Two-dimensional quasiperiodic structures in nonequilibrium systems

B. A. Malomed,[1)] A. A. Nepomnyashchiĭ,[2)] and M. I. Tribel'skiĭ

Scientific-Research Institute of Organic Intermediate Products and Dyes, Moscow

(Submitted 30 December 1988)
Zh. Eksp. Teor. Fiz. **96**, 684–700 (August 1989)

Existence and stability conditions are determined for two-dimensional dissipative structures with the symmetry of a quasicrystal. The case when the symmetry of the quasicrystal axis is of twelfth order is investigated allowing for the presence of "resonant" triplets of the fundamental vectors **q** related by $\Sigma\mathbf{q} = 0$. The parameter space of the problem is divided, in the two special cases of eighth- and tenth-order symmetry axes, into regions in which various stable structures exist, including those where the amplitudes of some spatial harmonics vanish. An analysis is made of how corrections to the evolution equations which appear in higher orders of perturbation theory affect the structural stability of the boundaries between these regions and the partial lifting of the degeneracy of phason modes. The results obtained are used also to analyze the local stability of two-dimensional quasicrystalline phases which can form as a result of thermodynamic-equilibrium phase transitions of the weak crystallization type.

1. INTRODUCTION

The analogy between a change in a steady state of a dissipative system (dynamic phase transition) and a phase transition in a system in thermodynamic equilibrium is well known (see, for example, Ref. 1). In terms of this analogy the appearance of spatially periodic structures in a nonequilibrium system can be linked to a perturbation of the translational symmetry of thermodynamic-equilibrium states, an example of which is crystallization of quantum liquids (weak crystallization),[3] appearance of a charge-density wave,[4] a nematic–smectic A transition,[5] etc.

The theory of dissipative systems and equilibrium statistical physics have been concerned mainly with spatially periodic[6–10] or spatially disordered[11,12] structures. Recent experimental results demonstrate the existence of three-dimensional[13] and two-dimensional[14–16] quasiperiodic crystals. It is natural to expect these quasiperiodic structures to appear also in dissipative systems. This hypothesis is supported by the discovery of two-dimensional "quasicrystals" in biological optics.[17] Another example of a dynamic structure with the symmetry of a two-dimensional quasicrystal is represented by structures that appear in the phase space of some self-contained Hamiltonian systems (Ref. 18).[3)]

The present paper reports a study of the existence and stability conditions of two-dimensional dissipative structures with the symmetry of a quasicrystal. The possible existence of these structures has already been demonstrated in the literature (see, for example, Ref. 19). In some specific cases studies have also been made of their stability.[20,21] However, a complete analysis of such a problem has not yet been carried out (at least, not to our knowledge). Such an analysis is provided below. The results are presented in the following order. A description of a class of these quasiperiodic structures is given in Sec. 2 together with the conditions for their existence and stability against "internal" perturbations. Some specific examples of dissipative quasicrystals differing from one another in their symmetry are discussed in Secs. 3–5 and transitions between them as a result of changes in the parameters of a system are studied. The restrictions on the phases of individual spatial harmonics, which when superimposed form the investigated quasiperiodic structure, are discussed in Sec. 6. As a rule, such restrictions are associated with inclusion of higher terms of the Landau expansion.

2. TWO-DIMENSIONAL QUASIPERIODIC STRUCTURES AND THEIR STABILITY

We shall consider a dissipative effectively two-dimensional system described by a real scalar order parameter

$$u(\mathbf{r},t)=\int a_{\mathbf{k}}(t)\exp(i\mathbf{kr})\,d\mathbf{k},\qquad a_{\mathbf{k}}^{*}=a_{-\mathbf{k}},$$

where **r** and **k** are two-dimensional vectors. It is assumed that the spatially homogeneous state of this system, which corresponds to $u\equiv 0$, now becomes unstable in the presence of perturbations characterized by a finite wave number when the order parameter exceeds a certain threshold value. This situation is typical of a number of cases of instability of planar layers and phase boundaries. Examples of these are the convective instability,[6–10,19] the Couette flow instability,[19] the instability of gas flames,[22] and of laser evaporation of condensed matter,[23] a model Turing instability of biophysical systems,[24] etc.

The general evolution equation for the amplitudes $a_k(t)$ can be represented in the form[8,10]

$$\frac{da_{\mathbf{k}}}{dt}=\gamma(k)a_{\mathbf{k}}-2\int \alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}a_{\mathbf{k}_1}a_{\mathbf{k}_2}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_1\,d\mathbf{k}_2 -\frac{4}{3}\int \beta_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}a_{\mathbf{k}_1}a_{\mathbf{k}_2}a_{\mathbf{k}_3}\delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3-\dots,\qquad (1)$$

where the instability growth rate $\gamma(k)$ is approximated by the expression

$$\gamma(k)\approx\gamma_0-\gamma_2(k^2-k_0^2)^2 \qquad (2)$$

($\gamma_{0,2}$ and k_0 are certain constants), and the explicit form of the real matrix elements $\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ and $\beta_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}$ is governed by the actual formulation of the problem.

The right-hand side of Eq. (1) represents expansion in powers of the order parameter, analogous to the familiar expansion of Landau in his theory of equilibrium phase transitions.[3] Inclusion in Eq. (1) of terms which are not only quadratic but also cubic in $a_{\mathbf{k}}$ is due to the fact that in some specific applications the matrix element $\alpha_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ is either iden-

 0038-5646/89/080388-09$04.00

tically zero (this happens if the evolution equation contains even powers of a_k so that the invariance under the substitution $a_k \to -a_k$ breaks down; in some cases the existence of such invariance follows from the symmetry of the problem) or is a small quantity. For this reason it is found that, in spite of the smallness of the amplitudes a_k, both nonlinear terms in Eq. (1) can be of the same order of magnitude. This is precisely the situation in the case of weakly supercritical convection.[6,8] However, if α is not small and $\gamma_0 > 0$, such a system exhibits an explosive instability and Eq. (1) rapidly ceases to describe the real situation.[8]

We shall consider structures with a discrete set of Fourier components of the type

$$a_{\mathbf{k}} = \sum_{n=1}^{2N} a_n \delta_{\mathbf{k}-\mathbf{q}_n} \qquad \mathbf{q}_{n+N} = -\mathbf{q}_n, \qquad a_{n+N} = a_n^*. \tag{3}$$

The structure described by Eq. (3) is spatially periodic for $N = 1$ (banks), for $N = 2$ (rectangular cells), and also for $N = 3$ when the fundamental vectors $\mathbf{q}_n$ satisfy the relationships $|\mathbf{q}_1| = |\mathbf{q}_3| = |\mathbf{q}_5|$; $\mathbf{q}_1 + \mathbf{q}_3 + \mathbf{q}_5 = 0$ (hexagonal cells). In the remaining cases the function $u(\mathbf{r},t)$ is generally quasiperiodic along at least one of the spatial coordinates.

