Photoabsorption by atoms in external fields near the ionization threshold

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An expression is obtained for the photoabsorption cross section of atoms near the ionization threshold in an external static magnetic field. The results satisfactorily describe quasi-Landau resonances observed experimentally in the photoabsorption by hydrogen atoms in a magnetic field. A similar expression is obtained for the photoabsorption in an electric field.

1. INTRODUCTION

The present paper reports a study of the process of absorption of photons by hydrogen-like atoms subjected to a magnetic or an electric field in the case when transitions take place from lower states to highly excited states with energies close to the ionization threshold of a free atom (see Ref. 1).

Some time ago it was found experimentally[2] that in a magnetic field the photoabsorption cross section of atoms exhibits strong oscillations (as a function of the photon energy) near and above the ionization threshold. The existence of such oscillations, known as the quasi-Landau resonances, was confirmed by recent precision experiments.[3,4] Similar (but simpler) oscillations were found to be exhibited by the photoabsorption cross section above the ionization threshold of atoms in an electric field.[5,6]

The main difficulty in the direct investigation of problems of this kind is in constructing a wave function for the final highly excited state. In the case of a hydrogen atom in a magnetic field of intensity B, directed along the z axis, the problem reduces to the solution of the following Schrödinger equation, written in atomic units; (see, for example, Ref. 7):

$$\hat{H}\Psi=E\Psi, \quad \hat{H}=\frac{1}{2}\hat{p}^2+\frac{1}{2}\beta^2\rho^2-\frac{1}{r}, \tag{1}$$

where $r^2=x^2+y^2+z^2$; $\rho^2=x^2+y^2$; m is the azimuthal quantum number; $E=E-\beta m$ is the energy after subtraction of the usual Zeeman splitting;

$$\hat{p}^2=-\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial\rho^2}+\frac{m^2}{\rho^2},$$

and $\beta=B/(4.7\times10^5\ \text{T})$ is the magnetic field intensity in atomic units. The experiments reported in Refs. 3 and 4 were carried out in a field $B=6$ T, so that $\beta=1.28\times10^{-5}$ is a small parameter.

In studies of transitions to lower levels, characterized by $\beta^2\langle\rho^2\rangle\ll\langle1/r\rangle$, we can use perturbation theory in which the small parameter is β (Ref. 8). In the case of transitions to states with very high values of $\langle\rho^2\rangle$ when the opposite inequality is satisfied, it is permissible to apply perturbation theory to the Coulomb potential.[9] However, if in the case of the investigated state the Coulomb term is comparable with the paramagnetic term, $\langle1/r\rangle\approx\beta^2\langle\rho^2\rangle$, then such approximations are unacceptable. These states are characterized by large values of $\langle r\rangle$: $\langle r\rangle\propto1/\beta^{2/3}$, whereas the energy measured from the ionization threshold is low: $E\propto\langle1/r\rangle\propto\beta^{2/3}$. This is the case we shall consider in the present paper.

The Schrödinger equation corresponding to the Hamiltonian (1) was solved numerically in Refs. 10 and 11 by diagonalizing large matrices and it was found that the calculated photoabsorption cross section exhibits oscillations of the expected type below the ionization threshold.

It has been pointed out earlier[10–15] that quasi-Landau resonances are related in a way to the classical periodic orbits of the problem in hand. It was shown in Refs. 10–15 that the separation between neighboring peaks (representing the period of one oscillatory function) agrees well with the differences $\Delta E_n=E_{n+1}-E_n$, between the energies E_n which satisfy the "quantization condition":

$$S(E_n)=\oint \mathbf{p}\,d\mathbf{q}=2\pi n\hbar, \tag{2}$$

where $S(E)$ is the classical action calculated along a given periodic orbit passing through the Coulomb center and n is an integer.

Assuming that $\Delta E_n\ll E_n$, we find that

$$\Delta E_n=\frac{2\pi}{T_n}\hbar, \tag{3}$$

where $T_n=dS/dE\,|_{E=E_n}$ is the period of the orbit investigated.

It was assumed initially in Refs. 12–15 that there are real quantum states localized near periodic orbits and the condition (2) is the standard criterion of semiclassical quantization of one-dimensional motion. It was later understood[4,5] that this interpretation is incorrect, because in the case of a hydrogen atom in a magnetic field near the ionization threshold all such periodic orbits are classically unstable[1)] and any wave packet centered on such an orbit should spread out rapidly in a time governed by classical mechanics.

Our aim (in accordance with a brief communication published earlier[17]) is to show that the absorption of a photon by atoms in external fields can be described by simple semiclassical expressions and that the external field taken to a certain power plays the role of the Planck constant.

The main result obtained below is as follows. Near the ionization threshold the photoabsorption cross section can be represented in the form

a) in an external static magnetic field:

$$\sigma(E)=\sigma_{\text{Coul}}+\beta^{1/6}\,\text{Im}\sum_p\sigma_p^{(M)}\exp(iS_p/\beta^{1/4}); \tag{4}$$

b) in an external static electric field:

$$\sigma(E)=\sigma_{\text{Coul}}+\gamma^{1/4}\,\text{Im}\sum_p\sigma_p^{(E)}\exp(iS_p/\gamma^{1/4}). \tag{5}$$

In these expressions we have $\beta=B/(4.7\times10^5\ \text{T})$, and

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$\gamma = F/(5.1\times 10^9$ V/cm) are the intensities of the magnetic and electric fields expressed in atomic units (for the hydrogen atom); σ_{Coul} is the cross section of the pure Coulomb photoabsorption (without allowance for the external field). The summation in Eqs. (4) and (5) is carried out over all unstable periodic orbits passing through the Coulomb center. The quantity S_p is the reduced value of the classical action calculated along a given orbit; $\sigma_p^{(M)}$ and $\sigma_m^{(E)}$ are the amplitudes expressed below [see Eqs. (33)–(37)] in terms of elements of the classical monodromy matrix of the investigated periodic orbit and in terms of a wave function of the initial state. Near the ionization threshold the cross section σ_{Coul} can be regarded as independent of energy, whereas S_p, $\sigma_p^{(M)}$, and $\sigma_p^{(E)}$ depend only on the reduced energy ε, which is equal to $E/\beta^{2/3}$ for the magnetic field and $E/\gamma^{1/2}$ for the electric field. A finite value of the reduced energy corresponds to the case when the Coulomb energy is comparable with the energy in the external field and perturbation-theory methods are inapplicable.

For an atom in an electric field when the energy is positive, there is only one unstable periodic orbit which emerges from the Coulomb center against the field and returns back to the center. The corresponding sum in Eq. (5) contains only repeated replicas of this orbit, so that calculations are simple.

