Diamagnetism of a turbulent plasma

E. I. Avdeev, V. A. Dogel', and O. V. Dolgov

P. N. Lebedev Physical Institute, Academy of Sciences of the USSR

(Submitted 28 February 1989)
Zh. Eksp. Teor. Fiz. **96**, 885–891 (September 1989)

We consider the problem of the behavior of a large-scale magnetic field in a turbulent conducting medium. To describe the coupling between the magnetic field in the medium and an external random velocity field we introduce an effective magnetic susceptibility μ. In contrast to earlier studies we show that this quantity is independent of whether the turbulence is two- or three-dimensional. We are able to express μ as a number in the case of uniform turbulence: $\mu = 1/R_m$, where R_m is the magnetic Reynolds number, $R_m = u_0\lambda_c/\eta$. In the case of spatially inhomogeneous turbulence μ is a tensor and its form depends on the geometry of the problem.

1. INTRODUCTION

The behavior of magnetic fields in a conducting turbulent medium is one of the most interesting and up to now unsolved problems of magnetohydrodynamics.[1] This problem is most acute in astrophysics, for as in a cosmic plasma both these factors (turbulence and magnetic fields) are decisive when one considers a number of problems (cosmic-ray propagation, magnetic-field generation, molecular-cloud collapse, and so on; see in this connection Refs. 2–5). In particular, the behavior of large-scale fields in a turbulent plasma is of interest for a number of astrophysical problems. One of the important consequences of the study of this problem is the conclusion that a turbulent plasma has diamagnetic properties (in this case the turbulence is assumed to be mirror symmetric and the mean velocity of the fluid to be zero).[6] The diamagnetism in that case is different from the usual one used in macroscopic electrodynamics, since the diamagnetic properties are manifested only in the mean field obtained by averaging the actual field over a statistical ensemble of realizations of the velocity field.

We shall describe the effect of turbulence on a large-scale magnetic field by means of an effective magnetic permeability μ of the turbulent medium. The magnetic permeability is in the general case defined as an integral operator which connects the total field in the medium $\mathbf{B}(t,\mathbf{x})$ (i.e., the magnetic induction) with the field $\mathbf{H}(t,\mathbf{x})$—the external magnetic field which is maintained solely by external sources and exists when there is no medium present. When we want to describe the behavior of the average magnetic field in a turbulent medium, it makes sense to introduce the effective magnetic permeability as an operator which shows the connection between the average field in the medium and the external magnetic field:

$$\langle B_i(t,\mathbf{x})\rangle = \int dt' \int d^3\mathbf{x}'\, \mu_{ij}^{eff}\ (t,t',\mathbf{x},\mathbf{x}')H_j(t',\mathbf{x}'). \qquad (1)$$

We define the other symbols. We denote the quantity μ^{eff} simply by μ. As we shall consider nonmagnetic media, the magnetic induction $\mathbf{B}(t,\mathbf{x})$ is equal to the field strength $\mathbf{H}(t,\mathbf{x})$ in the same point; we shall denote the average field $\langle\mathbf{H}(t,\mathbf{x})\rangle$ by $\overline{\mathbf{B}}(t,\mathbf{x})$.

In the case of a uniform medium we have

$$\mu_{ij}(t,\ t',\ \mathbf{x},\ \mathbf{x}')=\mu_{ij}(t-t',\ \mathbf{x}-\mathbf{x}'), \qquad (2)$$

and we can write Eq. (1) in Fourier components as follows:

$$\overline{B}_i(\omega,\ \mathbf{k})=\mu_{ij}(\omega,\ \mathbf{k})H_j(\omega,\ \mathbf{k}). \qquad (3)$$

The magnetic permeability introduced in this way cannot be used to determine the magnetic energy density in the sense it is done in macroscopic electrodynamics where $W=\mu H^2/8\pi$. In a turbulent medium we have $W\gg\mu H^2/8\pi$.

