Dissipative processes in superfluid ^{3}He–^{4}He solutions

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(Submitted 21 May 1989)
Zh. Eksp. Teor. Fiz. **96**, 1286–1300 (October 1989)

Dissipative processes in low-concentration ^{3}He–^{4}He solutions are considered in the entire temperature interval in which the quasiparticle description is valid. All the kinetic coefficients and the second-sound damping factor are calculated by using the Boltzmann equations for a three-component quasiparticle gas. The roton–impuriton times that determine the various coefficients are obtained. The calculated values are compared with the experimental data.

INTRODUCTION

The main premises of the kinetic theory of superfluid ^{3}He–^{4}He solutions were developed by Khalatnikov and Zharkov.[1] Their results were used by Baym *et al.*[2–4] to develop a detailed theory of kinetic phenomena in a phonon–impuriton system for solutions at $T<0.6$ K, when the roton contribution can be neglected. For dilute solutions, however, the values calulated by Baym *et al.* differed from experiment by more than an order of magnitude. This, as well as the complexity of the computations, is why no theoretical study was made, in the twenty years following the papers of Baym *et al.*, of the general case of a three-component gas comprising phonons, rotons, and impuritons.

Recent experimental and theoretical investigations[5–8] have explained the physics of relaxation phenomena in a phonon–impuriton system at $T<0.6$ K. They serve as the basis for further development of a kinetic theory of solutions, valid for the general case of a three-component quasiparticle gas. This is the task of the present paper. Stimulating the development of such a theory, in particular, is the need to explain the many experimental data accumulated to date on the kinetic properties of superfluid solutions at $T>0.6$ K. The first after the long hiatus was a paper[9] in which the temperature range considered in Ref. 7 was expanded by including the roton contribution to the second-sound absorption coefficient.

The use of the initial equations of Ref. 1 and of the procedure developed in Refs. 6 and 7 has enabled us to calculate in the present paper all the kinetic coefficients and solve the problem of second-sound absorption in superfluid ^{3}He–^{4}He solutions in the entire temperature interval in which the quasiparticle description is valid.

1. SOLUTION OF BOLTZMANN EQUATIONS

We start with the Boltzmann equations for the quasiparticle distribution function f_i, which we express as a sum of a local equilibrium distribution function and a small increment:

$$\delta f_i = -f_{0i}' g_i,\quad i=1,\ 2,\ 3, \tag{1.1}$$

where the subscripts 1, 2, and 3 label respectively phonons, rotons, and impuritons, while f_{0i}' is the derivative of the equilibrium distribution function with respect to energy. The equations for g_i are calculated by a procedure described in Ref. 1. The method proposed in Refs. 6 and 7 reduces the Boltzmann equations to the form

$$I|g\rangle = \rho^{-1}|\varphi_\zeta^{(3)}\rangle \operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n) + |\varphi_\zeta^{(2)}\rangle \operatorname{div}\mathbf{v}_n + |\varphi_\eta\rangle_{\alpha\beta} V_{\alpha\beta} + (n_3T)^{1/2}|\varphi_D\rangle \frac{\nabla c}{c} + T^{-1/2}|\varphi_T\rangle \nabla T \equiv |h\rangle, \tag{1.2}$$

where

$$I = \begin{pmatrix} I_{11}+I_{12}+I_{13} & I_{12} & I_{13} \\ I_{21} & I_{22}+I_{23}+I_{21} & I_{23} \\ I_{31} & I_{32} & I_{33}+I_{31}+I_{32} \end{pmatrix} \tag{1.3}$$

is an operator matrix of linearized collision integrals,

$$|g\rangle = \begin{vmatrix} g_1 \\ g_2 \\ g_3 \end{vmatrix}\rangle, \tag{1.4}$$

$\mathbf{j}$ is the momentum per unit volume of the solution, $\rho = n_3m_3 + n_4m_4$ is the solution density, $\mathbf{v}g_n$ is the velocity of the normal component, $c = n_3m_3/\rho$ is the concentration,

$$V_{\alpha\beta} = \frac{\partial v_{n\alpha}}{\partial x_\beta} + \frac{\partial v_{n\beta}}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\operatorname{div}\mathbf{v}_n,$$

$|\varphi_\eta\rangle_{\alpha\beta}$ is a vector defining the first viscosity, $|\varphi_\zeta^{(2)}\rangle$ and $|\varphi_\zeta^{(3)}\rangle$ are vectors defining the second viscosity coefficients, while $|\varphi_D\rangle$ and $|\varphi_T\rangle$ are vectors defining the diffusion, thermal conductivity, and thermal diffusion vectors. The explicit forms of these vectors will be given below, together with the corresponding dissipation coefficients. The latter are expressed in terms of their scalar products defined as follows:

$$\langle\psi|\varphi\rangle = \sum_{i=1}^{3}\langle\psi_i|\varphi_i\rangle = -\sum_{i=1}^{3}\int \psi_i^*(\mathbf{p}_i)\varphi_i(\mathbf{p}_i) f_{0i}'\, d\Gamma_i. \tag{1.5}$$

To find $|g\rangle$ from Eq. (1.2) we must separate the subspace of the collision invariants of the Hermitian operator I, which corresponds to zero eigenvalues. We choose as the basis of this subspace the orthonormalized set of vectors

$$|J_N\rangle = \left(\frac{T}{n_3}\right)^{1/2} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}\rangle,$$

$$|J_\varepsilon\rangle = \frac{1}{(C_VT)^{1/2}} \begin{vmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \tilde{\varepsilon}_3 \end{vmatrix}\rangle,\quad |J_p\rangle = \rho_n^{-1/2} \begin{vmatrix} p_{z1} \\ p_{z2} \\ p_{z3} \end{vmatrix}\rangle, \tag{1.6}$$

where $C_V = C_1 + C_2 + C_3$ is the total specific heat per unit volume of the solution, $\rho_n = \rho_{1n} + \rho_{2n} + \rho_{3n}$ is the total density of the normal component, $\varepsilon_1 = u_1p_1$ is the photon energy, $\varepsilon_2 = \Delta_2 + (p_2 - p_0)^2/2\mu$ is the roton energy $E_3 = \Delta_3 + \varepsilon_3$ is the impuriton energy, $\varepsilon_3 = p_3^2/2m_3'$, and

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$\tilde{\varepsilon}_3 = \varepsilon_3 - 3T/2$. The vectors containing the x and y components of the momenta drop out of the calculations, since we assume henceforth that all the gradients are directed along the z axis. Here and elsewhere we consider the case of low concentrations. At the temperatures $T > 0.6$ K of interest to us, where the roton contribution is significant, low-concentration solutions are not degenerate. It may be necessary to allow for their degeneracy only at $T < 0.6$ K, when the roton contribution can be neglected, as has indeed been done in Refs. 6 and 7.

We define the projector-operators on the subspace of collision invariants and on the subspace orthogonal to it:

$$\mathscr{P}_c = |J_N\rangle\langle J_N| + |J_\varepsilon\rangle\langle J_\varepsilon| + |J_p\rangle\langle J_p|, \quad \mathscr{P}_n = 1 - \mathscr{P}_c. \qquad (1.7)$$

To obtain the vector $|g\rangle$ we introduce an operator inverse to I and defined only in the subspace of nonconserved quantities. Taking the foregoing into account, we can rewrite the Boltzmann equations (1.2) in the form

$$|g\rangle = (\mathscr{P}_n I \mathscr{P}_n)^{-1} |h\rangle. \qquad (1.8)$$

In the calculations that follow we assume the collision integrals in (3) to be given in explicit form.