An atom in a magnetic field near the ionization threshold is characterized by an infinitely large number of various unstable periodic orbits[3,4,10] and exact summation over all the orbits is in any case not simpler than the direct solution of the Schrödinger equation. However, if we are interested not in the exact value of the cross section at a given energy, but in the average cross section in a small energy interval ΔE (which is equivalent to experimental determination of the cross section using a method characterized by a finite resolution), the main contribution to the sum in Eq. (4) comes from the finite number of short-period orbits the motion along which is limited by the inequality (T and ΔE are measured in atomic units and we have $\Delta E \ll \beta^{2/3}$)

$$T < \frac{\text{const}}{\Delta E}. \tag{6}$$

The necessary parameters of such orbits can be found numerically.

The expressions derived below are in good agreement with the experimental data on the photoabsorption in electric and magnetic fields, and with the results of a direct numerical solution of the Schrödinger equation in such cases.

The method used to derive Eqs. (4) and (5) is close to the method of Refs. 18 and 19 where semiclassical expressions were obtained for highly excited wave functions of the ergodic stadium billiard type. This method is based on a quasiclassical representation of the Green's function in the form of a sum over all the classical orbits joining two fixed points.[20,21]

The paper is organized as follows. The expression for the photoabsorption cross section is obtained in Sec. 2 in terms of the exact Green's function of the Schrödinger equation and the semiclassical representation is considered. In the problem under discussion the existence of a Coulomb singularity modifies the standard expressions and Sec. 3 gives the results of matching the semiclassical expressions to the exact expression near the Coulomb center. The general properties of the semiclassical expressions for the absorption cross sections in external fields are discussed in Sec. 4. The results obtained for the photoabsorption in electric and magnetic fields are compared in Secs. 5 and 6 with the experimental data and with the results of direct numerical calculations, respectively.

2. GENERAL FORMALISM

It is well known (see, for example, Ref. 7) that the photoabsorption cross section expressed in atomic units is

$$d\sigma = \frac{4\pi^2\alpha}{\omega} Q,$$
$$Q = |\mathbf{l}\mathbf{V}_{fi}|^2 \frac{d^3p}{(2\pi)^3}\delta(E+I-\omega), \tag{7}$$
$$\mathbf{V}_{fi} = -i\int \Psi_f^*(x)\nabla\Psi_i(x)\,d^3x.$$

Here, $\mathbf{V}_{fi}$ is a matrix element of the electron velocity; Ψ_i is the wave function of the initial (bound) state of energy $-I$; Ψ_f is the wave function of the final state of energy E, normalized at large distances to a plane wave with a single coefficient; ω is the photon energy; $\mathbf{l}$ is the photon polarization vector; p is the momentum of the final electron; α is the fine-structure constant.

We introduce the Green's function for the Schrödinger equation in the usual way:

$$G(x', x; E) = \sum_n \frac{\Psi_n^*(x')\,\Psi_n(x)}{E - E_n + i0}, \tag{8}$$

where E_n and Ψ_n are the exact eigenvalues and eigenfunctions of the Schrödinger equation. We then have

$$\operatorname{Im} G(x', x; E) = -\pi \sum_n \delta(E-E_n)\,\Psi_n^*(x')\,\Psi_n(x), \tag{9}$$

and the quantity Q governing the photoabsorption cross section of Eq. (7) can be expressed in terms of the Green's function as follows:

$$Q = -\frac{1}{\pi}\int \operatorname{Im} G(x', x; E)\,(\mathbf{l}\nabla\Psi_i^*(x'))\cdot(\mathbf{l}\nabla\Psi_i(x))\,d^3x'\,d^3x. \tag{10}$$

We first consider the case of a magnetic field. In view of the invariance under rotation around the magnetic field direction, we have

$$G(x', x; E) = \frac{1}{2\pi(\rho_1\rho_2)^{1/2}} \sum_{m=-\infty}^{+\infty} \exp[im(\varphi'-\varphi)]\,G_m(q', q; E), \tag{11}$$

where m is the azimuthal quantum number; $G_m(q', q; E)$ is the Green's function for the two-dimensional equation (1), and q is a vector with the components (z,ρ).

Subsequent calculations will be based, as in Refs. 18 and 19, on the use of the general semiclassical representation of the Green's function $G(x', x; E)$ in the form of a sum over all the classical paths passing through two fixed points x'' and x' (see, for example, Refs. 20 and 21 and the references given there). In the quasiclassical approximation the Green's function of the Schrödinger equation can be represented as a sum of two terms:

$$G(x'', x'; E)=G_0(x'', x'; E)+G^{osc}(x'', x'; E), \quad (12)$$

where $G_0(x'',x';E)$ is the contribution of "short" classical paths, i.e., of paths for which the classical action is small compared with the Planck constant. Such paths usually exist only if the initial and final points are close to one another and, naturally, if there are no such paths, there is no corresponding contribution. The semiclassical approximation cannot be applied to these paths, but because of their local nature the potential in the Schrödinger equation can be expanded as a series in terms of a deviation from the point $(x''+x')/2$ and in the leading approximation the quantity G_0 is identical with the Green's function considered in the Thomas–Fermi approximation. In the problem under discussion the quantity G_0 is simply the Coulomb Green's function without allowance for the external field (i.e., in the case when $\beta\to 0$).

The term $G^{osc}(x'',x';E)$ represents the contribution of "long" paths when the action along the paths is large. It is $G^{osc}(x'',x';E)$ that can be represented in the semiclassical limit by a sum over all classical paths joining the points x'' and x'. In n-dimensional space, we have

$$G^{osc}(x'', x'; E)=\frac{1}{i\hbar(2\pi i\hbar)^{(n-1)/2}} \times \sum{}^{*}\Delta^{1/2}\exp\left(i\frac{S(x'', x'; E)}{\hbar}-i\nu\frac{\pi}{2}\right), \quad (13)$$

where Σ^* denotes summation over paths and each term in the sum is related to one of the classical paths of energy E connecting the points x'' and x'. Moreover,

$$S(x'', x'; E)=\int_{x'}^{x''}\mathbf{p}\,d\mathbf{q}$$

is the action along a path and Δ is a determinant composed of the second derivatives of the action:

$$\Delta=\frac{1}{|\dot{\mathbf{q}}'||\dot{\mathbf{q}}''|}\mathrm{Det}\left(\frac{\partial^2 S}{\partial y''\,\partial y'}\right).$$

In this expression the symbol y is used for coordinates in an $(n-1)$-dimensional plane perpendicular to a given path; the primes refer to the initial and final points of a path; $|\dot{\mathbf{q}}|$ is the modulus of the velocity; ν is an integer which appears if a given path has points at which the semiclassical approximation is invalid. This number is equal to the number of conjugate points (at which $\Delta\to\infty$) in combination with an additional phase associated with the reflection from the boundaries (as discussed below).