The magnitude of the magnetic permeability was first determined for the case of two-dimensional turbulence in Ref. 6 where it was shown that at the boundary of a turbulent sample the following relation holds (methodically correct calculations for the two-dimensional case were carried out in Ref. 7):

$$\overline{\mathbf{B}}_t \approx \frac{1}{R_m}\mathbf{H}_t, \qquad (4)$$

where $\overline{\mathbf{B}}_t$ is the tangential component of the field inside the sample and $\mathbf{H}_t$ the one outside it. Here R_m is the magnetic Reynolds number, $R_m = u_0\lambda_c/\eta$ (u_0 is the root mean square velocity, λ_c the correlation scale of the turbulent oscillations, η the magnetic viscosity, $\eta = c^2/4\pi\sigma$, and σ the conductivity of the medium). It is clear from Eq. (4) that a well conducting medium is diamagnetic, as we have $R_m \gg 1$ when $\overline{B}_t \ll H_t$. It also follows from Eq. (4) that in the case of two-dimensional turbulence the following relation holds:

$$\mu\approx 1/R_m. \qquad (5)$$

We shall attempt in what follows to generalize the result to the three-dimensional case. In Ref. 8 an equation was obtained which describes the behavior of the mean field in a turbulent medium. From the interpretation of that equation given in that paper the conclusion was reached that in three-dimensional turbulence μ is equal to

$$\mu \approx \frac{1}{R_m^{1/2}} \quad \text{for} \quad R_m\gg 1. \qquad (6)$$

However, it was assumed in the derivation of Eq. (6) that curl$\mathbf{H}$ (note that $\mathbf{H}(t,\mathbf{x})$ is the actual field in a point) is proportional to the induced and not to the total current, as occurs in reality [see Eq. (8) below]. For this reason the estimate obtained in Ref. 8 for μ is correct for the case when $\mu \approx 1$, but inapplicable when $\mu \ll 1$ although we note that the equations given in Ref. 8 are completely valid.

In the present paper we make an attempt at a consistent introduction of the magnetic permeability in the sense indicated above.

 0038-5646/89/090502-04$04.00

2. MAGNETIC PERMEABILITY OF A TURBULENT MEDIUM

We write down the material Maxwell equations in the magnetohydrodynamics approximation. To do this we denote the external field produced by the external current by $\mathbf{H}_0(t,\mathbf{x}) = \mathbf{H}^{\text{ext}}(t,\mathbf{x})$:

$$\operatorname{curl}\mathbf{H}_0 = \frac{4\pi}{c}\mathbf{j}^{ext}; \tag{7}$$

we denote by $\mathbf{H}(t,\mathbf{x})$ the actual field in the medium, which is determined by both the external currents and those induced in the medium:

$$\operatorname{curl}\mathbf{H} = \frac{4\pi}{c}\mathbf{j}^{tot} = \frac{4\pi}{c}(\mathbf{j}^{ext}+\mathbf{j}^{ind}). \tag{8}$$

Using also the equations

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \qquad \mathbf{j}^{ind}=\sigma\left(\mathbf{E}+\frac{1}{c}[\mathbf{uH}]\right), \tag{9}$$

where $\mathbf{u}$ is the velocity of the medium, we get the well known induction equation

$$\left(\frac{\partial}{\partial t} - \eta\Delta\right)\mathbf{H} = \operatorname{curl}[\mathbf{uH}] - \eta\,\Delta\mathbf{H}_0. \tag{10}$$

We average this equation over the velocity field and we get for the average field $\overline{\mathbf{B}}(t,\mathbf{x})$ an equation first obtained in Ref. 8 and since then often discussed in the literature.[2–5,9] On the right-hand side of Eq. (10) we have a term describing the external field sources. One usually studies the case when the field sources are outside the region of space considered and we can thus drop the last term on the right-hand side of Eq. (10). The outside sources are in that case taken into account by introducing into the problem either boundary or initial conditions. However, in our case it is necessary, in accordance with Eq. (3), to retain this term in order to find the magnetic permeability.

One sees easily that the solution of Eq. (10) for the average field $\overline{\mathbf{B}}(t,\mathbf{x})$ can be written as a power series in the velocity u and that this series will be characterized by a dimensionless parameter, the so-called Strouhal number,

$$S=u_0\tau_c/\lambda_c, \tag{11}$$

where τ_c is the correlation time of the turbulence.