For simplicity, we shall consider only the structures for which all the vectors $\mathbf{q}_n$ have absolute value k_0 and the angle between any two adjacent vectors $\mathbf{q}_n$ and $\mathbf{q}_{n+1}$ is the same and amounts to π/N. Such structures are analogs of two-dimensional quasicrystals investigated in Ref. 25 (some examples of structures with different angles between the fundamental vectors are given in Secs. 3 and 4 below). Substituting Eq. (3) into Eq. (1), we can show that for $N \neq 3l$, where l is a positive integer, the evolution equations become

$$\frac{da_n}{dt} = \left[\gamma_0 - 4\beta(0)\,|a_n|^2 - 8\sum_{m=1}^{N} \beta(\theta_{nm})\,|a_m|^2 \right] a_n, \tag{4}$$

where $\theta_{nm} \equiv \pi(n-m)/N$ is the angle between the vectors $\mathbf{q}_n$ and $\mathbf{q}_m$: $\beta(\theta_{nm}) \equiv \beta_{\mathbf{q}_n\mathbf{q}_m\mathbf{q}_n-\mathbf{q}_m}$; $n = 1,2,\ldots,N$.

The function $\beta(\theta)$ should satisfy the following obvious relationships:

$$\beta(\theta) = \beta(-\theta) = \beta(\pi-\theta) = \beta(\theta+2\pi j).$$

Equation (4) is meaningful only if N is not too large, so that the angle between two adjacent fundamental vectors considerably exceeds the relative width of a wave packet of interacting modes occurring in Eq. (1), i.e.

$$\pi/N \gg |\gamma_0/(\gamma_2 k_0^2)|^{1/2}.$$

If $N = 3l$ then the right-hand side of Eq. (4) should be supplemented by a term quadratic in amplitudes and amounts to $4\alpha a_{n+2l} a_{n+4l}$. Here, $\alpha \equiv \alpha_{\mathbf{q}_n \mathbf{q}_{n+2l} \mathbf{q}_{n+4l}}$.

We shall first consider the case $N \neq 3l$. We assume $\beta(0) > 0$ [if $\beta(0) < 0$, the amplitudes increase without limit, which corresponds to the case when Eq. (4) cannot be used to describe the resultant dissipative structure]. If we assume that

$$a_n = A_n e^{i\varphi_n}/(2\beta^{1/2}(0)),$$
$$A_{n+N} = A_n, \quad \varphi_{n+N} = -\varphi_n, \tag{3'}$$

we can rewrite Eq. (4) in the form

$$\frac{dA_n}{dt} = \left(\gamma_0 - \sum_{m=1}^{N} T_{m-n} A_m^2 \right) A_n, \tag{5}$$

$$\frac{d\varphi_n}{dt} = 0. \tag{6}$$

Here, $n = 1,\ldots,N$; $T_0 = 1$; $T_{m-n} \equiv 2\beta(\theta_{nm})/\beta(0)$ if $m \neq n$.

The system (5) can be represented in the form

$$dA_n/dt = -\partial F/\partial A_n, \tag{7}$$

where the Lyapunov function F is described by

$$F(A_1,\ldots,A_N) = -\frac{1}{2}\gamma_0 \sum_{n=1}^{N} A_n^2 + \frac{1}{4} \sum_{m,n=1}^{N} T_{m-n} A_m^2 A_n^2. \tag{8}$$

Since

$$\frac{dF}{dt} = \sum_{n=1}^{N} \frac{\partial F}{\partial A_n}\frac{dA_n}{dt} = -\sum_{n=1}^{N} \left(\frac{dA_n}{dt} \right)^2 \leqslant 0,$$

the only possible limiting cases corresponding to the state of such a dissipative system in the limit $t \to \infty$ are stationary states. Only the states which correspond to local minima of the Lyapunov function are stable.

Apart from the trivial solution characterized by $A_n = 0$, where $n = 1,\ldots,N$, corresponding to a spatially homogeneous state, Eq. (5) has a set of stationary solutions differing in the indices $\{n\}$ for which $A_n \neq 0$. The solution with the maximum possible number of nonzero components is

$$A_n \equiv A = \left(\gamma_0 \Big/ \sum_{m=0}^{N-1} T_m \right)^{1/2}; \qquad n = 1,\ldots,N. \tag{9}$$

The condition for "soft" branching of this structure (i.e., the condition of its existence when $\gamma_0 > 0$) reduces to

$$\beta(0) + 2\sum_{n=1}^{N-1} \beta(\pi n/N) > 0. \tag{10}$$

For $N \geqslant 4$, the solution described by Eq. (9) represents a "two-dimensional quasicrystal." It is clear from Eq. (6) that the quasicrystalline state is degenerate [in terms of Eq. (1)] with respect to arbitrary changes in the N phases φ_n. Two of these N degrees of freedom correspond to spatial translations, whereas the other $N-2$ represent "phason" modes.[26] The phason mode degeneracy may be lifted partly or completely in higher orders of perturbation theory, i.e., when the evolution equation includes the higher terms of the expansion on the right-hand side in powers of a_k (as shown in Secs. 5 and 6 below).

Next we consider the stability of quasiperiodic structures. Within the framework of the system of equations (5)–(6), a local instability of the stationary solution (9) is governed by the spectrum of growth rates σ of amplitude perturbations $\delta A_m = \Delta_m \exp(\sigma t)$, because all phase perturbations are neutrally stable: according to Eq. (6), they correspond to zero values of the increments. The spectrum of amplitude perturbations is found by solving the eigenvalue problem

$$\sum_{m=1}^{N} M_{n-m}\Delta_m=\sigma\Delta_n; \qquad n=1,\ldots,N, \tag{11}$$

where

$$M_{n-m}\equiv-2A^2T_{n-m}. \tag{12}$$

The solution of this problem is given in the Appendix. It follows from this solution that the spectrum of the growth rates is of the form

$$\sigma_l=\sum_{m=1}^{N} M_m\cos\left(l\frac{2\pi}{N}m\right); \qquad l=0,1,\ldots,N-1. \tag{13}$$

The structure (9) is stable against this type of perturbation if $\sigma_l<0$ for all values of l.

Equation (13) allows us to write down the explicit mobility conditions for any value of N. However, an analysis of these expressions in the case of high values of N is very difficult. We therefore consider only the three simplest cases for which $N=4$, $N=5$, and $N=6$.

We conclude this section by noting that a general approach to an analysis of the stability of two-dimensional quasicrystals on the basis of a grid model was proposed recently.[27] It was concluded in Ref. 27 that a minimum of the effective Hamiltonian (in our case the Lyapunov function) can be reached in a quasicrystalline lattice only for the three values of N given above. It should be noted that although in what follows we limit ourselves to a discussion of these values of N, the arguments used in Ref. 27 are clearly inapplicable to the case when the effective Hamiltonian can be expanded in powers of a small order parameter.

3. OCTAGONAL STRUCTURES ($N=4$)

If $N=4$, the system (5) contains only two independent parameters representing a nonlinear interaction of the fundamental amplitudes:

$$T_1=T_3\equiv2\beta(\pi/4)/\beta(0),\ T_2\equiv2\beta(\pi/2)/\beta(0).$$

In this case, in addition to the trivial solution $A_n=0$ which is unstable when $\gamma_0>0$, the system (5) has five more stationary solutions listed below.