The preexponential factor Δ in Eq. (13) can be expressed conveniently in terms of elements of the monodromy matrix. This can be done by considering one of the classical paths joining points q'' and q' and introducing in the vicinity of this path a coordinate system selecting the x axis along the path and the y axis at right-angles to it at a point x^*. We shall linearize the classical equations of motion near this path. Then, $y(t)$ is described by certain linear second-order equations and we can determine the matrix $M_{ij}(t)$ linking $y(t)$ and $\dot{y}(t)$ with $y(0)$ and $\dot{y}(0)$:

$$\begin{bmatrix}\dot{y}(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}M_{11}(t) & M_{12}(t)\\ M_{21}(t) & M_{22}(t)\end{bmatrix}\begin{bmatrix}\dot{y}(0)\\ y(0)\end{bmatrix}. \quad (14)$$

If T is the time of motion from the initial point q' to the final point q'', the matrix $M_{ij}(T)$ can be called the monodromy matrix of a given path.

We can easily show[20,21] that in two-dimensional space we have

$$\Delta=\frac{1}{|\dot{\mathbf{q}}_i||\dot{\mathbf{q}}_f|M_{12}(T)}, \quad (15)$$

where $|\dot{\mathbf{q}}_i|$ and $|\dot{\mathbf{q}}_f|$ are moduli of the velocity vector at the initial and final points, respectively.

Equations (13) and (15) allow us to determine the semiclassical contribution to the Green's function made by a given path. This can be done starting with the quantities calculated from classical mechanics, i.e., the action S, the velocities at the initial and final points, and one element of the monodromy matrix.

We now return to the problem of an atom in a magnetic field described by Eq. (1). We can show that the centrifugal term in Eq. (1) obtained in the limit $\beta\to 0$ contributes not only to the phase ν, but also to the magnitude S, and Δ is found for classical paths corresponding to the Hamiltonian

$$H=\frac{1}{2}(p_\rho^2+p_z^2)-\frac{1}{(\rho^2+z^2)}+\frac{\beta^2}{2}\rho^2. \quad (16)$$

We carry out the following scaling transformation:

$$q(t)=\beta^{-2/3}\tilde{q}(\tau), \quad p(t)\equiv\dot{q}(t)=\beta^{1/3}\tilde{p}(\tau), \quad \tau=\beta t. \quad (17)$$

After substitution of $E=\varepsilon\beta^{2/3}$ the Hamiltonian expressed in new variables is independent of β. In the case of this transformation the action and the elements of the monodromy matrix transform as follows:

$$S=\beta^{-1/3}\tilde{S}, \quad M_{11}=\tilde{M}_{11}, \quad M_{22}=\tilde{M}_{22}, \quad M_{12}=\beta^{-1}\tilde{M}_{12}, \quad M_{21}=\tilde{M}_{21}\beta, \quad \Delta=\beta^{1/3}\tilde{\Delta}. \quad (18)$$

The quantities on the right-hand sides of these relationships are called reduced. They can be calculated from the Hamiltonian of Eq. (16) when $\beta=1$ and they depend on the reduced energy $\varepsilon=E/\beta^{2/3}$. The range of energies near the ionization threshold of interest to us corresponds to finite values of ε.

After these transformations, Eq. (13) for G^{osc} becomes ($\hbar=m=1$, $n=2$):

$$G^{osc}(q'', q'; E)=\frac{\beta^{1/6}}{i(2\pi i)^{1/2}}\sum{}^{*}\tilde{\Delta}^{1/2}\exp\left(i\frac{\tilde{S}(q'', q'; E)}{\beta^{1/3}}-i\nu\frac{\pi}{2}\right). \quad (19)$$

We must stress once again that the reduced values $\tilde{\Delta}$ and $\tilde{S}$ depend only on $\varepsilon=E/\beta^{2/3}$, which is assumed to be finite. We can describe G^{osc} fully if we identify the values of the final and initial points q'' and q'.

The expression for the photoabsorption cross section (10) includes integrals of the Green's function containing the initial wave function. We are interested in the case when the initial state is one of the lowest states with all quantum numbers. The wave functions of such states are concentrated

in a region of the order of the Bohr radius or, after the scaling transformation of Eq. (17), in a region of the order of $\beta^{2/3}$. This means that in the limit $\beta \to 0$ in the reduced equation both q'' and q' are located near the Coulomb center where semiclassical expressions are generally invalid and we have to match the results to the exact Coulomb functions.

3. MODIFICATION OF SEMICLASSICAL EXPRESSIONS NEAR A COULOMB CENTER

We now go over from cylindrical coordinates (ρ,z) to parabolic coordinates μ, ν; we know that $\mu \geqslant 0$, $\nu \geqslant 0$ from the expressions

$$z = {}^1/_2(\mu^2 - \nu^2), \quad \rho = \mu\nu. \tag{20}$$

After additional transformation of the time t into a variable τ is related to t by

$$\frac{dt}{d\tau} = \mu^2 + \nu^2, \tag{21}$$

we find that the classical equations of motion corresponding to the Hamiltonian (16) are transformed into equations corresponding to the Hamiltonian

$$H = {}^1/_2(p_\mu^2 + p_\nu^2) + {}^1/_2\mu^2\nu^2(\mu^2+\nu^2) - \varepsilon(\mu^2+\nu^2) - 2 \tag{22}$$

with vanishing energy.

We can easily show that the action and the quantity Δ remain unchanged in the old and new coordinates. Since the Hamiltonian of Eq. (22) has no singularity at $\mu = \nu = 0$, it follows that for q'', $q' \to 0$ the functions $S(q'',q')$ and $\Delta(q'', q')$ tend to the finite limits S_0 and Δ_0. It is clear from Eq. (22) that if $\mu = \nu = 0$, we have $|\dot{\mu}| = 2$ and

$$\Delta_0 = \frac{1}{4m_{12}}, \tag{23}$$

where m_{12} is an element of the monodromy matrix of Eq. (14) expressed in parabolic coordinates.

If we allow for the centrifugal term, we find that the complete Schrödinger equation [without separation of the factor $\rho^{1/2}$, as in Eq. (11)] becomes

$$\left[-\frac{1}{2}\left(\frac{1}{\mu}\frac{\partial}{\partial\mu}\mu\frac{\partial}{\partial\mu} + \frac{1}{\nu}\frac{\partial}{\partial\nu}\nu\frac{\partial}{\partial\nu}\right) + m^2\left(\frac{1}{\mu^2}+\frac{1}{\nu^2}\right) + \frac{1}{2}\mu^2\nu^2(\mu^2+\nu^2) - \varepsilon(\mu^2+\nu^2) - 2\right]\Psi = 0. \tag{24}$$

The centrifugal term is large if μ and/or ν is close to zero. In this range we can ignore the term with E and $\mu^2\nu^2$. The corresponding equations are easily solved and we can show that for each passage of a given path across the $\rho = 0$ axis (or the $\mu = 0$ and $\nu = 0$ axes in terms of parabolic coordinates) in each part of the path entering and leaving the Coulomb center an allowance for the centrifugal term reduces to the appearance in Eq. (19) of an additional phase $\nu_m^0 = 2m + 1$.