2. COLLISION INTEGRALS

We write the collision integrals I_{ii} of like particles by using the corrent τ approximation[6,7]

$$I_{ii} = -(1 - \mathscr{P}_{ci})\tau_{ii}^{-1}, \quad i = 2, 3, \qquad (2.1)$$

where $\mathscr{P}_{ci}$ is the projection operator on the subspace of the collision invariants of the Hermitian operator I_{ii}. The times τ_{22} and τ_{33} are given in Ref. 10 and in Refs. 1 and 11, respectively. We write the phonon–phonon collision integral in the form

$$I_{11} = -(1 - \mathscr{P}_{\parallel})\tau_{\parallel}^{-1} - (1 - \mathscr{P}_{c1})\tau_{11}^{-1}. \qquad (2.2)$$

This relation takes into account the presence of a three-phonon fast longitudinal and relatively slow transverse relaxation in the phonon system, with corresponding times $\tau_{\parallel}$ and τ_{11}. Below we assume $\tau_{\parallel} \ll \tau_{11}$. Explicit expressions are given for τ_{11} and $_{\parallel\parallel}$ in Refs. 10 and 12, and for the operator $\mathscr{P}_{\parallel}$ in Ref. 6.

The τ-approximation is not valid for the collision integrals I_{ij} $(i \neq j)$ because of the substantial dependence of the transition probabilities on the momenta. In accord with Refs. 6, 7, and 10 we have

$$I_{1j} = -\sum_{l=0}^{\infty} t_{(l)j}^{-1} \mathscr{P}_l, \qquad (2.3)$$

where $\mathscr{P}_l$ is the projection operator on a Legendre-polynomial basis, and the coefficient $t_{l(j)}^{-1}$ determines the corresponding collision frequency. The frequencies $t_{l(3)}^{-1}$ for $l > 0$ were obtained in Ref. 3, and the operator $t_{(0)3}^{-1}$ that describes inelastic phonon–impuriton scattering is given in Ref. 7. In accord with Refs. 13 and 14 and the approximation (2.3) for the phonon–roton collisions, all the times $t_{(l)2}$ can be regarded as equal.

Roton scattering by an impuriton was described in Ref. 1 with the aid of a δ-function potential adjusted by an interaction constant. A subsequently proposed[15] impuriton–impuriton interaction model makes it possible to take into account here, in similar fashion, the long-range part of the roton–impuriton interaction, shown below to be the principal one. According to the conservation laws, the momentum transfer in a roton–impuriton collision is $q \sim m_3^* v_3 \sim (m_3^* T)^{1/2}$ where $v_i = \partial\varepsilon_i/\partial p_i$ is the quasiparticle velocity. Such momentum transfers correspond to an interaction at distances $r \propto q^{-1}$ that are shown by numerical estimates to exceed the interatomic distances. We can therefore expect roton–impuriton collision to be described by a hydrodynamic Hamiltonian containing phonon–roton[10] and phonon–impuriton[2] interactions. Such a Hamiltonian leads to interaction via exchange of virtual phonons between the quasiparticles.

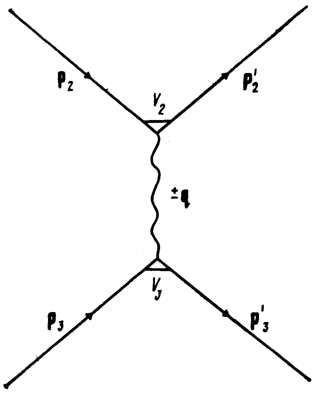

FIG. 1. Diagram describing roton–impuriton interaction via exchange of virtual phonons.

A diagram corresponding to second-order perturbation theory and describing the interaction through virtual-phonon exchange is shown in Fig. 1. The explicit expressions for the vertices V_3 and V_2 can, according to Refs. 2 and 10, be written in the form

$$V_2 = \left(\frac{u_1 q}{2\rho_4 V}\right)^{1/2} \left[\frac{\rho_4}{u_1}\frac{\partial \Delta_2}{\partial \rho_4} + \frac{\mathbf{q}(\mathbf{p}_2 + \mathbf{p}_2')}{2q}\right] \delta_{\mathbf{p}_2, \mathbf{p}_2' \pm \mathbf{q}}, \qquad (2.4)$$

$$V_3 = \left(\frac{u_1 q}{2\rho_4 V}\right)^{1/2} \left\{\frac{\rho_4}{u_1}\frac{\partial \Delta_3}{\partial \rho_4} + \frac{\delta m}{m_3^*}\frac{\mathbf{q}(\mathbf{p}_3 + \mathbf{p}_3')}{2q}\right\} \delta_{\mathbf{p}_3, \mathbf{p}_3' \pm \mathbf{q}}, \qquad (2.5)$$

where V is the volume of the solution and $\delta m = m_3^* - m_3$. This diagram corresponds to a transition probability

$$\tilde{W}(\mathbf{p}_2, \mathbf{p}_3 | \mathbf{p}_2', \mathbf{p}_3')$$

$$= (2\pi)^4 \delta(\varepsilon_2 + \varepsilon_3 - \varepsilon_2' - \varepsilon_3') \delta(\mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}_2' - \mathbf{p}_3') A_{23}^2, \qquad (2.6)$$

where

$$A_{23} = -n_4^{-1} \Delta_2 a_2 a_3, \qquad a_2 = \frac{\rho_4}{\Delta_2}\frac{\partial \Delta_2}{\partial \rho_4}, \qquad a_3 = \frac{n_4}{u_1^2}\frac{\partial \Delta_3}{\partial \rho_4},$$

A_{23} is the transition amplitude, and ρ_4 is the ^{4}He density in the solution. Terms containing the small parameter v_3/u_1 were disregarded in the derivation of (2.6).

The I_{23} matrix elements for the calculations that follow can be written in the form

$$\langle \varphi_i | I_{23} | \psi_j \rangle = -\frac{1}{2} \int \{\varphi^*(\mathbf{p}_i') - \varphi^*(\mathbf{p}_i)\} \{\psi(\mathbf{p}_j') - \psi(\mathbf{p}_j)\}$$

$$\times W(\mathbf{p}_2, \mathbf{p}_3 | \mathbf{p}_2', \mathbf{p}_3') T f_{02}' f_{03}' \, d\Gamma_2 \, d\Gamma_3 \, d\Gamma_2' \, d\Gamma_3', \quad i, j = 2.3. \qquad (2.7)$$

They determine the various kinetic coefficients.