1. Banks:

$$A_1=\gamma_0^{1/2}, \quad A_2=A_3=A_4=0. \tag{14a}$$

2. Square cells:

$$A_1=A_3=\left(\frac{\gamma_0}{1+T_2}\right)^{1/2}, \quad A_2=A_4=0. \tag{14b}$$

3. A lattice of rectangular cells with a ratio of side equal to $\tan(\pi/8)$ (a unit cell in the **k** space is a rhombus with the vertex angle $\pi/4$, so that such structures are called rhombic in Refs. 10 and 21):

$$A_1=A_2=\left(\frac{\gamma_0}{1+T_1}\right)^{1/2}, \quad A_3=A_4=0. \tag{14c}$$

4. An octagonal quasiperiodic structure:

$$A_1=A_2=A_3=A_4\equiv A=\left(\frac{\gamma_0}{1+2T_1+T_2}\right)^{1/2}. \tag{14d}$$

5. A structure quasiperiodic in the direction of the wave vector $\mathbf{q}_2$ and periodic in the direction $\mathbf{q}_4$:

$$A_1=A_3=\left(\gamma_0\frac{1-T_1}{1+T_2-2T_1^2}\right)^{1/2}, \quad A_2=\left(\gamma_0\frac{1-2T_1+T_2}{1+T_2-2T_1^2}\right)^{1/2},$$

$$A_4=0. \tag{15}$$

All these structures appear as a result of "soft" branching of the trivial solution $A_n=0$ at the point $\gamma_0=0$: they exist only if $\gamma_0>0$, i.e., they appear in the instability region of the trivial solution, and their amplitude increases with γ_0 as $\gamma_0^{1/2}$.

In studies of the stability of the structures described by Eqs. (14) and (15) in terms of the system (5)–(6) it is sufficient to consider pure amplitude perturbations, as was done in the case of Eq. (9). It is convenient to divide such perturbations into external, corresponding to the values of n characterized by $A_n=0$, and internal, corresponding to the values of n characterized by $A_n\neq0$.

It follows from Eq. (5) that the dispersion equation for external perturbations is of the form

$$\sigma_{n(\text{ext})}=\gamma_0-\sum_{m=1}^{N} T_{m-n}A_m^2. \tag{16a}$$

The dispersion relationship for internal perturbations is obtained by solving the appropriate secular equation, whose order is equal to the number of nonzero values of A_n in the stationary solution of the system (5), the stability of which is being investigated. In this case when the secular equation is of higher order, the stability conditions can be determined conveniently by applying the Routh–Hurwitz criterion, which guarantees the absence of roots with a positive real part in the case of the relevant polynomial. Bearing this point in mind and using the expressions labeled (14), we obtain the following stability criterion.

Banks:

$$T_1>1, \quad T_2>1. \tag{16b}$$

Square cells:

$$T_1>(1+T_2)/2, \ T_2<1. \tag{16c}$$

Rectangular cells:

$$T_1<1, \quad T_2>1. \tag{16d}$$

Octagonal quasiperiodic structure [Eq. (14d)]:

$$T_1<(1+T_2)/2, \quad T_2<1. \tag{16e}$$

The solution (15) is unstable throughout the range of its existence.

The conditions labeled (16) should be supplemented by the obvious inequalities ensuring the existence of a given structure, i.e., ensuring that the argument of the square root in the relationship (14) is positive. The whole diagram of stable stationary states in the (T_1,T_2) plane appears as shown in Fig. 1. The thick lines represent the boundaries of the regions of existence of the investigated structures. The denominators in the expressions (14) vanish on these lines. Since in this approximation the amplitude of a structure should be small, it follows that near the boundaries of the

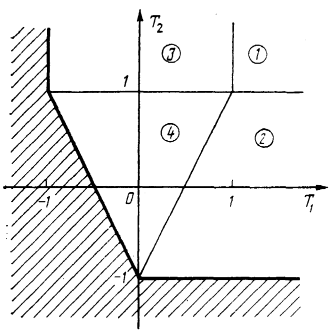

FIG. 1. Diagram of states obtained for the case when $N = 4$. The numbers in the circles denote the regions of stability of the various structures:. *1*) banks [Eq. (14a)]; *2*) square cells [Eq. (14b)]; *3*) rectangular cells [Eq. (14c)]; *4*) octagonal quasiperiodic structure [Eq. (14d)]. The thick lines are the boundaries of the ranges of existence; the shaded region represents an explosive instability.

various regions the initial equation (1) loses its validity and we have to refine this equation by including higher nonlinearities. The shaded region Fig. 1 represents those values of T_1 and T_2 which correspond to explosive instability of the system.

It should be stressed that the boundaries between the various regions shown in Fig. 1 are structurally unstable and become "smeared out" when higher nonlinearities are included. We illustrate this by considering the example of the boundary $T_2 = 1$ between regions corresponding to stable banks ($T_2 > 1$) and square cells ($T_2 < 1$). The system (5) then becomes

$$dA_1/dt = A_1(\gamma_0 - A_1^2 - T_2 A_3^2), \quad dA_3/dt = A_3(\gamma_0 - A_3^2 - T_2 A_1^2). \tag{17a}$$

For $T_2 = 1$, Eq. (17a) has a continuum of neutrally stable solutions

$$A_1 = \rho \cos\psi, \quad A_3 = \rho \sin\psi, \tag{18a}$$

where $\rho^2 = \gamma_0$ and ψ is an arbitrary phase. However, this expression becomes invalid in terms of higher order if the amplitudes $A_{1,3}$ are included in Eq. (17a). In general, the system (17a) written down to within terms of the order of A^7 becomes

$$dA_1/dt = A_1[(\gamma_0 - A_1^2 - T_2 A_3^2) - (C_0 A_1^4 + C_1 A_1^2 A_3^2 + C_2 A_3^4) - (D_0 A_1^6 + D_1 A_1^4 A_3^2 + D_2 A_1^2 A_3^4 + D_3 A_3^6)],$$

$$dA_3/dt = A_3[(\gamma_0 - A_3^2 - T_2 A_1^2) - (C_0 A_3^4 + C_1 A_1^2 A_3^2 + C_2 A_1^4) - (D_0 A_3^6 + D_1 A_3^4 A_1^2 + D_2 A_3^2 A_1^4 + D_3 A_1^6), \tag{17b}$$

where the C_j and D_j are constants.

The system (17b) still has a stationary solution of the type (18a), but we now have

$$\rho^2 = \gamma_0 - \gamma_0^2 C_0 + \frac{\gamma_0^2}{4}(C_0 - C_1 + C_2)\sin^2(2\psi) + O(\gamma_0^3), \tag{18b}$$

where ψ and T_2 are related by

$$T_2 = 1 + \gamma_0 B + \gamma_0^2 K \sin^2(2\psi) + O(\gamma_0^3),$$

where

$$B \equiv (C_0 - C_2) + \gamma_0[C_0(C_2 - C_0) + (D_0 - D_3)],$$

$$K \equiv {}^1/_4[(D_3 - D_2 + D_1 - D_0) + (C_0 - C_2)(C_0 - C_1 + C_2)].$$

Depending on the sign of the constant K, the branching pattern assumes the form shown in Fig. 2a ($K > 0$) or that shown in Fig. 2b ($K < 0$). For $K > 0$, then in a narrow range of T_2 there is a hysteresis between structures in the form of banks and squares; however, for $K < 0$, then in a narrow range of T_2 a stable structure is intermediate and it is described by Eq. (18): it represents a superposition of two perpendicular systems of banks with different amplitudes.

