parameter of our perturbation theory was the ratio ρ/v, which is by no means small for $g \to \infty$. Nevertheless, a Wigner crystal is formed, and it is possible to show that the method of steepest descent, applied in the neighborhood of the maximum of the wave function, gives in the spinless case the correct value of the index α, viz., $\alpha = 2$. In the Appendix it is shown that the coordinate wave function in the spin case for $g \to \infty$ is given by Eq. (17), in which $\Phi_0(x_{-N}, \ldots, x_N)$ coincides with the wave function of spinless particles for $g = \infty$. This implies that for a comparison with the conformal approach it is possible to use Eqs. (31), (32) with $\alpha = 2$.

A pointer to the possible conformal invariance at large distances is the power-law asymptotic form of the correlation functions. It is found that, at least in the spinless case, the effective long-wavelength theory is indeed conformal invariant.[10] This gives a powerful method of determining the critical indices, since they should then be expressed in terms of the conformal dimensions of the primary operators. As shown in Refs. 13 and 14, the latter can be found by calculating the volume corrections or temperature corrections to the energy of the system.

Let ε_0 be the energy density of the ground state, let $E_0(L)$ be the energy of the system in a "box" of length L with periodic boundary conditions, and let $E_\varphi(L)$ be the energy of the first excited state on one of the branches of the excitations. To each such branch there corresponds a primary operator φ with anomalous dimension h_φ, and

$$E_\varphi(L) - E_0(L) = 2\pi v h_\varphi L^{-1}, \quad L \to \infty, \quad (34)$$

while for the central charge c of the conformal theory we have

$$E_0(L) - L\varepsilon_0 = -\pi v c (6L)^{-1}, \quad L \to \infty. \quad (35)$$

The sound velocity v appears in these formulas for the following reason. At first sight, a theory with dispersion law $\varepsilon(q) = v(q)$ is not conformally invariant, since space and time (or momentum and energy) are not on an equal footing in it. In fact, however, to restore the conformal invariance it is sufficient to change the units of measurement along one of the axes by a factor of v, in order that the dispersion law take the form $\varepsilon = q$ with $v = 1$.

By calculating the right-hand side of (34), we immediately find the critical index of the correlator $\langle\varphi\varphi\rangle$, equal to $2h_\varphi$. The formula (35) for specific one-component systems always gives $c = 1$ (Refs. 15, 16), and there exists only one sound velocity v. In order to find all the critical indices, it is sufficient to classify those excitation elements which become gapless in the limit $L \to \infty$.

Here it is necessary to note two differences from the abstract conformal theory. First, in the asymptotic series, as a rule, there appear terms with different powers of R [see, e.g., (30)]. The point is that the operators of physical interest, such as, e.g., the local density of particles, are not in fact primary operators, i.e., do not possess a definite conformal dimension. Nevertheless, they can be represented in the form of a certain linear combination of different primary operators, and this gives in the correlator a sum of terms with different exponents. Second, such terms can contain oscillating factors of the form $\cos(m\pi\rho R)$ in (30). The appearance of these factors is caused by the fact that the corresponding excitation branch has a gap, equal to $m\pi\rho$, in the spectrum of the momentum operator.

Recently, in Ref. 19, expressions analogous to (34), (35) were obtained for the volume corrections to the energy of multicomponent integrable systems solvable by a hierarchy of Bethe substitutions. Below, we shall discuss a two-component system (corresponding to spin $\frac{1}{2}$). The right-hand sides of these formulas consist of two terms, each of which coincides with the right-hand side of the expressions (34), (35) with different values of the parameters v and h_φ but with the same $c = 1$. Starting from this fact, the authors of Ref. 19 put forward the hypothesis that the initial two-component system should decompose, in the long-wavelength limit, into two noninteracting conformally invariant subsystems with unit central charge.

For the correlator $W_{1,1}(R)$, the general relations of Ref. 19, applied to a system of fermions with the potential (11), give

$$W_{1,1}(R) \propto \sum_{m_1,m_2} C_{m_1,m_2} \exp[2\pi i\rho R(m_1+m_2/2)] \times (\rho R)^{-2h_1(m_1,m_2)-2h_2(m_1,m_2)}. \quad (36)$$

The numbers m_j are integers, and the dimensions h_j are determined from the formulas

$$h_1(m_1, m_2) = (Z_{11}m_1 + Z_{21}m_2)^2, \quad h_2(m_1, m_2) = (Z_{12}m_1 + Z_{22}m_2)^2. \quad (37)$$

Here, $Z_{\beta\gamma}$ is the so-called dressed-charge matrix (see the Appendix). We must find it in the limit $g \to \infty$. This limit is studied in the Appendix; the limiting matrix has the form

$$Z^{(\infty)}=\begin{pmatrix}1 & 0\\ 1/2 & 1/\sqrt{2}\end{pmatrix}. \tag{38}$$

By substituting (38) into (37) and writing out the terms of the series (36), one can convince oneself that the result coincides with the series (30) for $\alpha=2$. Thus, the hypothesis of the conformal invariance of the two-component system is correct, at least in the limit $g\to\infty$.