3. FIRST-VISCOSITY COEFFICIENT

In accord with the definition of the momentum-flux tensor, the first-viscosity coefficient in our notation takes the form

$$\eta=-\langle\varphi_\eta|(\mathscr{P}_n I\mathscr{P}_n)^{-1}|\varphi_\eta\rangle \tag{3.1}$$

where the ith component of the ket-vector $|\varphi_\eta\rangle$ is equal to

$$\varphi_{\eta i}=(3p_{zi}v_{zi}-p_iv_i)/2\sqrt{3}. \tag{3.2}$$

We rewrite (3.1) in the form

$$\eta=-\langle\varphi_\eta|\chi_\eta\rangle, \tag{3.3}$$

where the ket-vector $|\chi_\eta\rangle$ is obtained from the equation

$$\mathscr{P}_n I\mathscr{P}_n|\chi_\eta\rangle=|\varphi_\eta\rangle. \tag{3.4}$$

We seek the components $|\chi_{\eta i}\rangle$ in the form

$$|\chi_{\eta i}\rangle=|\chi_{\eta i}{}^\perp\rangle-\tau_{\eta i}|\varphi_{\eta i}\rangle \tag{3.5}$$

subject to the orthogonality condition

$$\langle\varphi_{\eta i}|\chi_{\eta i}{}^\perp\rangle=0. \tag{3.6}$$

From (3.3)–(3.6) we have

$$\eta=\sum_{i=1}^{3}\langle\varphi_{\eta i}|\varphi_{\eta i}\rangle\tau_{\eta i}. \tag{3.7}$$

The problem has been reduced to finding the characteristic times that determine the partial contribution of the particles of species i to the first viscosity coefficient. Substituting (3.5) in (3.4) we obtain a set of three equations for $\tau_{\eta i}$:

$$I_{ii}|\chi_{\eta i}{}^\perp\rangle+\sum_{j\neq i}I_{ij}|\chi_{\eta i}{}^\perp+\chi_{\eta j}{}^\perp\rangle=|\varphi_{\eta i}\rangle+\sum_{j\neq i}I_{ij}|\tau_{\eta i}\varphi_{\eta i}+\tau_{\eta j}\varphi_{\eta j}\rangle,$$
$$i=1,2,3. \tag{3.8}$$

Account must be taken next of the distinctive properties of the phonon collision operator that rapidly establishes an equilibrium only for phonons that move in a specified direction. We therefore obtain $|\chi_{\eta 1}^\perp\rangle$ from the first of the three equations (3.8) and substitute it in the other two. This yields a system of three equations, the first of which defines the vector $|\chi_{\eta 1}^\perp\rangle$ which, however, is not contained in the other two. We take the scalar product of each of the three equations with the corresponding bra-vector $\langle\varphi_{\eta i}|$ and take the orthogonality of (3.6) into account. The solution of the resultant three equations yields the desired times $\tau_{\eta i}$, which can be written in the form

$$\tau_{\eta i}^{-1}=\sum_{j=1}^{3}\tau_{ij\eta}^{-1}, \tag{3.9}$$

$$\tau_{ii\eta}^{-1}=-\langle I_{ii}\rangle_{\varphi_{\eta i}}, \tag{3.10}$$

$$\tau_{ij\eta}^{-1}=-\langle I_{ij}\rangle_{\varphi_{\eta i}},\quad i\neq j,\quad i\neq 1, \tag{3.11}$$

$$\tau_{1j\eta}^{-1}=-\langle RI_{1j}\rangle_{\varphi_{\eta 1}}/\langle R\rangle_{\varphi_{\eta 1}},\quad j\neq 1. \tag{3.12}$$

Here $R=(-\tau_\parallel^{-1}+I_{12}+I_{13})^{-1}$, and angle brackets with subscripts denote here and below a normalized matrix element:

$$\langle I_{ii}\rangle_{\varphi_{\eta i}}=\langle\varphi_{\eta i}|I_{ii}|\varphi_{\eta i}\rangle/\langle\varphi_{\eta i}|\varphi_{\eta i}\rangle.$$

We have left out of (3.10) a term $\langle I_{i1}RI_{1i}\rangle_{\varphi\eta_i}$ $(i=2,3)$ describing the relaxation of the i quasiparticles on phonons. This relaxation can be disregarded here and below, for in our case the dominant interaction, described by (3.10), is between the i quasiparticles.

Equations (3.9)–(3.12) were derived without the use of matrix elements of the type

$$\langle\varphi_{\eta i}|I_{ij}|\chi_{\eta j}{}^\perp\rangle,\quad j\neq 1, \tag{3.13}$$

$$\langle\varphi_{\eta i}|I_{ij}|\varphi_{\eta j}\rangle,\quad i\neq j. \tag{3.14}$$

The matrix elements (3.13) with $i\neq j$, and (3.14), are smaller by the parameters $\mu T/p_0^2$ and $T/m_3^* u_1^2\ll 1$ than those retained. The matrix elements (3.13) with $i=j$ vanish identically in the τ-approximation and are small if the transition frequencies are smooth functions of the roton and impuriton momenta. The situation is similar also in classical kinetics (see, e.g., Ref. 16), where terms such as (3.13) and (3.14) are likewise ignored. In the case of specific collision integrals, however, such as the operator I_{11}, matrix elements of type (3.13), which contain the vector $|\chi_{\eta 1}^\perp\rangle$, must be taken into account. This was done in fact in the above calculations and resulted in the appearance in (3.12) of the operator R that substantially changes the kinetics of the phonon system.[6]

Substituting (3.1) in (3.7) and taking the definition (1.5) into account we have

$$\eta=\sum_{i=1}^{3}{}^1\!/_5\rho_{in}\langle v_i^2\rangle_{p_i}\tau_{\eta i}. \tag{3.15}$$

The frequencies (3.12) are calculated by the procedure of Ref. 6, which yields

$$\tau_{1j\eta}^{-1}=\langle R_2t_{(2)j}^{-1}\rangle_{p_1}/\langle R_2\rangle_{p_1},\qquad j=2,3, \tag{3.16}$$

where, in general

$$R_l=-(\tau_\parallel{}^{-1}+t_{(l)2}^{-1}+t_{(l)3}^{-1})^{-1}.$$

According to approximations (2.1) and (2.2) we have from (3.10)

$$\tau_{11\eta}=\tau_{11},\qquad\tau_{22\eta}=\tau_{22},\qquad\tau_{33\eta}=\tau_{33}. \tag{3.17}$$

The times $\tau_{31\eta}$ and $\tau_{21\eta}$ are calculated from the collision integrals of Refs. 2 and 13. Their contribution to (3.15) is negligible and they are therefore omitted here.

The frequencies $\tau_{23\eta}^{-1}$ and $\tau_{32\eta}^{-1}$ are obtained from (3.11) and (2.7). The result is

$$\tau_{23\eta}^{-1}={}^8\!/_3A^2T^{1/2}n_3,\qquad\tau_{32\eta}^{-1}={}^8\!/_5A^2T^{1/2}n_2, \tag{3.18}$$

where $A_2=A_{23}^2m_3^*(2\mu\pi^{-3})^{1/2}$, with n_i the number of quasiparticles per unit volume. In the derivation of (3.18) it is assumed, as in Ref. 10, that the roton momentum is close to p_0, and it was recognized that $\mu/m_3^*\ll 1$. In the appropriate limiting cases Eq. (3.15) agrees with the results of Refs. 6 and 10.

Comparison shows that the first-viscosity coefficients calculated from (3.15) have the same order of magnitude and the same temperature dependence as the experimental data.[17,18] The calculated values, however, are smaller than the measured ones. The maximum disparity (by a factor 2–3) occurs where the phonons make a substantial contribution. The disparity decreases at higher temperatures, and the agreement is good where the phonon contribution can be neglected. Possible causes of this disparity will be discussed at the end of the paper.