The existence of a finite limit to Eq. (19) when $q'', q' \to 0$ does not mean that this semiclassical expression is valid in this range also. The correct form of the Green's function can be found by matching the semiclassical expression to the exact Coulomb solution near the center.

We omit details and give only the final result:

$$G^{osc}(x'', x'; E) = \sum_{m=-\infty}^{+\infty} G_m^{osc}(q'', q'; E)\exp[im(\varphi' - \varphi'')],$$

$$G_m^{osc}(q'', q'; E) = \sum{}' (|p_i^{(\rho)}||p_f^{(\rho)}|)^{1/2}[\Psi_{\mathbf{p}_i}^{(m)}(q')]^*[\Psi_{\mathbf{p}_f}^{(m)}(q'')]e^{-i\pi}G_0,$$

$$G_0 = \frac{\beta^{1/6}}{i(2\pi i)^{1/2}}\Delta_0^{1/2}\exp\left(i\frac{S_0}{\beta^{1/3}} - i\frac{\pi}{2}\nu_0\right), \tag{25}$$

where—as before—summation is carried out over all classical paths emerging from and returning to the Coulomb center; $\mathbf{p}_i$ and $\mathbf{p}_f$ are the initial and final momenta on a path; $p_i^{(\rho)}$ and $p_f^{(\rho)}$ are the projections of these momenta along the axis ρ; $\Psi_p^{(m)}(q)$ is the mth term in the expansion of the function $\Psi_p^{(-)}(r)$ as a Fourier series in terms of the azimuthal angle; $\Psi_p^{(-)}(r)$ is the standard solution of the Schrödinger equation in a Coulomb field which at large distances represents a plane wave with a single coefficient in combination with a converging spherical wave (see, for example, Ref. 7).

Equation (25) is the basis of the semiclassical calculation of the photoabsorption cross section of an atom in a magnetic field.

The problem of a hydrogen atom in a static electric field reduces to the solution of the Schrödinger equation with the following Hamiltonian[7] (in atomic units):

$$\hat{H} = \frac{1}{2}\hat{p}^2 - \frac{1}{r} + \gamma z, \tag{26}$$

where the electric field is assumed to be directed along the z axis; $\gamma = F/(5.1\times 10^9\ \text{V/cm})$ is the electric field intensity in atomic units.

As in the case of an atom in a magnetic field, we consider transitions to highly excited states for which the Coulomb energy is comparable with the potential energy in an external field. In the case of these states the energy is close to the ionization threshold: $E = \gamma^{1/2}\varepsilon$, where ε is a finite quantity.

In principle, this problem is simpler than that of an atom in a magnetic field, since parabolic coordinates make it possible to separate the variables and reduce the equation expressed in terms of partial derivatives to a coupled system of ordinary differential equations.[7] This makes it much easier to carry out numerical calculations,[5] but does not identify the nature of oscillations of the photoabsorption cross section above the ionization threshold, so that we can still proceed as in the case of a magnetic field.

The photoabsorption cross section in an electric field is described by Eqs. (7)–(10), where $G(x'',x';E)$ is the Green's function in an electric field corresponding to the Hamiltonian of Eq. (26). In the semiclassical approximation the function $G(x'',x';E)$ is given by Eqs. (12) amd (13). We can show that the only classical path that contributes to Eq. (13) in the case when $E > 0$ is that emerging from the Coulomb center along the z axis against the field and reflected back to the center. In the case of an atom in a magnetic field we reduce the problem to the two-dimensional form by utilizing conservation of the azimuthal quantum number and we ignore the centrifugal term, as in Eq. (16). In the case of the investigated path we have $\rho = 0$ and this cannot be done, so that in order to find Δ of Eq. (14) we have to use

the three-dimensional semiclassical approximation. It follows from Eq. (14) that if the path follows the z axis and the motion along the x and y axes is independent, then

$$\Delta=\frac{1}{|\dot{q}''||\dot{q}'|}\left[\frac{\partial^2 S}{\partial x''\,\partial x'}\right]\left[\frac{\partial^2 S}{\partial y''\,\partial y'}\right]. \tag{27}$$

Each of the second derivatives can be expressed, as before, in terms of the corresponding element of the monodromy matrix, as in Eq. (15).

In describing the motion along and close to the z axis it is convenient to adopt coordinates similar to the parabolic coordinates of Eqs. (20) and (21). We assume that

$$z=\tfrac{1}{2}\mu^2, \quad x=\mu\nu_1, \quad y=\mu\nu_2. \tag{28}$$

It follows from Eq. (26) that μ, ν_1, and ν_2 are described by

$$\tfrac{1}{2}\mu''=2+\varepsilon\mu^2-\mu^4/2, \tag{29}$$

$$\nu_1''=2\varepsilon\nu_1, \quad \nu_2''=2\varepsilon\nu_2, \tag{30}$$

where a prime denotes the derivative with respect to the available τ, which is linked to t by $dt/d\tau=\mu^2$.

Further calculations are analogous to the case of a magnetic field. The final expression for the contribution to the Green's function by a path emerging and returning along the z axis is

$$G^{osc}(x'',x';E)=\sum k^{1/2}(\Psi_k^{(-)}(x'))^{*}k^{1/2}(\Psi_k^{(-)}(x''))\,4e^{-i\pi/2}G_0, \tag{31}$$

$$G_0=\frac{\gamma^{1/4}}{i(2\pi i)4m_{12}}\exp\left(i\frac{S}{\gamma^{1/4}}-i\frac{\pi}{2}\nu\right),$$

where summation is carried out over all multiple passages along a given path; $S(\varepsilon)$ is the action corresponding to Eq. (29); $m_{12}=\sinh(\lambda T)/\lambda$ represents an element of the monodromy matrix corresponding to Eq. (30); $\lambda=(2\varepsilon)^{1/2}$; $T(\varepsilon)$ is the period of the path described by Eq. (29); the vector $\mathbf{k}$ is directed along the z axis; $\Psi_k^{(-)}(x')$ is the same standard Coulomb function as in Eq. (25).