Similar effects occur also on the remaining boundaries: allowance for higher nonlinearities "smears out" these boundaries and, depending on the sign of the relevant constants, we can observe either hysteresis or a continuous transition between structures of different symmetries via an intermediate structure which is a superposition of the other structures.

More complex and richer in opportunities is a situation which appears in the vicinity of the point with the coordinates (1, 1), where the stability boundaries of three structures meet (Fig. 1). However, a detailed analysis of this situation is outside the scope of the present paper.

We complete the analysis of the diagram of states by noting, as demonstrated by direct calculation, that the absolute minimum of the Lyapunov function (8) corresponds to each structure in the region where it is stable.

We now consider the stability of an octagonal "quasicrystal." We note that the conditions (16) are only necessary. A complete analysis of the local stability of such a structure against all possible small perturbations should be based on linearization of the initial equation (1) near the solution described by Eq. (14d). Such a complete investigation of the stability of an octagonal quasicrystalline structure was reported in an earlier paper by the present authors.[21] It showed that the conditions of Eq. (16) should be supplemented by two additional conditions

$$\frac{\gamma_0^{1/2}}{\alpha} \geqslant \frac{2(1+2T_1+T_2)^{1/2}}{\tilde{T} - (1+2T_1+T_2)}, \tag{19}$$

$$T(\theta) \geqslant 1 + 2T_1 + T_2, \tag{20}$$

where the following notation is introduced:

$$\tilde{T} = 2[\beta(\pi/12) + \beta(\pi/6) + \beta(\pi/3) + \beta(5\pi/12)]/\beta(0),$$

$$T(\theta) = 2[\beta(\theta) + \beta(\pi/4 - \theta) + \beta(\pi/2 - \theta) + \beta(3\pi/4 - \theta)]/\beta(0).$$

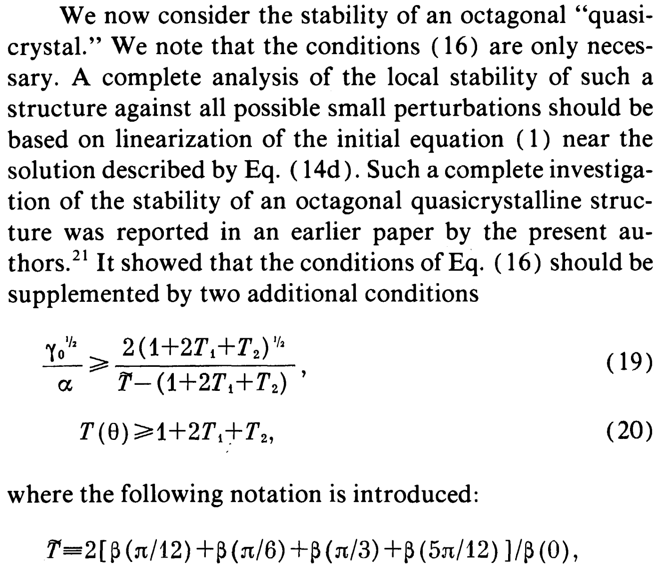

FIG. 2. "Smeared out" boundaries between walls and square cells in seventh-order perturbation theory; $\delta T_2 \equiv T_2 - (1 + \gamma_0 B)$. The thick curves correspond to stable structures and the thin ones to unstable structures: *1*) banks; *2*) square cells; *3*) intermediate structure [Eq. (18)]; a) $K > 0$; b) $K < 0$.

The inequality (20) should be satisfied for all values of θ.

In the case of square and rectangular cell structures an expansion of the spectrum of possible perturbations on transition from the system (5) to the initial equation (1) also gives rise to similar additional stability criteria.[8,10]

It should be noted that for $\alpha\neq 0$, the stability region of an octagonal quasiperiodic structure of Eq. (14d) may overlap the region of stability of a hexagonal structure.[21]

4. DECAGONAL STRUCTURES ($N=5$)

For $N=5$, the system (5) also has just two independent nonlinear interaction parameters:

$$T_1=T_4\equiv 2\beta(\pi/5)/\beta(0),\quad T_2=T_3\equiv 2\beta(2\pi/5)/\beta(0).$$

The exhaustive list of possible stationary solutions of the system (5) is as follows.

1. Banks:

$$A_1=\gamma_0^{1/2},\quad A_2=A_3=A_4=A_5=0. \tag{21a}$$

2. Rectangular cells of the first type:

$$A_1=A_2=[\gamma_0/(1+T_1)]^{1/2},\quad A_3=A_4=A_5=0. \tag{21b}$$

3. Rectangular cells of the second type:

$$A_1=A_3=[\gamma_0/(1+T_2)]^{1/2},\quad A_2=A_4=A_5=0. \tag{21c}$$

4. A decagonal quasiperiodic structure:

$$A_1=A_2=A_3=A_4=A_5=A\equiv[\gamma_0/(1+2T_1+2T_2)]^{1/2}. \tag{21d}$$

5. Three "nonsymmetric" structures of the type

$$A_1=A_3=[\gamma_0(1-T_1)/Z_1]^{1/2},\ A_2=[\gamma_0(1+T_2-2T_1)/Z_1]^{1/2},$$
$$A_4=A_5=0,\quad Z_1\equiv 1+T_2-2T_1^2; \tag{21e}$$

$$A_1=A_5=[\gamma_0(1-T_2)/Z_2]^{1/2},\ A_3=[\gamma_0(1+T_1-2T_2)/Z_2]^{1/2},$$
$$A_2=A_4=0,\quad Z_2\equiv 1+T_1-2T_2^2; \tag{21f}$$

$$A_1=A_4=[\gamma_0(1-T_2)/Z_3]^{1/2},\quad A_2=A_3=[\gamma_0(1-T_1)/Z_3]^{1/2},$$
$$A_5=0,\quad Z_3=1+T_1+T_2-T_1^2-T_1T_2-T_2^2. \tag{21g}$$

The ranges in which these structures exist are governed by the conditions which require that the respective arguments of the square roots should be positive. It should be noted that the structures described by Eqs. (21e) and (21f) are quasiperiodic in the direction $\mathbf{q}_2$ [structure (21e)] or $\mathbf{q}_3$ [structure (21f)], and are periodic in the perpendicular directions. The structure described by Eq. (21g) is quasiperiodic along all directions.

An investigation of the stability of these solutions against small perturbations can be made exactly as in the preceding cases. Such an investigation yields the following stability criteria.