4. DIFFUSION AND THERMODIFFUSION

The impuriton diffusion-flux density **G** is determined by the impurity mass and is given by

$$\mathbf{G}=\int m_3\mathbf{v}_3\delta f_3\,d\Gamma_3. \tag{4.1}$$

Using (1.1), (1.2), (1.5), and (1.8) we rewrite (4.1) in the form

$$G_z=\frac{m_3}{m_3^*}\left\langle p_{z3}\right|(\mathscr{P}_nI\mathscr{P}_n)^{-1}\left|(n_3T)^{1/2}\varphi_D\frac{1}{c}\frac{\partial c}{\partial z}+T^{-1/2}\varphi_T\frac{\partial T}{\partial z}\right\rangle, \tag{4.2}$$

where

$$|\varphi_D\rangle\equiv\begin{vmatrix}\varphi_{D1}\\ \varphi_{D2}\\ \varphi_{D3}\end{vmatrix}=\frac{1}{m_3^*\rho_n}\left(\frac{T}{n_3}\right)^{1/2}\begin{vmatrix}-\rho_{3n}p_{z1}\\ -\rho_{3n}p_{z2}\\ \rho_{4n}p_{z3}\end{vmatrix} \tag{4.3}$$

is the vector that determines the diffusion,

$$|\varphi_T\rangle=T^{-1/2}\begin{vmatrix}\varepsilon_1v_{z1}-\bar{S}T\rho_n^{-1}p_{z1}\\ \varepsilon_2v_{z2}-\bar{S}T\rho_n^{-1}p_{z2}\\ \tilde{\varepsilon}_3v_{z3}-\bar{S}T\rho_n^{-1}p_{z3}\end{vmatrix} \tag{4.4}$$

is the vector that determines the thermodiffusion and thermal conduction $\rho_{4n}=\rho_{in}+\rho_{2n}$, $S=S_4+\bar{S}_3$, $\bar{S}_3=2/3C_3$, where $S_4=S_1+S_2$ is the entropy of pure ^{4}He. Relations (1.7) and (4.3) yield

$$\langle p_{z3}|\mathscr{P}_n=\langle p_{z3}|-\langle p_{z3}|J_p\rangle\langle J_p|=m_3^*(n_3/T)^{1/2}\langle\varphi_D|. \tag{4.5}$$

Substituting (4.5) in (4.2) we get

$$G_z=\rho\langle\varphi_D|(\mathscr{P}_nI\mathscr{P}_n)^{-1}|\varphi_D\rangle\frac{\partial c}{\partial z}+\frac{\rho c}{Tn_3^{1/2}}\langle\varphi_D|(\mathscr{P}_nI\mathscr{P}_n)^{-1}|\varphi_T\rangle\frac{\partial T}{\partial z}. \tag{4.6}$$

Comparing the definition

$$G_z=-\rho D\left(\frac{\partial c}{\partial z}+\frac{k_T}{T}\frac{\partial T}{\partial z}\right) \tag{4.7}$$

with (4.6), we have for the diffusion coefficient D and for the thermodiffusion coefficient Dk_T

$$D=-\langle\varphi_D|(\mathscr{P}_nI\mathscr{P}_n)^{-1}|\varphi_D\rangle, \tag{4.8}$$

$$Dk_T=-cn_3^{-1/2}\langle\varphi_D|(\mathscr{P}_nI\mathscr{P}_n)^{-1}|\varphi_T\rangle. \tag{4.9}$$

The matrix elements in (4.8) and (4.9) are obtained by the same method as the first-viscosity coefficient. The difference is that since the conservation laws lead to the relation

$$I_{ii}|\varphi_{Di}\rangle=0, \tag{4.10}$$

$$I_{ij}|p_{zj}\rangle=-I_{ij}|p_{zi}\rangle, \tag{4.11}$$

it follows that the system of three equations for τ_{Di}, in contrast to the system for $\tau_{\eta i}$, is linearly dependent. D is then determined by two linear combinations containing τ_{Di} and obtained from two linearly independent equations of the system. In addition, (4.11) makes it possible to reduce off-diagonal matrix elements of the form (3.14) to the diagonal elements taken into account here. As a result we obtain for the diffusion coefficient

$$D=u_{2N}^2\tau_D, \tag{4.12}$$

where

$$\begin{gathered}\tau_D=\tau_\varkappa^{-1}d_D^{-1},\ u_{2N}^2=n_3T/\rho_n,\\ \tau_\varkappa^{-1}=\frac{\rho_{1n}}{\rho_n}(\tau_{23D}^{-1}+\tau_{21D}^{-1}+\tau_{12D}^{-1})+\frac{\rho_{2n}}{\rho_n}(\tau_{13D}^{-1}+\tau_{12D}^{-1}+\tau_{21D}^{-1}),\\ d_D=\tau_{13D}^{-1}\tau_{23D}^{-1}+\tau_{12D}^{-1}\tau_{23D}^{-1}+\tau_{21D}^{-1}\tau_{13D}^{-1},\\ \tau_{ijD}=\tau_{jiD}\rho_{in}/\rho_{jn},\\ \tau_{1jD}^{-1}=\langle R_1t_{(1)j}^{-1}\rangle_{p_1}/\langle R_1\rangle_{p_1},\quad j\neq1,\quad \tau_{23D}^{-1}=-\langle I_{23}\rangle_{p_{z2}}.\end{gathered} \tag{4.13}$$

From the last expression of (4.13) and from (2.7) we get

$$\tau_{23D}^{-1}=(4m_3^*T/p_0^2)A^2T^{1/2}n_3. \tag{4.14}$$

Note that according to (3.18) and (4.14) the roton–impurity times contained respectively in the first-viscosity and diffusion coefficients are not similar and differ numerically by an order of magnitude, owing to the specific features of the roton-dispersion law. In fact, the frequency τ_{23D}^{-1} (4.13) is proportional to the ratio of the squared change of the roton momentum due to collision with the impuriton to the squared momentum of the roton. At the same time, according to (3.11) the frequency $\tau_{23\eta}^{-1}$ is proportional to the ratio of the squares of the mechanical and thermal velocities of the roton. By virtue of the specific nature of the roton dispersion law, the first of these ratios is of order

$$m_3^*T/p_0^2\ll1, \tag{4.15}$$

and the second of order unity. As a result, (4.14) has the additional factor (4.15) which is absent from (3.18). The ratio (4.15) can be treated as the ratio of the impuriton mass to the roton "thermal mass" $\rho_{2n}/n_2=p_0^2/3T$. In this sense, collision of a roton with an impuriton is similar to the collision of a light particle with a heavy one.

For a phonon–impuriton system ($T<0.6$ K) relation (4.12) agrees with the result of Ref. 7. For a roton–impuriton system expression (4.12) is the same, apart from the definition of the time, as the result of Ref. 1 in which m_3 is replaced by m_3^*. (The mass flow in Ref. 1 differs from (4.1) in that m_3 is replaced by m_3^*.)

In general, the diffusion coefficient depends not only on the frequencies of the collisions between the thermal excitations and the impuritons, but also on the phonon–roton relaxation. If the relaxation is rapid (τ_{12D} and $\tau_{21D}\to0$) we get from (4.12)

$$D=\left(\frac{\rho_{4n}}{\rho_n}\right)^2u_{2N}^2\left(\frac{\rho_{1n}}{\rho_n}\tau_{13D}^{-1}+\frac{\rho_{2n}}{\rho_n}\tau_{23D}^{-1}\right)^{-1}. \tag{4.16}$$

In the opposite limiting case of a slow onset of equilibrium between the phonons and rotons, Eq. (4.12) yields

$$D=u_{2N}^2\left(\frac{\rho_{1n}}{\rho_n}\tau_{13D}+\frac{\rho_{2n}}{\rho_n}\tau_{23D}\right), \tag{4.17}$$

which corresponds, in contrast to (4.16), to an independent activation of the relaxation mechanism.