4. SEMICLASSICAL EXPRESSIONS FOR THE PHOTOABSORPTION CROSS SECTIONS IN EXTERNAL FIELDS

The final expressions for the photoabsorption cross section can be written down by introducing quantities A_p^m defined by the overlap integral of the initial wave function $\Psi_i(x)$ and the function $\Psi_p^{(m)}(x)$:

$$A_p^{(m)}=(|p|)^{1/2}\int d^3x e^{im\varphi}[\Psi_p^{(m)}(x)]^{*}(1\nabla\Psi_i(x)), \tag{32}$$

where φ is the azimuthal angle of the vector $\mathbf{r}$. It should be noted that if $|\mathbf{p}|\to 0$, the value $A_p^{(m)}$ tends to a finite limit.[7]

Substituting Eqs. (25) and (31) into Eq. (10), we obtain an expression for the photoabsorption cross section of an atom in an external field near the ionization threshold in the form of semiclassical sums of Eqs. (4) and (5), where the amplitudes $\sigma_p^{(M)}$ and $\sigma_p^{(E)}$ governing the contribution of each classical orbit, beginning and ending at the Coulomb center, are as follows:

a) in the case of a magnetic field, we have

$$\sigma_p^{(M)}=\frac{(2\pi)^{1/2}}{\omega}\alpha(\sin\theta_1\sin\theta_2)^{1/2}\left(\sum_m A_{p_1}^{(m)*}A_{p_2}^{(m)}\right)$$

$$\times\frac{1}{|m_{12}|^{1/2}}\exp\left(-i\frac{\pi}{2}\nu_m+i\frac{\pi}{4}\right), \tag{33}$$

b) in the case of an electric field, we obtain

$$\sigma_p^{(E)}=\frac{2}{\omega}\alpha\left(\sum_m A_{p_0}^{(m)*}A_{p_0}^{(m)}\right)\frac{1}{|m_{12}|}\exp\left(-i\frac{\pi}{2}\nu\right), \tag{34}$$

where θ_1 and θ_2 are the angles between a classical orbit and the z axis. The phases ν_m and ν are described by

$$\nu_m=\nu_0+\nu_1+(2m+1)\nu_2+\nu_3+\nu_4, \quad \nu=\nu_0+\nu_1+\tfrac{1}{2}\nu_3, \tag{35}$$

where ν_0 is the number of conjugate points (at which we have $m_{12}=0$) on a path; ν_1 is the number of reflections from the boundary of the allowed region (number of points at which the velocity modulus vanishes); ν_2 is the number of intersections of the orbits along the $\rho=0$ ($z\neq 0$) axis; ν_3 is an even number equal to the number of times that an orbit enters and leaves the Coulomb center:

$\nu_4=2T_0m/\pi$, where $T_0=(dS/d\varepsilon)_{\varepsilon=0}$.

This term appears because of the shift of the energy in Eq. (1) by an amount equal to the Zeeman splitting.

The quantity $\mathbf{p}_0$ in Eq. (34) is the momentum directed along the field. In the above expressions an element of the monodromy matrix of Eq. (14) calculated using the parabolic coordinates is denoted by m_{12}. The reaction S in Eqs. (4) and (5) is along a given orbit. Near the ionization threshold the values of $A_p^{(m)}$ can be regarded as constant, whereas m_{12} and S are functions of the reduced energy ε, which is $E/\beta^{2/3}$ for a magnetic field and $E/\gamma^{1/2}$ for an electric field.

As pointed out above, the contribution of short classical orbits is equal to the purely Coulomb contribution in the absence of an external field. We can easily show that

$$\sigma_{\text{Coul}}=\frac{\alpha}{2\pi\omega}\sum_m |A_p^{(m)}|^2\,do_p, \tag{36}$$

where $A_p^{(m)}$ is defined in Eq. (32) and do_p is an element of a solid angle with the vector $\mathbf{p}$.

In view of conservation of the azimuthal quantum number the sums over m in Eqs. (33), (34), and (36) contain a finite number of terms governed by the initial wave function. Summation in these semiclassical expressions is carried out over all the classical orbits beginning and ending at the Coulomb center. In view of the specific nature of the Coulomb potential, any orbit entering the Coulomb center is reflected back with the same momentum (which follows directly if we use the parabolic coordinates). Therefore, any path which begins and ends at the Coulomb center must be periodic and must pass through the Coulomb center. The semiclassical sums include also contributions due to multiple passes along the same periodic orbit. Let us assume that $S^{(n)}$, $\nu^{(n)}$, and $\Delta^{(n)}$ are quantities corresponding to the nth passage along one path. In the case of two degrees of freedom, we have

$$S^{(n)}=nS, \quad \nu^{(n)}=n\nu, \quad \Delta^{(n)}=\Delta\frac{\lambda_1-\lambda_2}{\lambda_1^n-\lambda_2^n}, \tag{37}$$

where S, ν, and Δ are quantities calculated for one trip along a given path; λ_1 and $\lambda_2 = 1/\lambda_1$ are eigenvalues of the monodromy matrix of Eq. (14).

Depending on the properties of the eigenvalues of the monodromy matrix, we can divide periodic orbits into three groups:

a) neutral, if $\lambda_1 = \pm 1, \lambda_2 = \pm 1$;

b) stable (elliptic) if $\lambda_1 = e^{i\varphi}, \lambda_2 = e^{-i\varphi}$;

c) unstable (hyperbolic) if $\lambda_1 = \pm e^{u}, \lambda_2 = \pm e^{-u}$.

Studies of the problem of an atom in a magnetic field by numerical methods have demonstrated (see, for example, Ref. 10) that if $E = 0$ then all (at least those found so far) periodic orbits are unstable (as in the case of ergodic systems) and we shall confine our attention to this case.

Before considering specific applications of these expressions, we need to determine how many periodic orbits must be included in Eq. (4). Formally, the summation in Eq. (4) is over all classical periodic orbits passing through a Coulomb center. The time of motion along these paths can be as long as we please and exact allowance for all the paths in the case of nonintegrable systems is a difficult task (see, for example, Ref. 22), which is equivalent in its complexity to the direct solution of the Schrödinger equation.

In the case of chaotic systems it is physically reasonable to calculate not the exact values of any quantities for a fixed energy but quantities which are averaged over the energy (or smoothed out).[18] Formally, this means that instead of the value of the cross section $\sigma(E)$, we can calculate the quantity $\overline{\sigma(E_0)}$, defined as follows:

$$\overline{\sigma(E_0)} = \int f(E-E_0, \Delta E)\,\sigma(E)\,dE, \tag{38}$$

where the smoothing-out function $f(E - E_0, \Delta E)$ has a maximum at $x = 0$, falls rapidly in the range $|x| > \Delta E$, and integrates to $\int f(e, \Delta E)de = 1$.

