Theory of beat-wave acceleration of particles in inhomogeneous plasmas

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At a certain distance from the leading edge of a two-frequency laser pulse, the phase of the nonlinear plasma wave excited by this pulse is very sensitive to variations in the plasma density. Under these conditions, the requirements on the homogeneity of the plasma are least stringent.

Laser methods for accelerating particles in plasmas have been attracting increasing interest in recent years.[1,2] The most active research effort is on the acceleration of electrons in the fast plasma wave excited in a low-density plasma by two-frequency laser light.[3] When the difference between the frequencies of the laser waves is approximately equal to the electron plasma frequency, a longitudinal plasma wave is excited in the plasma.[4] This wave is used to accelerate electrons. The concept is called a "plasma beat-wave accelerator" (PBWA). At present, a plasma wave with an acceleration rate greater than 1 GeV/m has been excited over a distance of about 2 mm (Refs. 5 and 6).

Further progress in the development of a plasma beat-wave accelerator depends on increasing the acceleration length, which should be a matter of meters, so it will also be necessary to produce a plasma with an exceedingly homogeneous density of 10^{16}–10^{17} cm^{-3} over this distance. Even slight changes in the plasma density will disrupt the resonance between the driving system and the plasma waves and can substantially reduce the amplitude of these waves. Variations in the plasma were the reason for the lack of success in the detection of plasma waves in the experiments of Ref. 7 and the reason for the small size of the region in which these waves were excited in the experiments of Ref. 6. The method of multiphoton ionization of gases has made it possible to produce plasmas with a density which is homogeneous within 2% over a distance of 8 mm (Ref. 8).

Of greater danger to the acceleration of particles, however, are the changes in the phase velocity of the plasma wave which result from variations in the plasma. Such changes can cause a particle to leave the accelerating phase and to go into a decelerating phase.[9,10]

The effect of plasma variations on the excitation of plasma waves in beat-wave accelerators has been discussed from various standpoints in several places. Horton and Tajima,[11] for example, have examined inhomogeneities propagating at the velocity of the laser pulse. Periodic variations in the plasma density associated with a sound wave from (for example) stimulated Brillouin scattering were studied in detail by Darrow *et al.*[12] in the linear approximation. Immobile random variations in the plasma density (fluctuations) were discussed in Refs. 9 and 13, where the permissible variations—variations which would have no substantial effect on the particle acceleration—were determined.

It has been assumed in all of these studies that the laser pulse is fairly long and that the effect of the plasma inhomogeneity does not depend on the position of the particle being accelerated with respect to the leading edge of the laser pulse. In Ref. 14, where we used the linear approximation to describe the plasma waves, we showed that the amplitude and phase effects associated with an inhomogeneity depend strongly on the position of the particle with rspect to the leading edge of the laser pulse. Karttunen and Salomaa[10] reached the same conclusion by numerical calculations, in both linear and nonlinear approximations.

In the present paper we examine the excitation of plasma waves by a square two-frequency laser pulse in a plasma in which there are small static density variations. We show that at certain distances from the leading edge of the pulse the nonlinearity of the plasma wave causes the phase of this wave to become relatively insensitive to plasma variations, and the wave amplitude becomes fairly large. It is this region which is the most suitable for beat-wave acceleration of particles in an inhomogeneous plasma

1. INITIAL RELATIONS

We consider a wave packet (pulse) formed by two electromagnetic waves which are propagating along the X axis and which differ in frequency by a small amount ω, which is approximately equal to the plasma frequency $\omega_p = (4\pi e^2 N(x)/m)^{1/2}$, where $N(x)$ is the density of plasma ions, which varies along the coordinate x. This ions are assumed to be singly ionized and immobile. The resultant field of the waves is an amplitude-modulated wave (a beat wave), whose envelope is propagating at a velocity v_g, which is close to the velocity of light, c. If dispersion is ignored, and assuming a given packet, the complex amplitudes of the electromagnetic waves, $E_{1,2}$, are known functions which depend on the combination of variables $x - v_g t$.