Figure 2 shows a comparison of the calculated and mea-

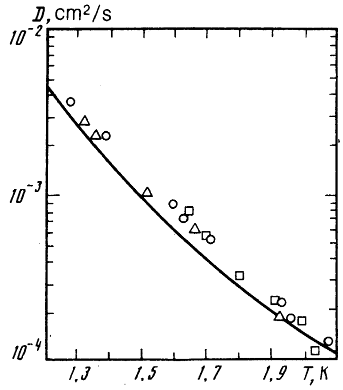

FIG. 2. Temperature dependence of the diffusion coefficient: ○—data of Ref. 1; △—data of Ref. 20 for a molar density; $x = 1.39\cdot 10^{-4}$; □—data on Ref. 20 for $x = 1.23\cdot 10^{-3}$. Solid line—calculated from (4.18).

sured values of the diffusion coefficient for $T > 1.2$ K. The main contribution in this temperature region is made by rotons, and we have here from (4.12)

$$D = (\rho_{4n}/\rho_n)^2 (T/m_3^*) \tau_{32D}, \qquad (4.18)$$

where $\tau_{32D} = \tau_{23D}\rho_{3n}/\rho_{2n}$. The agreement of the experiment with the theory favors the assumption that the roton–impuriton interaction considered in Sec. 2, via exchange of virtual phonons, makes the main contribution. Unfortunately, there are at present no experimental data for D at $T < 1.2$ K, so that the calculation using the general equation (4.12) cannot be compared with the measured data.

The thermodiffusion coefficient is calculated by starting with Eq. (4.9). It is convenient here to represent the vector (4.4) in the form

$$|\varphi_T\rangle = |\varphi_\varkappa\rangle + |\psi_\varkappa\rangle + \frac{m_3^*}{T} n_3^{1/2} \left(\frac{\bar{S}_3 T}{\rho_{3n}} - \frac{S_4 T}{\rho_{4n}} \right) |\varphi_D\rangle, \qquad (4.19)$$

where the vector

$$|\varphi_\varkappa\rangle = T^{-1/2} \left| \begin{matrix} \varepsilon_1 v_{z1} - S_1 T \rho_{1n}^{-1} p_{z1} \\ \varepsilon_2 v_{z2} - S_2 T \rho_{2n}^{-1} p_{z2} \\ \tilde{\varepsilon}_3 v_{z3} - S_3 T \rho_{3n}^{-1} p_{z3} \end{matrix} \right\rangle \qquad (4.20)$$

defines the partial thermal conductivity, and

$$|\psi_\varkappa\rangle = \frac{1}{\rho_{4n} T^{1/2}} \left(\frac{S_1 T}{\rho_{1n}} - \frac{S_2 T}{\rho_{2n}} \right) \left| \begin{matrix} \rho_{2n} p_{z1} \\ -\rho_{1n} p_{z2} \\ 0 \end{matrix} \right\rangle. \qquad (4.21)$$

Calclations similar to those for the first-viscosity and diffusion coefficients yield

$$k_T = \frac{c m_3^*}{T} \left[\frac{\bar{S}_3 T}{\rho_{3n}} - \frac{S_4 T}{\rho_{4n}} + \frac{\rho_{1n}\rho_{2n}}{\rho_{3n}\rho_{4n}} \left(\frac{S_1 T}{\rho_{1n}} - \frac{S_2 T}{\rho_{2n}} \right) \frac{\tau_{13D}^{-1} - \tau_{23D}^{-1}}{\tau_\varkappa^{-1}} \right]. \qquad (4.22)$$

In the limiting cases of a phonon–impurity or roton–impurity systems relation (4.22) goes over into the results of Ref. 1. In the general case k_T contains a term that depends on the relaxation processes in a quasiparticle gas.

5. THERMAL CONDUCTIVITY

In accordance with the definition of the heat-flux density

$$\mathbf{Q} = \sum_{i=1}^{3} \int \varepsilon_i \mathbf{v}_i \delta f_i \, d\Gamma_i, \qquad (5.1)$$

we obtain from (1.1) and (1.2)

$$Q_z = \langle \varphi_T | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_T \rangle \frac{\partial T}{\partial z} + \frac{T n_3^{1/2}}{c} \langle \varphi_T | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_D \rangle \frac{\partial c}{\partial z}. \qquad (5.2)$$

The thermal conductivity, as usual, is so defined that at $\mathbf{G} = 0$ the heat flux $\mathbf{Q} = -\varkappa \nabla T$. Equations (4.6) and (5.2) yield

$$\varkappa = -\langle \varphi_T | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_T \rangle - D^{-1} \langle \varphi_T | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_D \rangle^2. \qquad (5.3)$$

Substituting (4.19) in (5.3) we get

$$\varkappa = \sum_{i=1}^{3} \varkappa_i - [\langle \psi_\varkappa | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \psi_\varkappa \rangle + D^{-1} \langle \psi_\varkappa | (\mathscr{P}_n I \mathscr{P}_n)^{-1} | \varphi_D \rangle^2], \qquad (5.4)$$

where

$$\sum_{i=1}^{3} \varkappa_i = \sum_{i=1}^{3} \langle \varphi_{\varkappa i} | \varphi_{\varkappa i} \rangle \tau_{\varkappa i} \qquad (5.5)$$

is the partial thermal conductivity that determines the additive contributions of the phonons, rotons, and impuritons to the total thermal conductivity $\varkappa$, and is similar in this sense to the result for a mixture of classical gases (see, e.g., Ref. 16). As to the second term in (5.4), which we designate by $\varkappa_{123}$, it is due to the specific features of the gas of thermal excitations, the number of which is not conserved. The coefficient $\varkappa_{123}$ determines a nonadditive contribution to $\varkappa$ and has no classical counterpart.

According to (4.20), for a linear phonon dispersion law we have $\varphi_{\varkappa_1} = 0$ and, in accord with Ref. 10, $\varkappa_1 = 0$. The times $\tau_{\varkappa i}$ are obtained from (3.9)–(3.11), in which the subscript η should be replaced by $\varkappa$. Calculations similar to those used to obtain the first-viscosity and diffusion coefficients, yield

$$\varkappa = \varkappa_2 + \varkappa_3 + \varkappa_{123}, \qquad (5.6)$$

where

$$\varkappa_2 = \frac{\Delta_2^2 n_2}{3\mu T} \left[1 - \frac{3 S_2 T^2 \mu}{\Delta_2^2 n_2} \frac{S_2}{\rho_{2n}} \right] (\tau_{22}^{-1} + \tau_{23\varkappa}^{-1})^{-1}, \qquad (5.7)$$

$$\varkappa_3 = \frac{5}{2} n_3 \frac{T}{m_3^*} (\tau_{33}^{-1} + \tau_{32\varkappa}^{-1})^{-1}, \qquad (5.8)$$

$$\varkappa_{123} = \frac{1}{T} \left(\frac{S_1 T}{\rho_{1n}} - \frac{S_2 T}{\rho_{2n}} \right)^2 \frac{\rho_{1n}\rho_{2n}}{\rho_n} \tau_\varkappa, \qquad (5.9)$$

$$\tau_{23\varkappa}^{-1} = -\langle I_{23} \rangle_{\varphi_{\varkappa 2}} = \tau_{23\eta}^{-1}, \qquad \tau_{32\varkappa}^{-1} = {}^4/_3 A^2 T^{1/2} n_2. \qquad (5.10)$$

In the limiting case when c and $\tau_\parallel$ vanish, expression (5.7) becomes the result given in Ref. 1, provided S_2/ρ_{2n} in (5.7) is replaced by S_4/ρ_{4n}. This difference is due to the correct τ-approximation used here, according to which a term that is an eigenvector of the operator I_{22}, and has a zero eigenvalue, has been separated in $|\varphi_{T2}\rangle$. This case, however, does not lead to a significant numerical discrepancy, since the contribution of the second term in the square brackets of (5,7) is small here compared with unity.