For example, the function $f(E - E_0, \Delta E)$ can be in the form of the Gaussian function

$$f(E-E_0, \Delta E) = \frac{1}{(2\pi)^{1/2}\Delta E}\exp\left[-\frac{(E-E_0)^2}{2(\Delta E)^2}\right] \tag{39}$$

or the piecewise-constant function

$$f(E-E_0, \Delta E) = \begin{cases} 1/\Delta E & \text{for } |E-E_0| < {}^1/_2\Delta E, \\ 0 & \text{for } |E-E_0| > {}^1/_2\Delta E. \end{cases} \tag{40}$$

We average the semiclassical expression (25) in accordance with Eq. (38). We can readily show that in the limit $\beta \to 0$ and for a fixed function $f(x, \Delta E)$, the main contribution to the sum comes from those paths which obey the inequality

$$T < \frac{\text{const}}{\Delta E}, \tag{41}$$

where $T = dS/dt$ is the time of motion along the path. For example, if we assume that the action is a linear function of the energy, $S(E) = S(E_0) + T(E_0)(E - E_0)$, and if we average $\exp(iS)$ using the function (39), we obtain

$$\langle \exp(iS) \rangle = \exp[iS(E_0)]\exp\left[-\frac{(T\Delta E)^2}{2}\right]. \tag{42}$$

The inequality (41) means that, after averaging, the main contribution to the semiclassical sum over the paths comes from a finite number of orbits for which the time of motion is limited.

5. PHOTOABSORPTION IN AN ELECTRIC FIELD

In this section we give specific expressions for the process of absorption of a photon by a hydrogen atom in an electric field under conditions close to the ionization threshold and we compare the results with Ref. 5.

The general expression for this process is given by Eqs. (5) and (34). As pointed out already, the sum in Eq. (5) contains only one periodic orbit described by Eq. (29) and its multiple replicas. The action $S(\varepsilon)$ and the period $T(\varepsilon)$ occurring in Eq. (31) describing $m_{12}(\varepsilon)$ follow directly from Eq. (29):

$$S(\varepsilon) = \oint (4 + 2\varepsilon\mu^2 - \mu^4)^{1/2}\,d\mu,$$
$$T(\varepsilon) = \oint (4 + 2\varepsilon\mu^2 - \mu^4)^{-1/2}\,d\mu. \tag{43}$$

The above quantities are readily expressed in terms of complete elliptic integrals of the first and second kind $K(m)$ and $E(m)$ (Ref. 23). However, for low values of m it is convenient to use expansions as a series in ε and the coefficients of this expansion are readily expressed in terms of the β function[23] directly from Eq. (43):

$$S(\varepsilon) = 4{,}9442 + 1{,}6944\varepsilon + 0{,}4635\varepsilon^2 + \ldots,$$
$$T(\varepsilon) = 1{,}8541 + 0{,}2118\varepsilon - 0{,}0579\varepsilon^2 + \ldots. \tag{44}$$

In deriving the explicit expressions it is necessary to calculate the quantity $A_{\mathbf{p}}^{(m)}$ of Eq. (34), which can be expressed in terms of the initial wave function (32). When the initial states have small quantum numbers we can assume that their wave functions are governed by the pure Coulomb-type Schrödinger equation in the absence of an external field.

We confine ourselves to calculating the photoabsorption cross section in an electric field for initial states specified by the following parabolic quantum numbers[7]:

a) $n_1 = 1, \quad n_2 = 0, \quad m = 0,$ (45a)

b) $n_1 = 0, \quad n_2 = 1, \quad m = 0.$ (45b)

These states are the initial ones in the experiments described in Ref. 5. The wave functions can be selected to be[7]

$$\Psi_\sigma = 2^{-1/2}[R_{20}(r)\,Y_{00}(\theta) - \sigma R_{21}(r)\,Y_{10}(\theta)], \tag{46}$$

where $\sigma = +1$ corresponds to the function (45a), while $\sigma = -1$ corresponds to the function (45b); $R_{nl}(r)$ and $Y_{lm}(\theta, \varphi)$ are the standard radial and angular parts of the Coulomb wave functions (see, for example, Ref. 7). In the case of the initial states considered here the photoabsorption cross section found in the semiclassical approximation differs from zero only if the photon polarization is parallel to the field (i.e., for the π transition with $\Delta m = 0$). In this case the nonzero terms are those with $m = 0$ and we denote them by A_σ:

$$A_\sigma = p^{1/2}\int \Psi_{\mathbf{p}}^{(0)}(x)\,\frac{\partial}{\partial z}\,\Psi_\sigma(x)\,d^3x. \tag{47}$$

These integrals are readily calculated using an expansion of

the function $\Psi_k^{(-)}(x)$ as a series in spherical functions.[7] Bearing in mind that in our case we have $|p| \propto \beta^{1/3}\varepsilon^{1/2} \to 0$ in the limit $\beta \to 0$, it is sufficient to calculate Eq. (47) for $p = 0$. Standard procedures (see, for example, Ref. 7) yield

$$A_\sigma(\theta') = \pi^{1/2} 2^7 e^{-4} \left(\frac{1}{6^{1/2}} Y_{10}(\theta') + \frac{\sigma}{6 \cdot 2^{1/2}} \left[Y_{00}(\theta') + \frac{8}{5^{1/2}} Y_{20}(\theta') \right] \right), \tag{48}$$

where θ' is the angle between the vector **p** and the field direction.

The semiclassical expression of Eq. (34) includes $A_\sigma(\theta')$ at $\theta'' = 0$:

$$A_\sigma(0) = \pi 2^{9/2} e^{-4} \begin{cases} 5 & \text{for } \sigma = 1, \\ -1 & \text{for } \sigma = -1. \end{cases} \tag{49}$$

Hence, it follows that the contribution of the investigated periodic orbit to the photoabsorption cross section for a transition from an initial state described by Eq. (45a) is 25 times greater than for a transition from the state described by Eq. (45b).

The cross section of the Coulomb photoabsorption of Eq. (36) is the same for the states of Eqs. (45a) and (45b) and is easily obtained from Eq. (48) (bearing in mind that $\omega \approx 1/8$):

$$\sigma_{\text{Coul}} = \frac{43 \cdot 2^{13} \pi^2 e^{-8}}{15} \alpha = 0{,}567. \tag{50}$$

We finally have

$$\sigma(E) = \sigma_{\text{Coul}} \left(1 + g\gamma^{1/4} \sum_{n=1}^{\infty} \frac{\lambda}{\operatorname{sh}(\lambda T n)} \sin\left(\frac{S}{\gamma^{1/4}} n - \pi n \right) \right), \tag{51}$$

where $g = 3 \times 5^3/43$ for the initial state of Eq. (45a) and $g = 15/43$ for the initial state of Eq. (45b); $\lambda(\varepsilon) = (2\varepsilon)^{1/2}$; the functions $S(\varepsilon)$ and $T(\varepsilon)$ are defined by Eqs. (44) and (45); $\varepsilon = E/\gamma^{1/2}$.