The rf pressure force excites a plasma wave in the plasma.[4] The equation for the slowly varying amplitude of this wave has been discussed in many papers (e.g., Refs. 10, 15–17). It is

$$-2i\partial q/\partial\xi - q\Delta + {}^3/_2 q|q|^2 = a_1 a_2^*, \tag{1}$$

where

$$q(\xi, x) = eE_p(\xi, x)/2m\omega c, \quad \xi = (\omega/v_g)(x - v_g t),$$
$$\Delta = [\omega_p^2(x) - \omega^2]/\omega^2 = [N(x) - N_0]/N_0 = \delta N(x)/N_0.$$

Here q is the dimensionless amplitude of the plasma wave, Δ is a density variation or frequency deviation, $N_0 = m\omega^2/4\pi e^2$ is the resonant plasma density, $a_{1,2} = eE_{1,2}/2mc\omega_{1,2}$, and $\omega_{1,2}$ are the frequencies of the electromagnetic waves. The derivation of Eq. (1) used the assumptions $|q| < 1, |a_{1,2}| < 1$, and $|\Delta| < 1$.

We assume that the laser pulse has a sharp leading edge, whose spatial position is given by $x = v_g t$; correspondingly, we have $\xi = 0$. Transforming to real quantities in Eq. (1) ($a_{1,2} = A_{1,2}\exp(i\varphi_{1,2}), q = -ia\exp(i\varphi)$), and assuming

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that the amplitudes and phases of the electromagnetic waves remain constant behind the leading edge, we find a system of equations for the amplitude and phase of the plasma wave which is excited by the laser pulse[10,15–18]:

$$db/d\eta = -\cos\theta, \tag{2}$$

$$b\,d\theta/d\eta = b\delta - 4b^3 + \sin\theta, \tag{3}$$

where

$$b(\eta, x) = 3^{1/3} a (A_1 A_2)^{1/3}/2,\ \theta(\eta, x) = \varphi + \varphi_2 - \varphi_1,$$
$$\delta(x) = 2\Delta(x)(A_1 A_2)^{2/3}/3^{1/3},\ \eta = 3^{1/3}\xi(A_1 A_2)^{2/3}/4.$$

This system of equations has the first integral

$$b\sin\theta - b^4 + {}^1/_2\delta b^2 = \text{const}. \tag{4}$$

Behind the leading edge of the pulse, the variable η is negative and determines the position of a point which is moving along with the pulse. The variable $-\eta$ may also be interpreted as the time which has elapsed since the leading edge of the laser pulse crossed the given point x (the delay time[10]).

It follows from the vanishing of the field of the longitudinal wave at the leading edge of the laser pulse that the constant in (4) is zero and that we have

$$\sin\theta = b(b^2 - \delta/2). \tag{5}$$

Using (5), we can put Eq. (3) in the form

$$d\theta/d\eta = \delta/2 - 3b^2. \tag{6}$$

From (2) and (5) we find an equation for the wave amplitude:

$$db/d\eta = -[1 - b^2(b^2 - \delta/2)^2]^{1/2}. \tag{7}$$

In connection with the acceleration of particles by a beat wave, the analysis of Eq. (7) has focused on determining the maximum plasma-wave amplitude b_m as a function of the frequency deviation δ. It has been shown[18,19] that under the condition $\delta < \delta_c = 3.2^{1/3}$ the maximum amplitude is determined by the equation.

$$b_m(b_m^2 - \delta/2) = 1. \tag{8}$$

In the case $\delta < \delta_c$, this maximum amplitude increases with increasing δ ($b_m \approx 1 + \delta/6$) and reaches its maximum value $b_m = b_{mm} = 2^{2/3}$ at $\delta = \delta_c$. When δ_c is exceeded, the maximum amplitude falls abruptly to $b_{mm}/2$ and then falls off further with increasing δ. The dashed line in Fig. 1b shows $b_m(\delta)$.

In this connection, it has been suggested[15] that the efficiency with which plasma waves are excited be increased by using a homogeneous plasma with a density exceeding N_0 by an amount $N_0\delta_c[3^{1/3}(A_1A_2)^{2/3}/2]$.