1. Banks (21a):

$$T_1>1,\quad T_2>1. \tag{22a}$$

2. Rectangles (21b):

$$T_1<1,\quad T_2>1. \tag{22b}$$

3. Rectangles (21c):

$$T_1>1,\quad T_2<1. \tag{22c}$$

4. A decagonal "quasicrystal" (21d):

$$1+\omega_1T_1-\omega_2T_2>0,\quad 1+\omega_1T_2-\omega_2T_1>0,$$
$$\omega_1\equiv 2\cos\frac{2\pi}{5}\equiv\frac{5^{1/2}-1}{2},\qquad \omega_2\equiv 2\cos\frac{\pi}{5}\equiv\frac{5^{1/2}+1}{2}. \tag{22d}$$

5. Structure (21e):

$$1+\omega_1T_1-\omega_2T_2<0,\quad T_2<1. \tag{22e}$$

6. Structure (21f):

$$1+\omega_1T_2-\omega_2T_1<0,\quad T_1<1. \tag{22f}$$

Finally, a study of the stability of the structure (21g) demonstrates that it is always unstable.

The set of stability criteria given by the equations (22) is supplemented by restrictions that follow from the conditions for the existence of these structures, so that the diagram of states assumes the form shown in Fig. 3. As in the case when $N=4$, the boundaries between regions are structurally unstable and they "smear out" if we include higher nonlinearities.

The regions corresponding to stable rectangles or to a decagonal quasicrystal do not share boundaries. Therefore, when the parameters $T_{1,2}$ are varied, periodic cell structures first change to one-dimensional quasicrystals of the type described by Eqs. (21e) and (21f), which retain the spatial periodicity along one of the directions, whereas a two-dimensional quasiperiodic structure (21d) appears as a result of the next bifurcation.

A general analysis of the stability [based on the initial equation (1)] of the investigated stationary structures leads to the appearance of additional stability criteria similar to those described by Eqs. (19) and (20) and having the same physical meaning.

5. DODECAGONAL STRUCTURES ($N=6$)

If we substitute $\alpha\equiv 0$ in the initial equation (2), we find that the $N=6$ case is completely analogous to that obtained for $N=4$ and $N=5$. However, for $\alpha\neq 0$ a new type of nonlinear ("resonant") interaction appears for $N=6$, which gives rise to a number of qualitatively new effects. This oc-

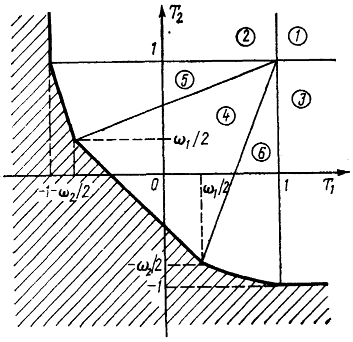

FIG. 3. Diagram of states for the case when $N=5$; *1*) banks [Eq. (21a)]; *2*) rectangular cells [Eq. (21b)]; *3*) rectangular cells [Eq. (21c)]; *4*) decagonal quasicrystal [Eq. (21d)]; *5*) structure described by Eq. (21e); *6*) structure described by Eq. (21f). The thick lines and the shaded region have the same meaning as in Fig. 1.

curs because for $N = 6$, a set of the fundamental vectors $\{\mathbf{q}_n\}$ can be used to compose two resonant triplets which combine the vectors linked by the relationship $\Sigma \mathbf{q} = 0$:

$$\mathbf{q}_1+\mathbf{q}_5+\mathbf{q}_9=0, \quad \mathbf{q}_2+\mathbf{q}_6+\mathbf{q}_{10}=0. \tag{23}$$

In this case the substitution of Eq. (3) into Eq. (1) yields the following equations [compare with Eqs. (5) and (6)]:

$$\frac{dA_n}{dt}=\left(\gamma_0+\sum_{m=1}^{6} T_{m-n}A_m^2\right)A_n+\nu A_{n+4}A_{n+8}\cos\Phi_n, \tag{24}$$

$$A_n\frac{d\varphi_n}{dt}=-\nu A_{n+4}A_{n+8}\sin\Phi_n, \tag{25}$$

where

$$\nu=2\alpha/[\beta(0)]^{1/2}, \quad \Phi_n\equiv\varphi_n+\varphi_{n+4}+\varphi_{n+8} \tag{26}$$

($\varphi_n \equiv \varphi_{n-12}$ for $n > 12$). We consider the specific case $\nu > 0$ (for $\nu < 0$, the problem reduces to the one under consideration if we make the substitution $\varphi_n \to \varphi_n + \pi$).

It follows from Eq. (25) that in the stationary solution corresponding to a dodecagonal quasicrystal ($A_1 = A_2 = ... = A_6 \equiv A$) the phases φ_n are related by two independent relationships of the type

$$\sin\Phi_n=0, \tag{27}$$

which are satisfied independently by the sublattices with even and odd indices. For this reason the number of independent phason modes increases by two.

The amplitude A is readily found from Eq. (24):

$$A=s\frac{\nu\pm(\nu^2+4\gamma_0 Q_0)^{1/2}}{2Q_0}, \tag{28}$$

where

$$Q_0\equiv1+2T_1+2T_2+T_3; \tag{29}$$

here, $T_1 = T_5 \equiv 2\beta(\pi/6)/\beta(0)$; $T_2 = T_4 \equiv 2\beta(\pi/3)/\beta(0)$; $T_3 \equiv 2\beta(\pi/2)/\beta(0)$; the quantity $s \equiv \cos\Phi_n = \pm 1$ has the same value for both sublattices.

The resonant interaction has the effect that the solution given by Eqs. (27) and (28) is characterized by hard excitation: it appears for $\gamma_0 < 0$, i.e., in the stability range of the trivial solution $A_n = 0$ and immediately reaches a finite amplitude $A_0 = \nu/2Q_0$ (the case when $Q_0 > 0$ is illustrated in Fig. 4). It is well known that in the class of periodic cell structures similar properties are exhibited by a hexagonal structure, which is also due to the resonant interaction of the modes.[6]

We consider the stability of a quasiperiodic structure described by Eqs. (27) and (28). The system of linearized equations splits into two independent subsystems representing amplitude and phase perturbations. In turn, the system of equations for the phases splits into two identical subsystems describing the evolution of perturbations with even and odd indices, each of which is analogous to the corresponding equations for the hexagonal structure. As in the case of hexagons, the branch with $s = -1$ is unstable, so that we shall assume $s = +1$.

The problem of amplitude perturbations is still described by Eq. (11), but the elements of the matrix $\hat{M}$ are different:

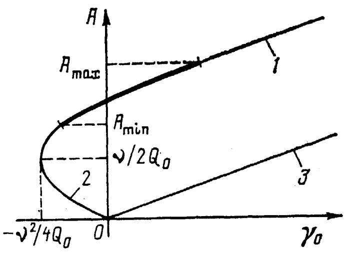

FIG. 4. Dependences of the amplitude of a dodecagonal quasicrystal ($N = 6$) on the supercriticality parameter γ_0 (bifurcation diagram). The thick curve represents the stable region; s is unity for branches *1* and *2* and $s = -1$ for branch *3*.

$$M_0=-\nu A-2A^2, \quad M_1=M_5=-2T_1A^2,$$
$$M_2=M_4=\nu A-2T_2A^2, \quad M_3=-2T_3A^2.$$

Equation (13) then yields the following expressions for the growth rates:

$$\sigma_0=\nu A-2Q_0A^2, \quad \sigma_1=\sigma_5=-2\nu A-2Q_1A^2,$$
$$\sigma_2=\sigma_4=-2\nu A-2Q_2A^2, \quad \sigma_3=\nu A-2Q_3A^2,$$

where Q_0 is defined by Eq. (29) and $Q_{1,2,3}$ by

$$Q_1\equiv1+T_1-T_2-T_3, \quad Q_2\equiv1+T_3-T_1-T_2,$$
$$Q_3\equiv1-2T_1+2T_2-T_3.$$

An analysis of the expressions for $\sigma_{0,3}$ shows that the structure under consideration is definitely unstable if at least one of the quantities Q_0 and Q_3 is negative. For $Q_{0,3} > 0$, the stability conditions become

$$A>A_{min}\equiv\max\{\nu/2Q_0;\ \nu/2Q_3\},$$

i.e., only the segment 1 of the amplitude curve (Fig. 4) can be stable.