We use in the present paper an equation which was obtained in Ref. 2 by the Klyatskin-Tatarskiĭ successive approximations method. To do this we assume the velocity field to be Gaussian and δ-correlated in time ($s\to 0$). In the second approximation we obtain the following equation:

$$\left(\frac{\partial}{\partial t} - \eta\Delta\right)\overline{B}_i = -\varepsilon_{ikl}\varepsilon_{lmn}\varepsilon_{n\alpha\beta}\varepsilon_{\beta\gamma\delta}\frac{\partial}{\partial x_k}\int dt'\,d^3\mathbf{x}' \times \frac{\partial G_T(t-t',\mathbf{x}-\mathbf{x}')}{\partial x_\alpha'}Q_{m\gamma}(t,t',\mathbf{x},\mathbf{x}')\overline{B}_\delta(t',\mathbf{x}') - \eta\,\Delta H_i^0, \tag{12}$$

where the function G_T and the tensor $Q_{m\gamma}$ are defined below [see Eqs. (25) and (26)].

It is in principle possible to obtain the analog of Eq. (12) in the two-dimensional case when $S\neq 0$ by using a diagram technique, since it becomes possible to sum an infinite number of terms of the series.[10]

We consider in more detail the case $\mathbf{u}(\mathbf{x}) = (u_x(x,y), u_y(x,y), 0)$. Here the turbulent motions do not affect the z-component of the magnetic field. It thus makes sense to consider the behavior of the magnetic field in the xy-plane. We introduce the vector potential $\mathbf{A} = (0,0,A(x,y))$ where $\mathbf{H}(x,y) = (\partial A/\partial y, -\partial A/\partial x, 0)$. The induction equation then will be of the form:

$$\left(\frac{\partial}{\partial t} - \eta\Delta\right)A = -(\mathbf{u}\nabla)A. \tag{13}$$

The presence of turbulent motions leads to a renormalization of the magnetic viscosity coefficient; it is this which yields a magnetic permeability different from unity.

We introduce the exact and the zeroth-order Green functions by the formulae

$$\left(\frac{\partial}{\partial t} - \eta\Delta\right)G^0(\mathbf{X},\mathbf{X}_0) = \delta(\mathbf{X}-\mathbf{X}_0), \tag{14}$$

$$\left(\frac{\partial}{\partial t} - \eta\Delta\right)G(\mathbf{X},\mathbf{X}_0) + u_i\partial_i G(\mathbf{X},\mathbf{X}_0) = \delta(\mathbf{X}-\mathbf{X}_0). \tag{15}$$

Here

$$\mathbf{X} = (t,x,y), \quad G^0(\mathbf{X},\mathbf{X}_0) = G^0(\mathbf{X}-\mathbf{X}_0) = G(\tau,\boldsymbol{\xi}) = \frac{\theta(\tau)}{4\pi\eta\tau}\exp\left(-\frac{\xi^2}{4\eta\tau}\right).$$

We shall solve the equation for the exact Green function by iteration:

$$G(\mathbf{X},\mathbf{X}_0) = G^0(\mathbf{X},\mathbf{X}_0) - \int d^3\mathbf{X}_1 G^0(\mathbf{X},\mathbf{X}_1)u_i(\mathbf{X}_1)\frac{\partial}{\partial x_{1i}}G^0(\mathbf{X}_1,\mathbf{X}_0) + \ldots \tag{16}$$

We shall assume the fluid to be incompressible and the velocity field to be Gaussian; in that case all odd velocity correlators will be equal to zero and the even ones can be split up into pairwise products of second order correlation tensors

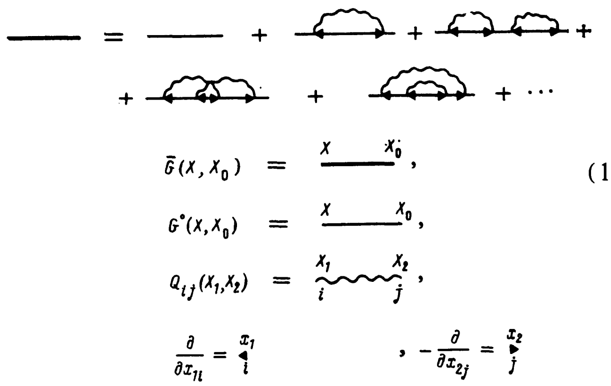
$$Q_{ij}(\mathbf{X}_1,\mathbf{X}_2) = \langle u_i(\mathbf{X}_1)u_j(\mathbf{X}_2)\rangle.$$