According to Eq. (7), the relationship between the amplitude b and the delay time $-\eta$ under the condition $\delta < \delta_c$ can be expressed in terms of the incomplete elliptic integral of the first kind,[10] $F(\varphi, k)$:

$$\eta = -\frac{b_m}{2c_0^{1/2}} F\left(2\,\mathrm{arcctg}\left[\frac{1}{c_0}\left(\frac{b_m^2}{b^2} - 1\right)\right]^{1/2},\ \left(\frac{c_0 - a_0}{2c_0}\right)^{1/2}\right) \tag{9}$$

where $c_0 = (2b_m^3 + 1)^{1/2}$ and $a_0 = 1 + b_m^3(1 - b_m^3/2)$.

The functional dependences $b(\eta)$ which follow from (9) were discussed in Refs. 15, 18, and 19.

2. FIELD OF PLASMA WAVE ACTING ON A PARTICLE IN AN INHOMOGENEOUS PLASMA

The idea underlying a plasma beat-wave accelerator is as follows: A particle (an electron) is injected with an initial velocity v_g at that point η at which the direction in which the field of the plasma wave is acting coincides with the direction in which the particle is moving (the particle is injected into the accelerating phase). In a homogeneous plasma, the particle is accelerated as it moves, and at the same time it is displaced with respect to the wave. The particle has to be extracted from the accelerator before it goes into the decelerating phase, where the field of the wave changes sign. The maximum energy can be acquired by particles which, as they move along with the wave, traverse the entire acceleration phase and in the process are phase-shifted by an amount π with respect to the wave. The path traversed by the particle (the acceleration length) is then[1,2] $l_a \approx 4\pi c\omega_1^2/\omega^3$ and in practice would have to be a matter of meters.

Because of the variations in the plasma, the amplitude and phase of the plasma wave vary. As a result, the particle may go out of the accelerating phase before it has traversed the entire acceleration length. A change in phase in the initial and final stages of the acceleration process is particularly effective, reducing both the number of accelerated particles and their maximum energy. The problem is to determine the permissible variations of the plasma density, i.e., those variations which will have no substantial effect on the acceleration process.

Let us consider a density variation with a length scale which is much shorter than the acceleration length l_a. The position of the particle being accelerated with respect to the leading edge of the laser pulse, η, can be assumed to remain constant as the particle passes through such a variation. The plasma density is taken into account in (5) and (7) through the frequency deviation δ. If the plasma variation is to have no substantial effect on the acceleration, we must require that the wave phase θ at the point η vary only slightly in comparison with π as δ varies. The particular interval of values of δ for which this condition holds depends on the coordinate η. The problem thus reduces to one of searching for those particle positions η for which the wave phase θ varies only slightly over a δ interval which is as wide as possible.

The possibility of this form of phase stabilization is indicated by a numerical solution of Eqs. (2) and (3). Figure 1 shows the phase θ and amplitude b of the plasma wave versus the deviation δ for four values of η. At $\eta = -1.9$, the phase varies only slightly over a fairly wide range of δ.

To find those positions of the particle undergoing acceleration for which the phase-stabilization effect occurs, we will analyze the solution of Eqs. (2) and (3) [or of Eqs. (5) and (7)].

For small values of $|\eta|$, which correspond to a particle which is close to the leading edge of the laser pulse, the nonlinearity of the plasma wave is manifested even more weakly. If this nonlinearity is ignored, we find from (5) and (6)