It is clear from the expressions for $\sigma_{1,2,4,5}$ that $Q_{1,2} > 0$ that these increments are always negative, i.e., the corresponding fundamental modes cannot give rise to an instability. However, if at least one of the coefficients $Q_{1,2}$ is negative, the conditions $\sigma_{1,2,4,5} < 0$ lead to an additional stability criterion which limits the amplitude of the investigated structure from above. For example, if Q_1 and Q_2 are negative, the corresponding restriction is of the form

$$A<A_{max}\equiv\min\{-\nu/Q_1;\ -\nu/Q_2\}.$$

If only one of these coefficients is negative, it is this coefficient that determines the value of $A_{\max}$.

The requirement that the upper and lower bounds be compatible leads to certain relationships between the coefficients $Q_{0,1,2,3}$. We can easily show that these relationships are

$$Q_1+2Q_0>0, \quad Q_2+2Q_0>0, \quad Q_1+2Q_3>0, \quad Q_2+2Q_3>0.$$

The resonant interaction in the $N = 6$ case alters the properties also of other structures formed on the same basis. In particular, there are no stationary solutions of Eqs. (24) and (25) in which the amplitude of the Fourier component corresponding to any one vector of a resonant triplet would vanish, and the amplitudes of the components corresponding to the other two vectors of the same triplet would be different from zero. For $\alpha \neq 0$ it therefore follows that among the stationary spatially periodic solutions of the problem only the following types of the structure are possible: banks ($A_1 \neq 0$, $A_{2,3,4,5,6} = 0$), rectangular cells of the type $A_1 = A_2 \neq 0$, $A_{3,4,5,6} = 0$, as well as square cells ($A_1 = A_4 \neq 0$, $A_{2,3,5,6} = 0$). All these structures are characterized by soft excitation, since in the case of these structures the quadratic term describing the resonant interaction drops out of Eqs. (24) and (25) and these equations reduce to Eqs. (5) and (6).

The last of the possible types of structure are those of the form $A_{2,4} = 0$ and $A_{1,3,5,6} \neq 0$. Such structures are periodic in one direction and quasiperiodic in a perpendicular direction (in the example selected here these directions are along the vectors $\mathbf{q}_3$ and $\mathbf{q}_6$, respectively). Such structures are characterized by hard excitation.

An investigation of the stability of all these stationary solutions can be carried out by the same methods as before. It should be pointed out that in this situation the ranges of stability of the structures characterized by soft and hard excitation may overlap and this gives rise to multistability and hysteresis, similar to that in the case of spatially periodic structures.[6]

6. PHASE RESTRICTIONS

The role of higher orders of perturbation theory in lifting the degeneracy of phase and modes has already been discussed in the literature. For example, it was pointed out in Ref. 28 that in the case of a two-dimensional quasicrystal with $N = 5$ the number of independent phason modes is four, whereas in Ref. 29 it is shown that a matrix element of the sixth order exhibits a resonance in the case of a triplet forming a hexagonal structure for $\alpha = 0$. This problem has been tackled also elsewhere (see, for example, Ref. 30). The present section gives a more detailed analysis of the problem.

As demonstrated in the preceding section, the existence of a resonant interaction in the $N = 6$ case results in partial lifting of the phason mode degeneracy. The state of neutral equilibrium then splits into stable ($s = 1$) and unstable ($s = -1$) states. We can easily show that similar effects should occur also for other values of N. The exceptional nature of the situation which appears for $N = 6$ (in the more general case for $N = 3l$) lies in the fact that the resonant interaction is predicted even if we use just Eq. (1). However, in the case of other values of N the lifting of the degeneracy of the phason variables requires inclusion of corrections in Eq. (1) and due to higher nonlinearities.

We now show this by considering the example $N = 5$. In this case the problem is characterized by a resonant quintet of vectors:

$$\sum_{m=1}^{5} \mathbf{q}_{2m-1} = 0 \tag{30}$$

[the sum of vectors with even indices does not form a quintet independent of Eq. (30) because, according to Eq. (3), we have $\mathbf{q}_{n+5} = -\mathbf{q}_n$)]. Therefore, the phason mode degeneracy is first lifted in the fourth order of perturbation theory. The corresponding term in the evolution equation for the amplitude of the nth Fourier component $a_n(t)$ is given by

$$C^{(5)} \prod_{m=1}^{4} a^{*}_{n+2m}. \tag{31}$$

It is important to note that in this approximation the real constant $C^{(5)}$ (representing a matrix element of the fifth order) is independent of n. This follows from the symmetry of the problem under cyclic transposition of the indices $\{n\}$. For this reason inclusion of such terms does not affect the gradient form of the evolution equations which can still be represented as

$$da_n/dt = -\partial F/\partial a_n^{*} \tag{32}$$

[see Eq. (7)]. The correction to the Lyapunov function corresponding to Eq. (31) is of the form (see Ref. 30)

$$\delta F^{(5)} = C^{(5)} \left(\prod_{m=1}^{5} A_m \right) \cos \Phi^{(5)}, \tag{33}$$

where

$$\Phi^{(5)} = \sum_{m=1}^{5} \varphi_{2m-1}.$$

The stable state corresponds to a local minimum of the Lyapunov function. The amplitudes A_m of the stable stationary states are found by minimization of the lowest ($\sim A^4$) terms of the expansion of F in powers of A [see Eq. 8]. The phase restrictions appear only as a result of minimization of the term (33). For this reason we can assume that the amplitudes in Eq. (33) are known since they are given by Eq. (21d).

Extrema of Eq. (33) occur at

$$\Phi^{(5)} = \pi l, \tag{34}$$

where the integer l has two inequivalent values $l = 0$ and $l = 1$. Depending on the sign of $C^{(5)}$, one of these values corresponds to the stable stationary state and the other to the unstable stationary state.

If the symmetry of the problem does not permit the existence of terms with even powers in the evolution equations, i.e., if $C^{(5)} \equiv 0$, then the degeneracy of the phason variables is lifted for $N = 5$ if the expansion of F in powers of a includes terms of the tenth order. The corresponding correction is then

$$\delta F^{(10)} = C^{(10)} \left(\prod_{m=1}^{5} A_m^2 \right) \cos(2\Phi^{(5)}). \tag{35}$$

Extrema of Eq. (35) correspond to the values of $\Phi^{(5)}$ which are given by

$$\Phi^{(5)} = \pi l/2. \tag{36}$$

Then a quartet of inequivalent values of l (0;1;2;3) splits into two pairs (0;2) and (1;3). Depending on the sign of $C^{(10)}$, one of these pairs corresponds to a minimum and the other to a maximum of the Lyapunov function.