Averaging of Eq. (16) and integration by parts on the right-hand side transforms the series into

$$\overline{G} = G^0 - \iint d^3\mathbf{X}_1\,d^3\mathbf{X}_2 G^0\langle u_i(\mathbf{X}_1)u_j(\mathbf{X}_2)\rangle(\partial_i\partial_j G^0)G^0 + \ldots \tag{17}$$

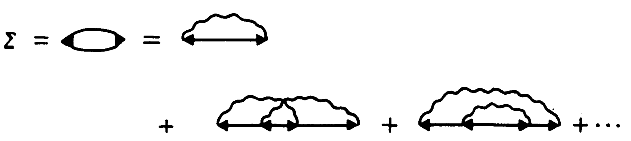
We introduce the following notation: $\overline{G}(\mathbf{X},\mathbf{X}_0)$ is the average Green function, $G^0(\mathbf{X},\mathbf{X}_0)$ the zeroth order Green function, and $Q_{ij}(\mathbf{X}_1,\mathbf{X}_2)$ the velocity correlator.

As a result the series (17) can be written in diagram form

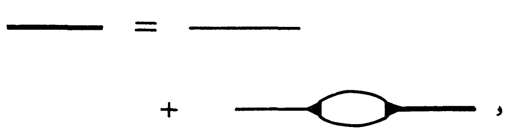


(18)

We denote the sum of all strongly coupled diagrams (the analog of the mass operator in field theory) by $\Sigma(\mathbf{X}_1,\mathbf{X}_2)$



The average Green function is then a solution of the Dyson equation



(19)

which in analytical form is

$$\bar{G}(\mathbf{X},\mathbf{X}_0)=G^0(\mathbf{X},\mathbf{X}_0)+\iint d^3\mathbf{X}_1\,d^3\mathbf{X}_2 G^0(\mathbf{X},\mathbf{X}_1)\Sigma(\mathbf{X}_1,\mathbf{X}_2)\bar{G}(\mathbf{X}_2,\mathbf{X}_0). \tag{20}$$

Let $A_0(\mathbf{X})$ be the potential distribution when there is no turbulence. We then multiply Eq. (20) by $A_0(\mathbf{X}_0)$ and afterwards integrate both sides of the equation over $d^3\mathbf{X}_0$ and act upon it with the operator $(\partial/\partial t-\eta\Delta)$. As a result we get

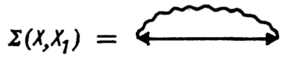
$$(\partial/\partial t-\eta\Delta)\bar{A}(\mathbf{X})=\int d^3\mathbf{X}_1\Sigma(\mathbf{X},\mathbf{X}_1)\bar{A}(\mathbf{X}_1). \tag{21}$$

We used here the fact that

$$(\partial/\partial t-\eta\Delta)A_0(\mathbf{X})=0,$$

$$\int d^3\mathbf{X}_0\bar{G}(\mathbf{X},\mathbf{X}_0)A_0(\mathbf{X}_0)=\bar{A}(\mathbf{X}).$$

We take the mass operator in the first perturbation-theory order



(22)

or, in analytical form,

$$\Sigma(\mathbf{X},\mathbf{X}_1)=-\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_{1j}}G^0(\mathbf{X}-\mathbf{X}_1)\right)Q_{ij}(\mathbf{X},\mathbf{X}_1). \tag{23}$$

Substituting (23) into (21) we get the following equation:

$$\left(\frac{\partial}{\partial t}-\eta\Delta\right)\bar{A}(t,\mathbf{x})=-\frac{\partial}{\partial x_i}\int dt'\,d^2\mathbf{x}'\,\frac{\partial G^0(t-t',\mathbf{x}-\mathbf{x}')}{\partial x_j'}Q_{ij}(t,t',\mathbf{x},\mathbf{x}')\bar{A}(t',\mathbf{x}'); \tag{24}$$

it is analogous to Eq. (12) given above.