The nonadditive part of the thermal conductivity $\varkappa_{123}$ depends according to (5.9) on the parameters of all three quasiparticle types. The time τ_{23D} contained in $\varkappa_{123}$ differs substantially from the roton-impuriton times contained in $\varkappa_i$. The latter, according to (5.10), turned out to be equal to the first-viscosity times, owing to the peculiarities of the roton spectrum, whereby $\varepsilon_2 \approx \Delta_2$ and $p_2 \approx p_0$. Note the symmetry of Eq. (5.9) with respect to permutation of the phonons and rotons $(1 \rightleftarrows 2)$. Therefore the conventional definition of (5.9) as the phonon part of the thermal conductivity is certainly arbitrary even in pure helium. The physical meaning of $\varkappa_{123}$ becomes clear if (5.9) is rewritten in the form

$$\varkappa_{123}^{-1}=\varkappa_{12}^{-1}+\varkappa_{13}^{-1}+\varkappa_{23}^{-1}, \tag{5.11}$$

where

$$\varkappa_{12}=\frac{1}{T}\left(\frac{S_1T}{\rho_{1n}}-\frac{S_2T}{\rho_{2n}}\right)^2\left[\left(\rho_{1n}^{-1}+\rho_{2n}^{-1}\right)\left(\tau_{12D}^{-1}+\tau_{21D}^{-1}\right)\right]^{-1}, \tag{5.12}$$

$$\varkappa_{13}=\frac{\rho_{1n}}{T}\left(\frac{S_1T}{\rho_{1n}}-\frac{S_2T}{\rho_{2n}}\right)^2\tau_{13D},$$

$$\varkappa_{23}=\frac{\rho_{2n}}{T}\left(\frac{S_2T}{\rho_{2n}}-\frac{S_1T}{\rho_{1n}}\right)^2\tau_{23D}. \tag{5.13}$$

In pure ^{4}He the second and third terms in (5.11) are zero, and for $\tau_{\parallel}$ expression (5.11) is easily reduced to the asymmetric form

$$\varkappa_{123}\equiv\varkappa_{12}=\frac{1}{T}\rho_{1n}u_1^4\left(1-\frac{TS_4}{\rho_{4n}u_1^2}\right)^2\tau_{12}, \tag{5.14}$$

given in Ref. 10. It was this asymmetric formulation that caused this part to be attributed to phonon thermal conductivity. If, in accordance with (5.11), the third term is left out and it is assumed that $\tau_{\parallel}=0$, then $\varkappa_{123}$ again reduces to the asymmetric "phonon" form given in Ref. 1, which is obtained by replacing the time τ_{12} in (5.14) by $(\tau_{12}^{-1}+\tau_{13}^{-1})^{-1}$. Owing to the presence of $\varkappa_{23}$, the coefficient $\varkappa_{123}$ cannot be attributed even formally to the phonon thermal conductivity.

Unfortunately, $\varkappa$ has not yet been determined in experiment. The only measurements were made of the effective thermal conductivity $\varkappa_{\text{eff}}$, which includes, according to Ref. 1, $\varkappa$ and also the coefficients D and k_T. For $T > 1.2$ K, where the main contribution to $\varkappa$ is made by terms containing the time τ_{23D}, the results of numerical calculations using the derived equations agree with experimental data.[19,20] At lower temperatures, where the phonon contribution is significant, the situation is the same as with the first visocosity: the calculated values are lower than the experimental values.[19-21] At low temperatures ($T<0.6$ K), in accordance with Refs. 6 and 7, the theory again agrees with experiment, although here too everything is determined mainly by the phonon contribution.

6. SECOND-VISCOSITY COEFFICIENTS

According to Ref. 10, the superfluid-motion equation is

$$\frac{\partial\mathbf{v}_s}{\partial t}+\nabla\left(\mu_0+\sum_{i=1}^{3}\int\frac{\partial\varepsilon_i}{\partial\rho_4}\delta f_i\,d\Gamma_i+\frac{v_s^2}{2}\right)=0, \tag{6.1}$$

where μ_0 is the chemical potential of the solution at $T=0$. Using (1.1), (1.2), and the definition (1.5) we get

$$\sum_{i=1}^{3}\int\frac{\partial\varepsilon_i}{\partial\rho_4}\delta f_i\,d\Gamma_i=\rho^{-2}\langle\varphi_\zeta^{(3)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\varphi_\zeta^{(3)}\rangle\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)$$
$$+\rho^{-1}\langle\varphi_\zeta^{(3)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\varphi_\zeta^{(2)}\rangle\operatorname{div}\mathbf{v}_n, \tag{6.2}$$

where

$$\langle\varphi_\zeta^{(3)}|=\langle(b_1-b)\varepsilon_1,\ (b_2-b)\varepsilon_2,\ (b_3-b)\tilde\varepsilon_3|+\langle\varphi_{\zeta r}^{(3)}|, \tag{6.3}$$

$$\langle\varphi_\zeta^{(2)}|=\left\langle\left(\frac{S_1}{C_1}-\frac{\bar S}{C_V}\right)\varepsilon_1,\ \left(\frac{S_2}{C2}\frac{\bar S}{C_V}\right)\varepsilon_2,\ \left(\frac{\bar S_3}{C_2}-\frac{\bar S}{C_V}\right)\tilde\varepsilon_3\right|$$
$$+\langle\varphi_{\xi r}^{(2)}|+\langle\varphi_\zeta^{(3)}|, \tag{6.4}$$

and the coefficients b_i and b are given by

$$b_i=-\frac{\rho_4}{C_i}\left(\frac{\partial S_i}{\partial\rho_4}\right)_{T,n_3}=\frac{\rho_4}{C_iT}\left\langle\varepsilon_i\middle|\frac{\partial\varepsilon_i}{\partial\rho_4}\right\rangle,\quad b=\frac{1}{C_V}\sum_{i=1}^{3}C_ib_i.$$

The vectors determining the second viscosity of the roton gas are

$$\langle\varphi_{\xi r}^{(3)}|=\left\langle 0,\ \rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}-b_2\varepsilon_2,\ 0\right|,$$

$$\langle\varphi_{\zeta r}^{(2)}|=\left\langle 0,\ \frac{p_2v_2}{3}-\frac{S_2}{C_2}\varepsilon_2,0\right|.$$

It was recognized in the derivation of (6.2) that

$$\rho_4\langle\partial\varepsilon_1/\partial\rho_4;\ \partial\varepsilon_2/\partial\rho_4;\ \partial\varepsilon_3/\partial\rho_4|\mathcal{P}_n=\langle\varphi_\zeta^{(3)}|,$$
$$^1/_3\langle p_1v_1;\ p_2v_2;\ p_3v_3|\mathcal{P}_n=\langle\varphi_\zeta^{(2)}|-\langle\varphi_\zeta^{(3)}|.$$