We can thus see that the inclusion of terms of the fifth (tenth) order in the expansion of the Lyapunov function in powers of the amplitude in the case when $N = 5$ gives rise to a new invariant of the problem, which takes the form of a sum of all the independent phases. This results in partial lifting of the degeneracy of the phason variables: the number of independent phason modes decreases by unity. It should be stressed that these phason restrictions exhaust the possibilities. The higher orders of perturbation theory do not give rise to new phase restrictions, because the problem does not have nontrivial resonant combinations of the fundamental vectors different from those given by Eq. (30).

We can easily see how to generalize the results obtained for the case of an arbitrary N. If N is a simple number added to 2, then (as before) we have just one phase restriction of the type given by Eq. (34) or (36), where $\Phi^{(5)}$ should be replaced with $\Phi^{(N)}$ and

$$\Phi^{(N)}=\sum_{m=1}^{N}\varphi_{2m-1}.$$

For $N=2^w$, where w is a positive integer, there are no phase restrictions at all. For $N=2^w p_1^{w_1} p_2^{w_2}\dots p_v^{w_v}$ and

$$N\sum_{m=1}^{v}p_m^{-1}<N-2,$$

where $p_m>2$ represents different simple numbers, there are

$$N\sum_{m=1}^{v}p_m^{-1}$$

restrictions of the $\Phi_n^{(p_m)}=\pi l$ type, where

$$\Phi_n^{(p_m)}=\sum_{j=0}^{p_m-1}\varphi_{n+2jN/p_m},\qquad n=1,2,\dots,N/p_m$$

[see Eq. (26)] if the evolution equations include even powers of the amplitude, or of the $\Phi_n{}^{(pm)}=\pi L/2$ type if these equations do not have such powers. For example, for $N=6$, there are two phase restrictions (see Sec. 5), for $N=9$ there are three restrictions, for $N=15$, there are eight restrictions, etc.

By analogy with Eqs. (34) and (36), the nontrivial values of l corresponding to a minimum of the Lyapunov function are governed by the sign of the matrix element $C^{(N)}$ (or $C^{(2N)}$). Therefore, such values of l are the same for all the phase restrictions.

For

$$N\sum_{m=1}^{v}p_m^{-1}\geqslant N-2$$

the number of phase restrictions is $N-2$. The degeneracy is lifted completely. The phason modes disappear and two arbitrary phases correspond to two degrees of freedom related to spatial translations. However, this situation corresponds to very similar values of N ($N\geqslant 3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\approx3.23\cdot10^9$) and is therefore of no practical interest.

We now consider the case when the local values of the combinations of Φ change at large (compared with k_0^{-1}) distances. In this case the Landau expansion for the Lyapunov function has gradient terms of the type $(\nabla\Phi)^2$, so that the Euler–Lagrange equations for the quantity Φ reduce to the standard sine–Gordon equation:

$$\Delta\Phi=\sin\Phi, \tag{37}$$

where Δ is a two-dimensional Laplacian (the coefficient in front of $\sin\Phi$ can be reduced to unity by a scaling transformation and its sign can be altered if necessary going over to $\Phi'=\Phi+\pi$).

It is well known that in addition to a spatially homogeneous solution $\Phi=\pi l$, Eq. (37) has a solution of the type

$$\Phi=4\,\mathrm{arctg}\,[\exp(\pm x)] \tag{38}$$

(kink). The extremal value given by Eq. (38) corresponds to a local minimum of the Lyapunov function obtained for two-dimensional variations of Φ. Therefore, the solution (38) is locally stable. Such a solution describes a topological defect of the domain wall type, separating two regions filled with quasicrystalline structures for which the values of Φ differ by $\Delta l=2$ [in the absence of the fourth powers in the evolution equations, we have to modify Eqs. (37) and (38) by replacing Φ with 2Φ]. There can also be more complex locally stable exact solutions of Eq. (37) describing an intersection of two perpendicular kinks, lattices of intersecting kinks, etc. (for details see, for example, Refs. 31 and 32).

An interesting question is the role of fluctuations in the formation of stability of dissipative structures.[33-35] From this point of view, we can regard the appearance of structures and transitions between them as first-order dynamic phase transitions. It should also be noted that the density of the Lyapunov function corresponding to Eq. (37) is identical with the density of the Hamiltonian of the two-dimensional sine–Gordon model. It is well known that such a model describes easy-plane[36] and weak[37,38] ferromagnets. In systems of this kind a transition to an ordered state is a first-order phase transition. Therefore, in the problem under consideration we can expect a second-order dynamic phase transition from a state with disordered phases to one with phase restrictions of the type given by Eqs. (34) and (36).

It is important to stress that because this problem includes expansions in powers of a small amplitude of the structure, the characteristic "energies" (i.e., the changes in the density of the Lyapunov function), corresponding to ordering of the amplitudes, are much higher than the corresponding quantities in the case of phase ordering. For example, in the $N=5$ case when the restrictions of Eq. (36) apply, the ratio of these "energies" is of the order of $A^{-6}\propto\gamma_0^{-3}$, where γ_0 is a small parameter (supercriticality) defined in accordance with Eq. (2). For this reason the "temperature" of a transition associated with phase ordering should be considerably lower than the "temperature" of a transition to an amplitude-ordered state.

We conclude by noting that the present results also apply to the existence and stability of various quasicrystalline states in problems of thermodynamic equilibrium phase transitions of the weak crystallization type.

The authors are grateful to A. B. Giveptal', G. M. Zaslavskiĭ, V. L. Pokrovskiĭ, and A. A. Chernikov for discussing the results given above and their valuable comments.

APPENDIX

We consider the problem of the eigenvalues of Eqs. (11) and (12). An important feature of this problem is the cyclic nature of the matrix $\hat{M}$: the next row is obtained from the preceding one by cyclic transposition of its elements. We shall use this property of the matrix $\hat{M}$ to find the spectrum of its eigenvalues.

Let us assume that $\Delta_n^{(l)}$ is an eigenvector of the matrix $\hat{M}$ corresponding to an eigenvalue σ_1. We can easily show that the vector $\Delta_{n+1}^{(l)}$ ($\Delta_{N+1}^{(l)} \equiv \Delta_1^{(l)}$), obtained as a result of cyclic transposition of the components of the vector $\Delta_n^{(1)}$, is also an eigenvector of $\hat{M}$ corresponding to the same eigenvalue σ_l. If the matrix $\hat{M}$ has no multiple eigenvalues, we have $\Delta_{n+1}^{(l)} = C_l \Delta_n^{(l)}$, where C_l is a certain constant. Applying this relationship N times and bearing in mind that $\Delta_{n+N}^{(l)} \equiv \Delta_n^{(l)}$, we obtain $C_l^N = 1$, i.e.,

$$C_l = \exp\left(i\frac{2\pi l}{N}\right), \qquad \Delta_n^{(l)} = \Delta_1^{(l)} \exp\left(i\frac{2\pi l n}{N}\right),$$

where $l = 0,1,2,...,N-1$.