We shall in what follows assume the turbulence to be uniform and isotropic and also reflectionally invariant. In that case we can, since we assume the velocity field to be Gaussian, characterize the turbulence by a second-rank correlation tensor

$$Q_{ij}(t,t',\mathbf{x},\mathbf{x}')=\langle u_i(t,\mathbf{x})u_j(t',\mathbf{x}')\rangle=\varphi(\tau,\xi)[\delta_{ij}-\xi_i\xi_j/\xi^2], \tag{25}$$

where $\xi=\mathbf{x}-\mathbf{x}', \tau=t-t'$.

We introduce $G_T(\tau,\xi)$—the Green function of the diffusion equation with a renormalized diffusion coefficient

$$\left(\frac{\partial}{\partial t}-(\eta+D_T)\Delta\right)G_T(\tau,\xi)=\delta(\tau)\delta(\xi). \tag{26}$$

Here D_T is the turbulent diffusion coefficient

$$D_T=\varphi(0,0)\tau_c\approx u_0^2\tau_c.$$

In the Fourier representation Eq. (12) has the following form:

$$[-i\omega+k^2(\eta+\Phi(\omega,\mathbf{k}))]\bar{\mathbf{B}}(\omega,\mathbf{k})=\eta k^2\mathbf{H}(\omega,\mathbf{k}). \tag{27}$$

Here

$$\Phi(\omega,\mathbf{k})=\frac{2}{3}\iint d\tau d^3\xi G_T(\tau,\xi)\varphi(\tau,\xi)e^{i\omega\tau-i\mathbf{k}\xi}. \tag{28}$$

In accordance with the definition (3) made above we introduce the magnetic permeability for the turbulent medium:

$$\mu(\omega,\mathbf{k})=\frac{\eta k^2}{-i\omega+k^2[\eta+\Phi(\omega,\mathbf{k})]}. \tag{29}$$

For a quasi-stationary external field, and when we take into account the smallness of the damping time of the field in the medium $T_T\sim T_{\rm mol}/R_m$, where $T_{\rm mol}\sim L^2/\eta$, it makes sense to consider the remanent magnetic field, i.e, to put $\omega=0$. In the stationary limit we then have

$$\mu(k)=\frac{1}{1+\Phi(k)/\eta}. \tag{30}$$

Moreover, considering large-scale magnetic fields with characteristic dimensions $L\gg\lambda_c$ (or $k_0\lambda_c\ll1$) and expanding $\exp(i\mathbf{k}\xi)$ in (28) in a series, we get for $\Phi(k)$ the approximate expression

$$\Phi(k)\approx D_T^{(0)}-D_T^{(2)}\lambda_c^2k^2, \tag{31}$$

where

$$D_T^{(0)}=\frac{2}{3}\iint d\tau d^3\xi G_T(\tau,\xi)\varphi(\tau,\xi), \tag{32}$$

$$D_T^{(2)}=\frac{2}{9\lambda_c^2}\iint d\tau d^3\xi\xi^2 G_T(\tau,\xi)\varphi(\tau,\xi). \tag{33}$$

It is clear from Eqs. (32) and (33) that $D_T^{(0)}$ and $D_T^{(2)}$ are of the same order of magnitude: $D_T^{(0)}\sim D_T^{(2)}\sim D_T\sim u_0^2\tau_c$. As $u_0^2\tau_c/\eta=R_mS$, in the high conductivity case ($R_m\gg1$) and in the case when S is of the order of unity, which corresponds to real turbulence, we have

$$\mu(k)=\frac{1}{R_m}(1+\alpha\lambda_c^2k^2), \tag{34}$$

where α is a numerical coefficient of the order of unity depending on the form of the velocity correlator. The average field in a turbulent fluid therefore turns out to be diminished by a factor R_m as compared to fields produced by external currents. It is important to note that the value of μ is independent of whether the turbulence is two- or three-dimensional.