The sum over n converges rapidly for all values of $\varepsilon > 0$ apart from the range of very small ε, when we have $\lambda(\varepsilon) \to 0$. However, in this case the summation can be carried out analytically using the expression

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2} \quad \text{for } 0 < x < 2\pi. \tag{52}$$

We then have

$$\frac{\sigma(E) - \sigma_{\text{Coul}}}{\sigma_{\text{Coul}}} \to \gamma^{1/4} \frac{g}{T_0} \pi \left(\frac{1}{2} - \delta \right) \quad \text{for } E \to 0, \tag{53}$$

where δ is the fractional part of the number $(S_0/2\pi\gamma^{1/4} - 1/2)$; S_0 and T_0 are the first terms of the expansion in Eq. (44).

Figure 1 shows a plot of $\sigma(E)$ given by Eq. (51) for the case of an electric field of intensity 5714 V/cm, which was the value used in the experiments reported in Ref. 5. This plot is practically identical with the dependence $\sigma(E)$ obtained by direct numerical solution of the Schrödinger equation in an electric field,[5] which in turn agrees well with the experimental data.[5]

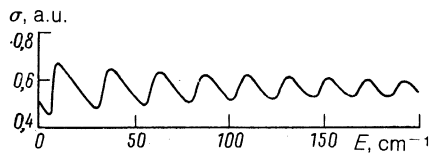

FIG. 1. Photoabsorption cross section in an electric field of intensity 5714 V/cm for the $\Delta m = 0$ transition from a state with the parabolic quantum numbers $n_1 = 1$, $n_2 = 0$, and $m = 0$ to highly excited states with energies above the ionization threshold.

6. PHOTOABSORPTION IN A MAGNETIC FIELD

An atom in a magnetic field is characterized by an infinite number of unstable periodic orbits which pass through a Coulomb center. Examples of such orbits are given in Refs. 3, 4, and 9. In this case it is not possible to obtain a closed expression of the type given by Eq. (51). As pointed out in Sec. 3, if we average the cross section in a narrow interval ΔE (which is equivalent to measurement of the cross section with a finite resolution), we find that the main contribution to the sum of Eq. (4) comes from a finite number of orbits for which the time of motion is emitted by the inequality of Eq. (41). The necessary parameters of such orbits are usually found numerically. Figure 2 shows the three simplest periodic orbits predicted for the case when $\varepsilon = 0$ and characterized by the shortest periods. Table I lists for these orbits the following quantities which occur in the semiclassical expression (33):

a) the classical action calculated along an orbit;

b) an element m_{12} of the monodromy matrix of Eq. (14) expressed in parabolic coordinates;

c) the square of the logarithm of the modulus of the largest eigenvalue of the monodromy matrix;

d) the angle of inclination of the orbit relative to the z axis when it emerges from the Coulomb center (for the paths under investigation the entry and return angles are the same).

All the quantities are presented in Table I in the form of the first three coefficients of a series expansion in terms of the reduced energy: $F = A + B\varepsilon + C\varepsilon^2$. In the case of the simplest path (No. 1) some quantities can be determined analytically. The action $S(\varepsilon)$ is readily obtained from Eq. (22) if we assume that $\mu = \nu$:

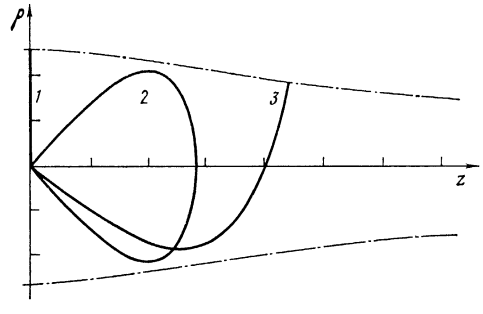

FIG. 2. Simplest periodic orbits for an atom in a magnetic field at an energy equal to zero. The chain curves represent the allowed limits of the range; the step along the axes is 0.5; the numbers alongside the curves are the same as the serial numbers of paths in Table I.

TABLE I. Energy dependences of the parameters of periodic orbits.

Parameter	A	B	C
		Orbit No. 1	
S	5,7229	2,0944	0,31365
m_{12}	2,82162	3,62	1,2
λ^2	1,7344	8,6	−0,2
θ	$\pi/2$	0	0
		Orbit No. 2	
S	8,580	4,9	0,3
m_{12}	11,01	15,3	−0,9
λ^2	8,39	13,8	−4,9
θ	0,940	0,59	−0,5
		Orbit No. 3	
S	10,21	8,11	0,26
m_{12}	24,44	47,1	−16,8
λ^2	14,07	21,4	−18,8
θ	0.75	0,76	−0,8

$$S=2\oint (2+2\varepsilon\mu^2-\mu^6)^{1/2}\,d\mu. \tag{54}$$

At low values of ε this expression can be expanded as a series in ε:

$$S=5{,}7829+2{,}0944\varepsilon+0{,}31365\varepsilon^2+\ldots. \tag{55}$$

Moreover, if $\varepsilon = 0$, we can find analytically the monodromy matrix for this path:

$$m_{12}(0)=\frac{\pi}{3\cdot 2^{1/2}}\left[\frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)}\right]^2\approx 2{,}8162,$$
$$\lambda^2(0)=(\ln(2+3^{1/2}))^2\approx 1{,}734378. \tag{56}$$

The remaining quantities in Table I were determined numerically by solving the classical equations corresponding to the Hamiltonian (22).

In determination of the cross sections we need to know not only the parameters of periodic orbits, but also the quantities $A_p^{(m)}$ of Eq. (32), dependent on the wave function of the initial state. In the dipole approximation we can expect transitions without a change in the azimuthal quantum number: $\Delta m = 0$ (π transitions) and transitions involving a change in this number by unity: $|\Delta m| = 1$ (σ transitions). We consider only the next three transitions from the initial $2p$ state for which the experimental data of Ref. 3 suggest that

1) $m_i=0 \xrightarrow{\pi} m_f=0$,

2) $m_i=1 \xrightarrow{\pi} m_f=1$, (57)

3) $m_i=1 \xrightarrow{\sigma} m_f=0+m_f=2$,

where m_i and m_f are the azimuthal numbers of the initial and final states, respectively. The conditions are similar for the other transitions.