We now know the explicit expression for the components vector $\Delta_n^{(l)}$ so that we can easily obtain Eq. (13) which gives σ_l. Finally, direct substitution of $\Delta_n^{(l)}$ and σ_l into Eqs. (11) and (12) shows that Eq. (11) is satisfied identically for these values of $\Delta_n^{(l)}$ and σ_l identically irrespective of whether the set defined by Eq. (13) has coincident values of σ_l.

[1)] P. P. Shirshov Institute of Oceanology, Academy of Sciences of the USSR.

[2)] Institute of the Mechanics of Continuous Media, Ural Division of the Academy of Sciences of the USSR.

[3)] A recent paper[39] reported an investigation of such forced two-dimensional flow of a viscous liquid characterized by a quasicrystalline symmetry.

[1]H. Haken, *Introduction to Synergetics: Nonequilibrium Phase Transitions and Self-Organization in Physics, Chemistry and Biology*, Springer Verlag, Berlin (1977).

[2]P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability, and Fluctuations*, Wiley-Interscience, New York (1971).

[3]L. D. Landau, Zh. Eksp. Teor. Fiz. **7**, 627 (1937); D. A. Kirzhnits and Yu. A. Nepomnyashchiĭ, Zh. Eksp. Teor. Fiz. **59**, 2203 (1970) [Sov. Phys. JETP **32**, 1191 (1971)]; Yu. A. Nepomnyashchiĭ, Teor. Mat. Fiz. **8**, 413 (1971); S. A. Brazovskiĭ, I. E. Dzyaloshinskiĭ, and A. R. Muragov, Zh. Eksp. Teor. Fiz. **93**, 1110 (1987) [Sov. Phys. JETP **66**, 625 (1987)].

[4]W. L. McMillan, Phys. Rev. B **12**, 1187 (1975).

[5]W. L. McMillan, Phys. Rev. A **9**, 1720 (1974).

[6]A. Schlüter, D. Lortz, and F. H. Busse, J. Fluid Mech. **23**, 129 (1965).

[7]A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 203 (1969).

[8]E. A. Kuznetsov and M. D. Spektor, Zh. Eksp. Teor. Fiz. **71**, 262 (1976) [Sov. Phys. JETP **44**, 136 (1976)]; Zh. Prikl. Mekh. Tekh. Fiz. No. 2, 76 (1980).

[9]J. E. Wesfreid and S. Zaleski (eds.), *Cellular Structures in Instabilities* (Proc. Meeting, Gif-sur-Yvette, France, 1983), Springer Verlag, Berlin (1984) [Lecture Notes in Physics, Vol. 210].

[10]B. A. Malomed and M. I. Tribel'skiĭ, Zh. Eksp. Teor. Fiz. **92**, 539 (1987) [Sov. Phys. JETP **65**, 305 (1987)].

[11]A. V. Gaponov-Grekhov and M. I. Rabinovich, Nonlinear Waves: Structures and Bifurcations, Vol. 2, Springer, New York (1989); Izv. Vyssh. Uchebn. Zaved. Radiofiz. **30**, 131 (1987).

[12]P. Coullet, C. Elphick, and D. Repaux, Phys. Rev. Lett. **58**, 431 (1987).

[13]D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).

[14]T. Ishimasa, H.-U. Nissen, and Y. Fukano, Phys. Rev. Lett. **55**, 511 (1985).

[15]L. Bendersky, Phys. Rev. Lett. **55**, 1461 (1985).

[16]N. Wang, H. Chen, and K. H. Kuo, Phys. Rev. Lett. **59**, 1010 (1987).

[17]N. Rivier, J. Phys. (France) **47**, Colloq. 3, C3-299 (1986).

[18]G. M. Zaslavskiĭ, M. Yu. Zakharov, R. Z. Sagdeev *et al.*, Zh. Eksp. Teor. Fiz. **91**, 500 (1986) [Sov. Phys. JETP **64**, 294 (1986)]; Pis'ma Zh. Eksp. Teor. Fiz. **44**, 349 (1986) [JETP Lett. **44**, 451 (1986)].

[19]H. L. Swinney and J. P. Gollub (eds.), *Hydrodynamic Instabilities in the Transition to Turbulence*, Springer Verlag, Berlin (1981).

L. M. Pismen, Phys. Rev. A **23**, 334 (1981); Dyn. Stability Syst. **1**, 97 (1986).

[21]B. A. Malomed, A. A. Nepomnyashchiĭ, and M. I. Tribel'skiĭ, Pis'ma Zh. Fiz. **13**, 1165 (1987) [Sov. Phys. Tech. Phys. **13**, 487 (1987)].

[22]G. I. Sivashinsky, Annu. Rev. Fluid. Mech. **15**, 179 (1983).

[23]M. I. Tribel'skiĭ, Izv. Akad. Nauk SSSR Ser. Fiz. **46**, 1127 (1982).

[24]A. M. Turing, Philos. Trans. R. Soc. London Ser. B **237**, 37 (1952).

[25]D. Levine and P. J. Steinhardt, Phys. Rev. Lett. **53**, 2477 (1984).

[26]P. A. Kalugin, A. Yu. Kitaev, and L. S. Levitov, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 119 (1985) [JETP Lett. **41**, 145 (1985)].

[27]L. S. Levitov, Europhys. Lett. **6**, 517 (1988).

[28]D. Levine, T. C. Lubensky, S. Ostlund *et al.*, Phys. Rev. Lett. **54**, 1520 (1985).

[29]M. Golubitsky, J.W. Swift, and E. Knobloch, Physica D (Utrecht) **101**, 249 (1984).

[30]P. Bak, Phys. Rev. Lett. **54**, 1517 (1985).

[31]G. Leibbrandt, Phys. Rev. B **15**, 3353 (1977).

[32]A. B. Borisov, G. G. Taluts, A. P. Tankeev, and G. V. Bezmaternykh, *Modern Problems in the Theory of Magnetism* [in Russian], Naukova Dumka, Kiev (1986), p. 103.

[33]J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[34]M. V. Feigel'man and I. E. Staroselsky, Z. Phys. B **62**, 261 (1986).

[35]H. van Beijeren and E. G. D. Cohen, Phys. Rev. Lett. **60**, 1208 (1988).

[36]U. Enz, Helv. Phys. Acta **37**, 245 (1964).

[37]A. K. Zvezdin, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 605 (1979) [JETP Lett. **29**, 553 (1979)].

[38]V. G. Bar'yakhtar, B. A. Ivanov, and A. L. Sukstanskiĭ, Zh. Eksp. Teor. Fiz. **78**, 1509 (1980) [Sov. Phys. JETP **51**, 757 (1980)].

[39]V. V. Beloshapkin, A. A. Chernikov, R. Z. Sagdeev, and G. M. Zaslavsky, Phys. Lett. A **133**, 395 (1988).

Translated by A. Tybulewicz