5. CONCLUDING REMARKS

Thus, we have found the asymptotic form of the spin-density correlator for a wide class of models, and, in one particular case, have found agreement with the predictions based on the hypothesis of conformal invariance in the long-wavelength limit. How one might manage a similar comparison in other, more interesting cases is not yet clear. The point is that the procedure for finding the anomalous dimensions from considerations of conformal invariance has been worked out, up to now, only for integrable models with local interaction, and, in the version presently existing, uses constructions that are only applicable to integrable systems. On the other hand, it is known that in the one-component case integrability is not, in fact, important, and the volume corrections (34), (35) to the energy, and hence the dimensions h_φ as well, can be expressed in terms of general thermodynamic characteristics of the system. It is important only to know the elementary gapless excitations, and they are arranged in qualitatively the same way for the majority of one-component models. It is this which constitutes the reason for the universal dependence of the critical index (1) on the sound velocity.

As we have already noted, in the two-component (spin-$\frac{1}{2}$) case the conformal invariance at large distances is less obvious because of the presence of two different sound velocities. We hope that our results will help in the analysis of what happens in two-component systems of general form, and then, possibly, in the consideration of these systems from the conformal point of view. This would give an effective method, independent of the integrability of the model, for finding the critical indices of correlation functions.

In this paper, we have studied only the spin correlator (6). No less interesting is the one-particle density matrix (5), which, in principle, can also be considered by our method. The calculations, however, are substantially more complicated, as in the case of spinless particles, for which the density matrix has a more complicated structure than the pair correlation function (cf. Refs. 9 and 8). In addition, an extra difficulty arises, associated with the fact that, instead of the relatively simple correlator (26), we need to know the average of a certain nonlocal operator of a cyclic permutation of n spins over the antiferromagnetic vacuum of the Heisenberg chain.

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APPENDIX

In this Appendix we gather together the information that we need about a system of spin-$\frac{1}{2}$ fermions with Hamiltonian

$$\hat{H}=-\sum_{i=1}^{N_1}\frac{\partial^2}{\partial x_i^{\,2}}+2g\sum_{i<j}^{N_1}\delta(x_i-x_j). \tag{A1}$$

According to Gaudin[21] and Yang,[22] the coordinate part of the N_1-particle wave function corresponding to total spin $N_1/2-N_2$ has the form

$$\Phi=\sum_P [Q,P]\exp\Big(i\sum_{j=1}^{N_1}\lambda^{(1)}_{P(j)}x_{Q(j)}\Big) \tag{A2}$$

in the sector Δ_Q (i.e., for $x_{Q(1)}<x_{Q(2)}<\ldots<x_{Q(N_1)}$; we have changed the labeling of the particles in comparison with that in the main part of the article), where Q and P are permutations of N_1 particles. The coefficients $[Q, P]$ are given by the formulas

$$[Q,P]=(-1)^{\eta(P)+\eta(Q_+)+\eta(Q_-)}\sum_R A_R\prod_{j=1}^{N_2}\Xi_P(\lambda^{(2)}_{R(j)},k_j), \tag{A3}$$

where

$$\Xi_P(\lambda^{(2)},k)=\prod_{j=1}^{k-1}\frac{\lambda^{(1)}_{P(j)}-\lambda^{(2)}+ig/2}{\lambda^{(1)}_{P(j+1)}-\lambda^{(2)}-ig/2}, \tag{A4}$$

$$A_R=\prod_{\substack{i<j\\ R(i)>R(j)}}\frac{\lambda^{(2)}_{R(j)}-\lambda^{(2)}_{R(i)}+ig}{\lambda^{(2)}_{R(j)}-\lambda^{(2)}_{R(i)}-ig}. \tag{A5}$$

Here the k_j are the labels of the variables $x_{N_1-N_2+1},\ldots,x_{N_1}$ after the permutation Q; Q_+ and Q_- are the permutations induced on the sets $1,2,\ldots,N_1-N_2$ and $N_1-N_2+1,\ldots,N_1$: the $\lambda^{(2)}_j$ are auxiliary quasimomenta.

The periodic boundary conditions lead to the following system of transcendental equations for the quasimomenta:

$$L\lambda^{(1)}_j=2\pi I^{(1)}_j-2\sum_{r=1}^{N_2}\operatorname{arctg}\frac{2(\lambda^{(1)}_j-\lambda^{(2)}_r)}{g}, \tag{A6a}$$

$$2\sum_{j=1}^{N_1}\operatorname{arctg}\frac{2(\lambda^{(2)}_r-\lambda^{(1)}_j)}{g}=2\pi I^{(2)}_r+2\sum_{s\neq r}\operatorname{arctg}\frac{\lambda^{(2)}_r-\lambda^{(2)}_s}{g}. \tag{A6b}$$