We consider a weakly inhomogeneous turbulence of a fluid with a characteristic inhomogeneity scale L with $L\gg\lambda_c$. The velocity correlator to first order in λ_c/L will then have the following form.[8]

$$Q_{ij}(t,\tau,\mathbf{x},\boldsymbol{\xi})=\langle u_i(\mathbf{x},t)u_j(\mathbf{x}+\boldsymbol{\xi},t+\tau)\rangle$$
$$=\varphi(\tau,\xi)\left[\delta_{ij}-\frac{\xi_i\xi_j}{\xi^2}\right]\left(F(\mathbf{x})+\frac{1}{2}\xi_i\frac{\partial F}{\partial x_i}\right). \tag{35}$$

The function $F(\mathbf{x})$ describes here the inhomogeneity of the turbulence. Expanding the integrand in Eq. (12) in powers of λ_c/L and considering the first-order terms we get the equation[8]

$$\frac{\partial\overline{\mathbf{B}}}{\partial t}=-\eta\,\mathrm{curl}\left(1+\frac{D_T}{\eta}F(\mathbf{x})\right)^{1/2}$$
$$\times\mathrm{curl}\left(1+\frac{D_T}{\eta}F(\mathbf{x})\right)^{1/2}\overline{\mathbf{B}}-\eta\Delta\mathbf{H}_0. \tag{36}$$

In the stationary limit $(\partial\overline{\mathbf{B}}(t,\mathbf{x})/\partial t=0)$ we can obtain an expression for the magnetic permeability operator

$$\hat{\mu}^{-1}(x)=\int d^3\mathbf{x}'\left\{\mathrm{curl}\left[1+\frac{D_T}{\eta}F(\mathbf{x}')\right]^{1/2}\right.$$
$$\left.\times\mathrm{curl}\left[1+\frac{D_T}{\eta}F(\mathbf{x}')\right]^{1/2}\Big/4\pi|\mathbf{x}-\mathbf{x}'|\right\}. \tag{37}$$

In contrast to the uniform case where μ is a number, in the case of an inhomogeneous medium the magnetic susceptibility becomes an operator, the actual form of which depends on the geometry of the problem.

It is interesting to compare the obtained value of μ with the magnetic permeability for an ideal London superconductor[11]

$$\mu_s\approx\lambda_L^2k^2. \tag{38}$$

Here λ_L is the London penetration depth. A comparison of Eqs. (38) and (34) shows that the behavior of the average field inside a highly conducting turbulent fluid is similar to the behavior of the field inside a superconductor. Since $\overline{B}\approx H_0/R_m\ll H_0$, it is clear from the analogy with the superconductor that the penetration depth of the external field into a turbulent plasma is of the order of the correlation radius λ_c. The difference is that when $k=0$ there is a remanent magnetic field $\overline{\mathbf{B}}=\mathbf{H}_0/R_m$ in a turbulent sample, while there is no such field in a superconductor.

3. CONCLUSION

In the present paper we have made an attempt to describe the behavior of the average magnetic field in a reflectionally invariant turbulent medium by means of the magnetic permeability. For uniform isotropic turbulence we found the following operator

$$\mu(\omega,\mathbf{k})=\frac{\eta k^2}{-i\omega+k^2[\eta+\Phi(\omega,\mathbf{k})]}.$$

For a quasi-stationary external field, recognizing that the damping time of the field in the medium $T_T\sim T_{mol}/R_m$ is small, it makes sense to consider the remanent average magnetic field (i.e., to put $\omega=0$). In that case in the case of high conductivity $(R_m\gg1)$ we have, independently of the dimensionality of the problem

$$\mu=\frac{1}{R_m}(1+\alpha\lambda_c^2k^2),\quad \text{i.e.}\quad \mathbf{B}\approx\frac{1}{R_m}\mathbf{H}_0.$$

In the case of an inhomogeneous turbulent fluid the magnetic permeability is a rather complicated integral operator.

In conclusion the authors express their gratitude to D. A. Kirzhnits and V. Ya. Faĭnberg for useful discussions.

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Translated by D. ter Haar