Comparing (6.1) and (6.2) with the hydrodynamic equation of superfluid motion, we obtain for the second-viscosity coefficients

$$\zeta_3=-\rho^{-2}\langle\varphi_\zeta^{(3)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\varphi_\zeta^{(3)}\rangle, \tag{6.5}$$
$$\zeta_4=-\rho^{-1}\langle\varphi_\zeta^{(2)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\varphi_\zeta^{(3)}\rangle. \tag{6.6}$$

The second viscosity coefficients ζ_1 and ζ_2 are determined similarly by using the momentum-conservation equations. We have ultimately

$$\zeta_1=-\rho^{-1}\langle\varphi_\zeta^{(3)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\varphi_\zeta^{(2)}\rangle. \tag{6.7}$$
$$\zeta_2=-\langle\varphi_\zeta^{(2)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\varphi_\zeta^{(2)}\rangle. \tag{6.8}$$

According to (6.5)–(6.8),

$$\zeta_1=\zeta_4, \tag{6.9}$$
$$\zeta_1^2\leqslant\zeta_2\zeta_3. \tag{6.10}$$

Equation (6.9) follows from the fact that the operator I is Hermitian and the vector $|\varphi_\xi\rangle$ is real; it reflects the Onsager symmetry principle for the kinetic coefficients. Relation (6.10) ensures that the dissipative function is not negative. In two-component systems (phonon–roton[10] or phonon–impurity[7]) the inequality (6.10) turns into an equality only if a slow establishment of energy equilibrium between the subsystems is considered.

Calculations using (6.5)–(6.8) yield

$$\zeta_1=\frac{1}{\rho}\sum_{i=1}^{3}C_iT(b_i-b)\left(b_i-b+\frac{\bar S_i}{C_i}-\frac{\bar S}{C_V}\right)\tau_{\zeta i}+\zeta_{r1}, \tag{6.11}$$

$$\zeta_2=\sum_{i=1}^{3}C_iT\left(b_i-b+\frac{\bar S_i}{C_i}-\frac{\bar S}{C_V}\right)^2\tau_{\zeta i}+\zeta_{r2}, \tag{6.12}$$

$$\zeta_3=\frac{1}{\rho^2}\sum_{i=1}^{3}C_iT(b_i-b)^2\tau_{\zeta i}+\zeta_{r3},\qquad \bar{S}_i\equiv S_i,\quad i=1,2. \tag{6.13}$$

The terms that determine the second viscosity of the roton gas are of the form

$$\zeta_{r1}=\frac{1}{\rho}\left\langle \rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}+\frac{1}{3}p_2v_2-\left(b_2+\frac{S_2}{C_2}\right)\varepsilon_2\middle|\rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}\right\rangle\tau_{\zeta r},$$

$$\zeta_{r2}=\left\langle \rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}+\frac{1}{3}p_2v_2-\left(b_2+\frac{S_2}{C_2}\right)\varepsilon_2\middle|\rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}+\frac{1}{3}p_2v_2\right\rangle\tau_{\zeta r},$$

$$\zeta_{r3}=\frac{1}{\rho^2}\left\langle \rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}-b_2\varepsilon_2\middle|\rho_4\frac{\partial\varepsilon_2}{\partial\rho_4}\right\rangle\tau_{\zeta r}.$$

The times in the second-viscosity coefficients (6.11)–(6.13) are equal to

$$\begin{gathered}\tau_{\zeta1}^{-1}=\tau_{23\zeta}d_\zeta,\quad \tau_{\zeta2}^{-1}=\tau_{13\zeta}d_\zeta,\quad \tau_{\zeta3}^{-1}=C_3^{-1}C_2\tau_{12\zeta}d_\zeta,\\ \tau_{\zeta r}^{-1}=\tau_{22}^{-1}+\tau_{23\eta}^{-1},\quad d_\zeta=\tau_{13\zeta}^{-1}\tau_{23\zeta}^{-1}+\tau_{12\zeta}^{-1}\tau_{23\zeta}^{-1}+\tau_{21\zeta}^{-1}\tau_{13\zeta}^{-1},\\ \tau_{ij\zeta}=\tau_{ji\zeta}C_i/C_j,\\ \tau_{1j\zeta}^{-1}=-\langle RI_{1j}\rangle_{p1}/\langle R\rangle_{p1},\quad \tau_{23\zeta}^{-1}=-\langle I_{23}\rangle_{\varepsilon_2}.\end{gathered} \tag{6.14}$$

From the last relation in (6.14) and from the results of Refs. 13 and 14 we have

$$\begin{gathered}\tau_{12\zeta}=\tau_{12\eta},\\ \tau_{13\eta}=\langle(\tau_{\parallel}^{-1}+t_{(0)2}^{-1})^{-1}{}_{p_1}m_3{}^{*}u_1{}^2/\langle(\tau_{\parallel}^{-1}+t_{(0)2}^{-1})^{-1}t_{(1)3}^{-1}\rangle_{p_1}T.\end{gathered} \tag{6.15}$$

From the last equation of (6.14) we calculate

$$\tau_{23\zeta}^{-1}=(16T^2/15\Delta_2{}^2)A^2T^{1/2}n_3. \tag{6.16}$$

Note that the time $\tau_{23\zeta}$ (6.16) differs from the times $\tau_{23\eta}$ (3.18) and τ_{23D} (4.14) and that $\tau_{23\zeta}\gg_{23D}\gg\tau_{23\eta}$. The frequency $\tau_{23\zeta}^{-1}$, according to the last expression of (6.1), is proportional to the ratio of the change of the roton energy by collision with an impuriton to the square of the roton energy. In this sense, the second viscosity is due to the slow establishment of energy equilibrium between the subsystems. Note that the question of the second-viscosity coefficients of solutions was first considered in Refs. 22 and 23, before the phonon-spectrum decay properties were known. At low temperatures, where the roton contribution can be neglected, relations (6.11)–(6.18) agree with the results of Ref. 7. Information on the second-viscosity coefficients can be obtained from sound-absorption experiments.

7. SECOND SOUND IN THE HYDRODYNAMIC REGIME

Second sound is known to be a collective mode in a quasiparticle gas. In the zeroth approximation in the small parameter ρ_n/ρ the problem of second-sound propagation in a solution reduces to solving three linearized kinetic equations

$$(\omega-\mathbf{kv}-iI)|\tilde{g}\rangle=0, \tag{7.1}$$

where ω and $\mathbf{k}$ are respectively the frequency and wave vector of the second sound, and

$$\omega-\mathbf{kv}=\begin{pmatrix}\omega-\mathbf{kv}_1 & 0 & 0\\ 0 & \omega-\mathbf{kv}_2 & 0\\ 0 & 0 & \omega-\mathbf{kv}_3\end{pmatrix}. \tag{7.2}$$

To obtain the dispersion law $k=k(\omega)$ we must find the poles of the resolvent

$$R_\omega=(\omega-\mathbf{kv}-iI)^{-1} \tag{7.3}$$

of Eq. (7.1). In the hydrodynamic limit it is necessary to project (7.3) on the basis $|J_N,\varepsilon,p\rangle$ and take into account that the inequality $\omega\tau\ll1$ holds for all relaxation times τ. The result is

$$\mathcal{P}_cR_\omega\mathcal{P}_c=\mathcal{P}_c\{\omega-\mathcal{P}_c\mathbf{kv}\mathcal{P}_c-i\mathcal{P}_c\mathbf{kv}(\tilde{\mathcal{P}}_nI\mathcal{P}_n)^{-1}\mathbf{kv}\mathcal{P}_c\}^{-1}\mathcal{P}_c. \tag{7.4}$$