$$\theta = \delta\eta/2, \quad b = -(2/\delta)\sin(\eta\delta/2). \tag{10}$$

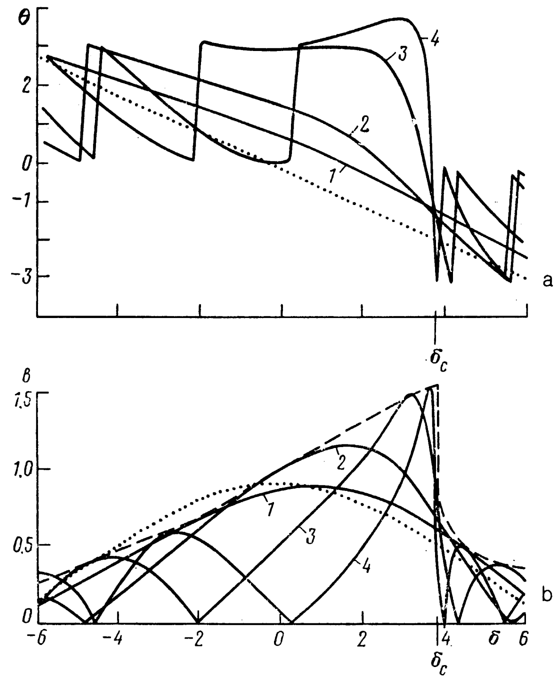

FIG. 1. The phase θ (a) and the amplitude b (b) of the plasma wave versus the deviation of the plasma density, δ, for various distances from the leading edge of the laser pulse: 1—$\eta = -0.9$; 2—$\eta = -1.2$; 3—$n = -1.9$; 4—$\eta = -2.5$. The dotted lines are results calculated from (10) at $\eta = -0.9$; the dashed line shows the function $b_m(\delta)$.

The behavior corresponding to (10) is shown by the dotted lines in Fig. 1, for the value $\eta = -0.9$, the same as for curves 1. A small change in phase corresponds to either small variations in the density or proximity of the particle to the leading edge of the laser pulse, where the wave amplitude is small ($b \approx -\eta$). The question of the field which is acting on a particle in a linear plasma wave was studied in more detail in Ref. 14.

As $|\eta|$ increases, the nonlinearity of the plasma wave becomes progressively more obvious. In the region $\delta < 0$ the nonlinearity gives rise to acceleration of the phase increase [see (6) and (10)], while at $\delta > 0$ the nonlinearity initially slows down the phase decrease and later causes a phase increase. As a result, the dependence of the phase on the deviation becomes progressively more nonlinear, and at certain values of η (curve 3 in Fig. 1) the phase depends only weakly on δ. The value of the phase is close to π, and simple approximate expressions can be derived for b and θ.

It follows from (5) that the amplitude b reaches a maximum as a function of η at $\theta = \pi/2$, while at $\theta = \pi$ it has a value of $(\delta/2)^{1/2}$ or vanishes. Since the amplitude is a linear function of η for $-1/3 < \delta/\delta_c < 2/3$ and under the condition $b < b_m$, according to (7), we can write the following at $\theta \sim \pi$:

$$b \approx T(\delta) + \eta. \tag{11}$$

Here $T(\delta)$ is the period of the variation in the function b, which is given, according to (9), by

$$T(\delta) = 2\frac{b_m}{c_0^{1/2}} K\left(\left[\frac{c_0 - a_0}{2c_0}\right]^{1/2}\right), \tag{12}$$

where $K(k)$ is the complete elliptic integral of the first kind.

For $\delta < (2/3)\delta_c$, the argument of the function K is small in comparison with unity, and from (11) we find

$$b \approx \alpha + \beta\delta + \eta, \tag{13}$$

where $\alpha = \pi/3^{1/4} \approx 2.4$ and $\beta = \alpha(8 + 3^{1/2})/96 \approx 0.24$. According to (5), at $\theta \sim \pi$ the phase can be expressed in terms of the amplitude (13) by means of the relation

$$\theta \approx \pi - b(b^2 - \delta/2). \tag{14}$$

The condition $\partial\theta/\partial\delta = 0$ can be used as a phase-stabilization condition. Using expressions (13) and (14), we find those values $\delta_\pm$ of the deviation at which this condition holds:

$$\delta_\pm = -\frac{\alpha + \eta}{\beta} + \frac{1 \pm [1 - 6\beta(\alpha + \eta)]^{1/2}}{6\beta^2}. \tag{15}$$

It follows from (15) that the minimum value of $|\eta|$ at which the condition $\partial\theta/\partial\delta = 0$ holds is $|\eta|_{\min} = |-\alpha + (6\beta)^{-1}| \approx 1.7$. For $|\eta| > |\eta|_{\min}$, there are two values of the deviation, $\delta_\pm$, for which the condition $\partial\theta/\partial\delta = 0$ holds. Figure 2 shows contour lines of the phase in the plane of $|\eta|$ and δ. The dotted line is the line on which the condition $\partial\theta/\partial\delta = 0$ holds; the solid line is the result which follows from (15).