The transitions of Eq. (57) correspond to the following integrals

$$A_1^{(m)}=k^{1/2}\int \Psi_k^{(m)}\frac{\partial}{\partial z}\Psi_1\,d^3x,$$
$$A_2^{(m)}=k^{1/2}\int \Psi_k^{(m)}\frac{\partial}{\partial z}\Psi_2\,d^3x, \tag{58}$$
$$A_3^{(m)}=k^{1/2}\int \Psi_k^{(m)}\frac{\partial}{\partial x}\Psi_2\,d^3x,$$

where $\Psi_1 = R_{21}(r)Y_{10}(\theta,\varphi)$ and $\Psi_2 = R_{21}(r)Y_{11}(\theta,\varphi)$ are the wave functions of the $2p$ states with $m = 0$ and $m = 1$, respectively. Such calculations give

$$A_1^{(0)}=-\frac{\pi^{3/2}2^8e^{-4}}{3}\left(Y_{00}+\frac{8}{5^{1/2}}Y_{20}\right),\quad A_2^{(0)}=\frac{\pi^{3/2}2^8e^{-4}}{15^{1/2}}Y_{21},$$
$$A_3^{(0)}=-\frac{\pi^{3/2}2^{11/2}e^{-4}}{3}\left(Y_{00}-\frac{4}{5^{1/2}}Y_{20}\right),\quad A_3^{(2)}=-\frac{\pi^{3/2}2^8e^{-4}}{15^{1/2}}Y_{22}. \tag{59}$$

Here, Y_{lm} depend on the angle between the vector k and the z axis, and the factor $e^{im\varphi}$ is dropped from Y_{lm}.

The Coulomb cross section follows from Eq. (36):

$$\sigma_{\text{Coul}}^{(1)}=\frac{69\cdot 2^{14}\pi^2e^{-8}}{45}\alpha\approx 0{,}607;\quad \sigma_{\text{Coul}}^{(2)}=\frac{2^{18}\pi^2e^{-8}}{15}\alpha\approx 0{,}422;$$
$$\sigma_{\text{Coul}}^{(3)}=\frac{13\cdot 2^{13}\pi^2e^{-8}}{5}\alpha\approx 0{,}515. \tag{60}$$

The oscillatory part of the semiclassical expression for the cross section is obtained from Eq. (33). We finally find that

$$\sigma^{(k)}=\sigma_{\text{Coul}}^{(k)}\left(1+\beta^{1/6}\sum_{\text{orb}}\sum_m g_m^{(k)}(\theta_1,\theta_2)\frac{1}{|m_\omega|^{1/2}}\right.$$
$$\left.\times\sum_{n=1}\left(\frac{\operatorname{sh}\lambda}{\operatorname{sh}\lambda n}\right)^{1/2}\sin\left[\frac{Sn}{\beta^{1/2}}-mT_0^{(k)}n-\frac{\pi}{2}\nu_m^{(k)}n+\frac{\pi}{4}\right]\right), \tag{61}$$

where the values $k = 1$, 2, and 3 correspond to the following transitions of Eq. (57):

$$g_0^{(1)}=\frac{45\pi^{1/2}}{69\cdot 2^{1/2}}(4\cos^2\theta_1-1)(4\cos^2\theta_2-1)(\sin\theta_1\sin\theta_2)^{1/2},$$

$$g_1^{(2)}=\frac{15\pi^{1/2}}{2\cdot 2^{1/2}}(\sin\theta_1\sin\theta_2)^{1/2}\cos\theta_1\cos\theta_2,$$

$$g_0^{(3)}=\frac{5\pi^{1/2}}{13\cdot 2^{1/2}}(1-2\cos^2\theta_1)(1-2\cos^2\theta_2)(\sin\theta_1\sin\theta_2)^{1/2},$$

$$g_2^{(3)}=\frac{5\pi^{1/2}}{13\cdot 2^{1/2}}4\sin^2\theta_1\sin^2\theta_2(\sin\theta_1\sin\theta_2)^{1/2},$$

$$\nu^{(1)}=3,\quad \nu^{(2)}=5,\quad \nu^{(3)}=7.$$

The summation in Eq. (61) is carried out over all different orbits emerging from and returning to the Coulomb center; θ_1 and θ_2 are the angles of emergence and return of a given orbit; $T_0^{(k)}=(dS/d\varepsilon)_{\varepsilon=0}$ is the second coefficient of the expansion of the action along a orbit k as a series in terms of the energy (Table I). Since for these transitions the value of m_f is even, the phases $\nu_m^{(k)}$ are independent of m, in accordance with Eq. (35).

The photoabsorption cross sections of a hydrogen atom in a magnetic field of 6 T intensity are plotted in Fig. 3 for all the investigated transitions allowing for three periodic orbits with the parameters listed in Table I (paths Nos. 2 and 3 exist for $z>0$ and $z<0$). For comparison with the results of Ref. 3, we carry out an approximate averaging of the cross section using Eq. (39) and $\Delta E=0.85\ \text{cm}^{-1}$, which can be done by multiplying the contribution of such periodic orbits, in accordance with Eq. (42), by a factor $f_k=\exp[-2(T_0^{(k)}\Delta E)^2/\beta]$. The numerical results are

$$f_1=0{,}82,\quad f_2=0{,}33,\quad f_3=0{,}05. \tag{62}$$

Figure 3 agrees well with the average experimental data of Ref. 3. If necessary, it is easy to find the parameters of a large number of periodic orbits and to calculate the photoabsorption cross section with a high degree of resolution.

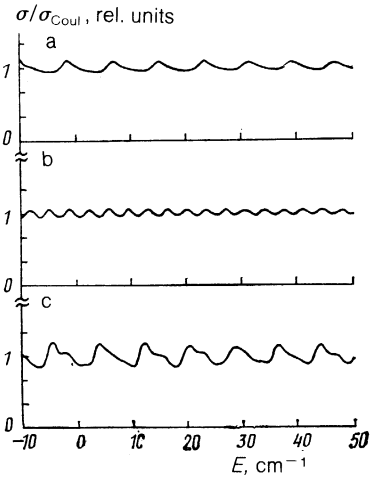

FIG. 3. Photoabsorption cross section in a magnetic field of intensity 6 T calculated allowing for three periodic orbits for the following transitions: a) $m_i=0\to m_f=0$; b) $m_i=1\to m_f=1$; c) $m_i=1\to m_f=0+m_f=2$. The cross section is averaged using the Gaussian function for which the half-width is 2 cm^{-1}.

It would be interesting to determine experimentally the differential photoabsorption cross section in an external field showing the individual periodic orbits.

7. CONCLUSIONS

All these calculations demonstrate that the method of semiclassical allowance for unstable periodic orbits employed in the present study provides a satisfactory description of the oscillations observed in the process of absorption of photons by atoms in external fields near the ionization threshold. The method is quite general and it can be applied to a wide range of problems (both integrable and nonintegrable) when unstable periodic orbits have a high value of the action along the orbit.

The author is deeply grateful to E.M. Solov'ev for numerous valuable discussions.

[1)] Moreover, the motion at zero energy looks chaotic and regions of regular motion are either completely absent or negligible.[16]

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