The matrix elements of the operator $\mathcal{P}_c\mathbf{k}\cdot\mathbf{v}\mathcal{P}_c$ are calculated from (1.5)–(1.7) and (7.2):

$$\langle J_N|\mathbf{kv}|J_p\rangle=ku_{2N},\quad \langle J_\varepsilon|\mathbf{kv}|J_p\rangle=ku_{2\varepsilon}, \tag{7.5}$$

where

$$u_{2\varepsilon}{}^2=T\bar{S}^2/\rho_nC_V.$$

The vectors in (7.4) are expressed in terms of the vectors that determine the dissipative coefficients:

$$\begin{gathered}\langle J_N|\mathbf{kv}\mathcal{P}_n=k\langle\varphi_D|,\quad \langle J_\varepsilon|\mathbf{kv}\mathcal{P}_n=kC_V{}^{-1/2}\langle\varphi_T|,\\ \langle J_p|\mathbf{kv}\mathcal{P}_n=2k(3\rho_n)^{-1/2}\langle\varphi_\eta|+k\rho_n{}^{-1/2}\langle\varphi_\zeta{}^{(2)}-\varphi_\zeta{}^{(3)}|.\end{gathered} \tag{7.6}$$

Using (7.5) and (7.6) we obtain the matrix of the operator in the curly brackets of (7.4), in the basis $|J_{N,\varepsilon,p}\rangle$. Equating the determinant of this matrix to zero we obtain the second-sound dispersion law

$$k=\omega/u_2+i\alpha_2, \tag{7.7}$$

where

$$u_2=(u_{2\varepsilon}{}^2+u_{2N}{}^2)^{1/2} \tag{7.8}$$

is the second-sound velocity and α_2 is the damping coefficient. Using the definitions (3.1) (4.8), (4.9), (5.3), (6.5)–(6.8). we can express the latter in the form

$$\alpha_2=\frac{\omega^2}{2\rho_nu_2{}^3}\left\{\frac{4}{3}\eta+\zeta_{II}+\frac{\rho_nu_{2\varepsilon}{}^2}{C_Vu_2{}^2}\varkappa+\rho_n\frac{u_{2N}{}^2}{u_2{}^2}D\left(1+k_T\frac{\rho_4\bar{S}}{\rho_3C_V}\right)^2\right\}, \tag{7.9}$$

where

$$\begin{gathered}\zeta_{II}=-\langle\varphi_\zeta{}^{(2)}-\varphi_\zeta{}^{(3)}|(\mathcal{P}_nI\mathcal{P}_n)^{-1}|\\ |\varphi_\zeta{}^{(2)}-\varphi_\zeta{}^{(3)}\rangle=\zeta_2-2\rho\zeta_1+\rho^2\zeta_3.\end{gathered} \tag{7.10}$$

Relations (7.8) and (7.9) are of the same form as the result obtained in Ref. 24 from the complete pheonomenological system of hydrodynamic equations.

Figure 3 shows the experimental[25] values of the second-sound absorption coefficient as well as those calculated from Eq. (7.9) using (3.15), (4.12), (4.22), (5.6), and (6.11)–(6.13). In this temperature and concentration region, the contribution of the diffusion is small. The contributions of the first three terms in (7.9) are approximately equal and are made by all quasiparticles of the solution. The term ζ_{II} is practically equal here to the sum of the second viscosities of the roton gas. In the region $0.6<T<1$ K the second-sound absorption is determined mainly by the thermal conductivity, which depends here on the phonon times. Just as in the cases of first viscosity and thermal conductivity, the calculated values are smaller here than the observed.

Analysis of the equations derived above and of all the experimental data shows that agreement can be achieved by assuming that the phonon-impuriton frequency τ_{13}^{-1} increases with temperature more slowly than T^4. This situa-

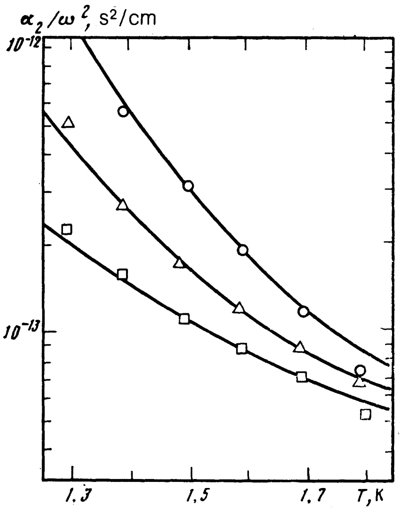

FIG. 3. Temperature dependence of the second-sound absorption coefficient in solutions with various molar concentrations. Points—data of Ref. 25: ○ − $x = 0$; △ − $x = 5\cdot10^{-3}$; □ − $x = 1\cdot10^{-2}$. Solid curves—calculated with Eq. (7.9).

tion is possible if the contribution of the three-phonon processes is assumed to decrease when the temperature is raised. The phonon–impuriton interaction constant can also depend on temperature. In addition it must be borne in mind that at higher temperatures the wavelength of the high-energy ($\varepsilon_1 \approx 7T$) phonons that determine the relaxation in the phonon gas[6] may not be large enough. This affects adversely the applicability of the hydrodynamic Hamiltonian used in Ref. 2 to obtain the times $t_{(l)3}$ that determine the phonon-impuriton collision frequency τ_{13}^{-1}.

CONCLUSION

We have calculated all the kinetic coefficients of low-concentration ^{3}He–^{4}He solutions in the entire interval range where the quasiparticle description is valid. Starting from the hydrodynamic Hamiltonian that contains the phonon-roton and phonon-impuriton interactions, we obtained the roton-impuriton times $\tau_{23\eta}$ (3.18), τ_{23D} (4.13), and $\tau_{23\zeta}$ (6.16) which were found to be different for different dissipation coefficients: $\tau_{23\zeta} \gg \tau_{23D} \gg \tau_{23\eta}$. The calculated values agree with the experimental data in the temperature and concentration region where the roton–impuriton interaction is strongest. It follows that this long-range part of the roton-impuriton interaction makes the main contribution.

According to (4.12), the diffusion coefficient depends on the time τ_{12D} needed to establish equilibrium between the phonons and the rotons. In the limiting cases $\tau_{12D} \to 0$ or $\tau_{12D} \to \infty$ we get the basically different results (4.16) and (4.17), which do not ultimately contain this time.

The thermodiffusion relation (4.22) depends not only on the thermodynamic properties of the solutions, but also on the relaxation processes in the quasiparticle gas. The total thermal conductivity $\varkappa$ is governed both by parallel (5.6) and sequential (5.11) applications of various thermal conductivity mechanism in the three-component quasiparticle gas.

The second-viscosity coefficients (6.11)–(6.13) are the result of energy equilibrium between quasiparticles of different species, and also of relaxation in the roton gas. The second-sound absorption coefficient (7.9) calculated in the hydrodynamic limit agrees in form with the result that follow from the complete phenomonological system of hydrodynamic equations. All the results were compared with all the available experimental data in the considered temperature and concentration regions.

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Translated by J. G. Adashko