As $|\eta|$ increases, the interval between δ_+ and δ_- expands,

$$\delta_0 = \delta_+ - \delta_- = (2/3)^{1/2}(\eta_{min} - \eta)^{1/2}\beta^{-3/2}, \tag{16}$$

but at the same time the difference between the values of the phase at δ_+ and δ_- increases

$$\Delta\theta = \theta(\delta_+) - \theta(\delta_-) = 2^{1/2}\cdot 3^{-3/2}(\eta_{min} - \eta)^{3/2}\beta^{-5/2}. \tag{17}$$

Specifying a permissible phase variation $\Delta\theta$, we can use (16) and (17) to determine both the position of the particle which is being accelerated with respect to the leading edge of the laser pulse, η, and the corresponding value of the plasma density variation, δ_0. Since the wave amplitude varies nearly linearly in this interval of δ [Fig. 1(b) and expression (11)], its average value is $\frac{1}{2}[b(\delta_+) + b(\delta_-)] = (6\beta)^{-1} \approx 0.7$.

To achieve the highest acceleration rate we would need to use a plasma wave with maximum amplitude. It can be seen from Fig. 1 that under the condition $|\eta| > |\eta|_{\min}$ the

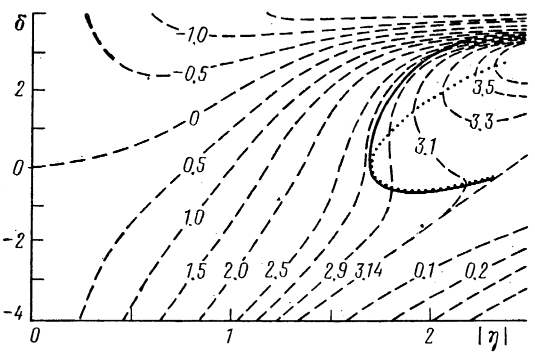

FIG. 2. Contour lines of the phase (dashed lines) in the plane of δ and $|\eta|$. The lines are labeled with the values of the phase. Dotted line—Curve on which the condition $\partial\theta/\partial\delta = 0$ holds; solid line—result calculated from approximate expression (14).

maximum amplitude approaches the value b_{mm} and corresponds to a deviation δ_m which is close to δ_c. Here, however, the phase of the plasma wave depends strongly on small variations of the plasma density. Specifically, we find from (5) under the condition $\partial b/\partial\delta = 0$.

$$(\partial\theta/\partial\delta)|_{\delta_m} = -(b_m^2/2)(2b_m^3+1)^{-3/4}(T/2-|\eta|)^{-1}|_{\delta_m},$$

where $T(\delta)$ is given by (12). Here we have used $b(\delta_m) \approx b_m(\delta_m)$, and in (9) we have carried out the expansion

$$b = b_m - (|\eta| - T/2)^2(2b_m^3+1)^{1/2}/2b_m.$$

At $b_m \simeq b_{mm}$, the magnitude of the elliptic integral is close to unity, and we can use the following approximation for T:

$$T \approx 2^{5/3}\cdot 3^{-1/2}\ln[3^{1/2}\cdot 2^{1/3}(b_{mm}-b_m)^{-1/2}].$$

Using this expression, calculating the derivative $\partial b/\partial\delta$, and equating it to zero, we find $b_{mm} - b_m = 3^{1/2}\cdot 2^{-5/3} \times (T/2 - |\eta|)$. As a result we find

$$\left.\frac{\partial\theta}{\partial\delta}\right|_{\delta_m} = \frac{-1}{2^{4/3}\cdot 3(b_{mm}-b_m)} = \frac{\delta_c}{4(\delta_m-\delta_c)}. \tag{18}$$

The phase of the wave at the maximum value of the amplitude thus becomes extremely sensitive to any variation in the density as δ_m approaches δ_c.