The different solutions of this system are parametrized by integer or half-integer numbers $I^{(1)}_j$, $I^{(2)}_r$. We shall be interested only in the ground state, which is always a singlet state, i.e., $N_1=2N_2$ (we shall assume N_1 to be even). In the thermodynamic limit, Eqs. (A6) are replaced by integral equations for the distribution functions $g_1(\lambda^{(1)})$ and $g_2(\lambda^{(2)})$ of the numbers $\lambda^{(1)}$ and $\lambda^{(2)}$ in the ground state:

$$2\pi g_1(\lambda^{(1)})=1+2\int_{-\Lambda_2}^{\Lambda_2}d\lambda^{(2)}K(2\lambda^{(1)}-2\lambda^{(2)})\,g_2(\lambda^{(2)}), \tag{A7a}$$

$$2\pi g_2(\lambda^{(2)})=-\int_{-\Lambda_2}^{\Lambda_2}d\mu^{(2)}K(\lambda^{(2)}-\mu^{(2)})\,g_2(\mu^{(2)})+2\int_{-\Lambda_1}^{\Lambda_1}d\lambda^{(1)}K(2\lambda^{(1)}-2\lambda^{(2)})\,g_1(\lambda^{(1)}), \tag{A7b}$$

where Λ_1 and Λ_2 are the Fermi momenta for the "charge" quasiparticles and the "spin" quasiparticles, respectively, and the kernel $K(\lambda)$ has the form

$$K(\lambda)=2g/(\lambda^2+g^2). \tag{A8}$$

It is known that in the ground state in zero magnetic field we have $\Lambda_2=\infty$ but Λ_1 is finite.

We have chosen this cumbersome notation, with indices 1 and 2, in order to stress that the two components are formally on an equal footing, and to have the possibility of writing the system (A7) in matrix form. To be precise, introducing the matrix

$$\hat{K}(\lambda)=\begin{pmatrix} 0 & 2K(2\lambda) \\ 2K(2\lambda) & -K(\lambda) \end{pmatrix}, \tag{A9}$$

we can represent (A7) in a compact matrix form. We shall also need the system of equations for the "dressed"-charge matrix $Z_{\beta\gamma}(\lambda^{(\gamma)})$ (Ref. 19):

$$Z_{\beta\tau}(\lambda^{(\tau)})-\frac{1}{2\pi}\sum_{\nu=1}^{2}\int_{-\Lambda_\nu}^{\Lambda_\nu} Z_{\beta\nu}(\lambda^{(\nu)})\hat{K}_{\nu\tau}(\lambda^{(\nu)}-\lambda^{(\tau)})\,d\lambda^{(\tau)}=\delta_{\beta\tau}. \tag{A10}$$

We denote $Z_{\beta\gamma}\equiv Z_{\beta\gamma}(\Lambda_\gamma)$; these are the quantities that appear in (37).

We recall that our problem is to find the wave function (A2) and the dressed-charge matrix in the limit $g\to\infty$. It is convenient to make the replacement $\lambda^{(\beta)}\to g\lambda^{(\beta)}$ in all the equations. Here it is important that the ratio Λ_1/g can be assumed to be equal to zero when $g_1\to\infty$, while the ratio Λ_2/g is, as before, infinite. In addition, the nondiagonal elements of the matrix $\hat{K}$ cease to depend on $\lambda^{(1)}$. The system (A6) becomes

$$L\lambda_j^{(1)}=2\pi I_j^{(1)}+2\sum_{s=1}^{N_1/2}\operatorname{arctg}(2\lambda_s^{(2)}), \tag{A11a}$$

$$N_1\operatorname{arctg}(2\lambda_r^{(2)})=2\pi I_r^{(2)}+2\sum_{s\neq r}\operatorname{arctg}(\lambda_r^{(2)}-\lambda_s^{(2)}), \tag{A11b}$$

where for $\lambda^{(1)}$ we have kept the previous scale, but $\lambda^{(2)}$ has been contracted by a factor of g. We see that Eq. (A11b) splits off from the system and that it is transformed into the Bethe equation for the antiferromagnetic Heisenberg chain.[23] In addition, Ξ_P (A3) ceases to depend on P, and

$$\sum_R A_R\prod_{j=1}^{N_1/2}\Xi_P(\lambda_{R(j)}^{(2)},k_j)$$

becomes the Bethe wave function for the Heisenberg chain. Therefore, (A2) in the limit $g\to\infty$ does indeed have the structure described by Eqs. (17) and (19), with

$$\Phi_0\propto\sum_P(-1)^{\eta(P)}\exp\Big(i\sum_{j=1}^{N_1}\lambda_{P(j)}^{(1)}x_j\Big).$$

It is also not difficult to find the solution of the system (A10) in this limit. It can be seen immediately that $Z_{12}(\lambda)=0$, and then $Z_{11}(\lambda)=1$, as for free spinless fermions. The integral equation for $Z_{22}(\lambda)$ splits off and coincides with the equation for the dressed-charge function in the Heisenberg chain, where, as is well known, we have $Z_{22}=Z_{22}(\Lambda_2)=1/\sqrt{2}$ (Ref. 6). Finally, for Z_{21} we have

$$Z_{21}=\frac{1}{\pi}\int_{-\infty}^{\infty}d\lambda\,Z_{22}(\lambda)K_0(2\lambda),$$

with $K_0(\lambda)=2/(\lambda^2+1)$. Using the known integral equations for the Heisenberg model,[23] we find $Z_{21}=\frac{1}{2}$.

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