Expressions (16) and (17) actually underestimate the deviation interval in which the phase variations do not exceed $\Delta\theta$. In actuality, this interval is wider, and its boundaries can be found approximately on the basis of the following considerations. On the side of small and negative values of δ, the region in which the phase variation is slight is bounded by the condition $\theta = \pi$, from which we find $\delta = -(\eta + \alpha)/\beta$. On the side of large deviations, for the values of η of interest here, the region in which the phase varies slightly is bounded by the value $\delta_m \lesssim \delta_c$, at which the amplitude is at a maximum, and we have $\partial b/\partial\delta = 0$.

For $|\eta| > 2.4$ (Fig. 1a), however, this interval of deviations narrows, and inside it the derivative $\partial\theta/\partial\delta$ reaches values on the order of unity. It can thus be concluded that only for particles which are at a distance $1.7 < |\eta| < 2.4$ from the leading edge of the laser pulse is the phase of the plasma wave least sensitive to plasma density variations.

3. DISCUSSION OF RESULTS

To make use of the results derived here, we write the major quantities involved in them in dimensional form, assuming for simplicity that the amplitudes of the laser waves are identical:

$$E = \frac{m\omega c}{e}\frac{4}{3^{1/2}}\left(\frac{v_E^2}{4c^2}\right)^{1/3} b, \quad \Delta = \frac{\delta N}{N_0} = \delta\frac{3^{1/3}}{2}\left(\frac{v_E^2}{4c^2}\right)^{2/3},$$
$$t = \frac{|\eta|}{\omega}\frac{4}{3^{1/3}}\left(\frac{v_E^2}{4c^2}\right)^{-2/3}, \tag{19}$$

where

$$v_E = eE_1/m\omega_1 \approx eE_2/m\omega_2,$$

and t is the delay time, which is a measure of the temporal lag of the particle behind the leading edge of the laser pulse.

As an example, we make some estimates for the POPE experiment, which is presently being carried out and which is described in Ref. 20, among other places. This experiment uses a two-frequency CO_2 laser (9.6 μm, 10.3 μm) with an intensity $\sim 10^{14}$ W/cm^2 and a pulse length of 100 ps. For these parameter values, expressions (19) become $E = 7\times 10^9 b$ V/m, $\Delta \approx 10^{-2}\delta$, and $t = 18|\eta|$ ps.

If the laser pulse is approximately square, then the amplitude of the plasma wave can reach a significant value $\sim 3\times 10^9$ V/m even in the early part of the pulse, say at $|\eta| = 0.6(\approx 10$ ps). According to (10), however, density variations of 1% ($\delta = 1$) lead to a phase variation $\Delta\theta \sim 0.3$ or $\sim\pi/10$.

Under the conditions of the phase stabilization which we are discussing here, at $|\eta| = 2$ ($t \sim 30$ ps) a phase variation $\sim\pi/10$ corresponds to a density variation $\sim 3.8\%$ according to (16). Using the expanded interval of δ for estimates, we find the permissible density variation to be 4.7%. The average amplitude of the plasma wave would be $\sim 6\times 10^9$ V/m in this case.

If we put the particle to be accelerated even further from the leading edge of the laser pulse, at $|\eta| = 3$ ($t \approx 50$ ps), then a phase variation $\sim\pi/10$ corresponds to a density variation of 0.7-1%, although the average amplitude of the accelerating field is still $\sim 6\times 10^9$ V/m.

The analysis above applies to square laser pulses with a sharp leading edge. In reality, the leading edge of a laser pulse has some finite duration. This circumstance is reflected in the process by which the plasma wave is excited; the primary effect is an increase in the duration of the linear stage. It may be that the nonlinear phase stabilization discussed above also arises for pulses with a gently sloping leading edge, but for longer delay times. Numerical calculations for pulses of various shapes support this conclusion, although the question requires a more detailed analysis.

It also follows from our analysis that the method proposed in Ref. 15 for increasing the amplitude of the plasma wave through an appropriate choice of plasma density has a serious drawback. As was pointed out in Ref. 10, and as our own estimates show [see (18)], under these conditions small plasma density variations will lead to large changes in the phase of the plasma wave.

In summary, in a beat-wave accelerator there is a certain position of the particle being accelerated with respect to the leading edge of the laser pulse at which the requirements on the homogeneity of the plasma are least stringent. These conditions are the most favorable for particle acceleration.

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