Dynamic quantization in the chiral Schwinger model

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This is a direct continuation of work previously reported in S. N. Vergeles, Zh. Eksp. Teor. Fiz. **95**, 397 (1989) [Sov. Phys. JETP **68**, 225 (1989)]. The method of dynamic quantization, proposed there, is now developed further and is used as a basis for the quantization of the chiral Schwinger model. In dynamic quantization, regularization is accomplished by imposing an infinite set of second-order constraints on the degrees of freedom in the deep ultraviolet region and by going over to the Dirac commutation relations. This leads to the conventional interpretation of gauge theories, which are regarded as anomalous. The chiral Schwinger model is found to be gauge and Lorentz invariant.

1. INTRODUCTION

In this paper, we shall apply the method of dynamic quantization[1] to the (right) chiral Schwinger model. We shall show that, in contrast to Feynman quantization, dynamic quantization is completely correct in this case, i.e., both gauge and Lorentz invariance are preserved. As a result, we obtain a spatially odd theory of free neutral Bose particles with the spectrum $\omega(k) = k$ where $k > 0$ [in the left theory $\omega(k) = -k$ where $k < 0$].

We present the essence of the method of dynamic quantization. We consider the asymptotically free gauge theory in Minkowski space. Apart from the gauge field, the theory can include the chiral fermion (Weyl) field. The set of boson and fermion fields will be denoted by $\Phi(x)$.

It is well known that, in the case of Feynman quantization, such theories are anomalous and, in the usual sense, contradictory. The program of Faddeev and Shatshvili[2] for the quantization of anomalous theories (as it applies to the Feynman method) lies outside the framework of the conventional interpretation of gauge theories. This approach is very interesting to us, but we shall not examine it here. Readers interested in the quantization of the chiral Schwinger model by the Faddeev-Shatashvili method should refer to Refs. 3–8.

Let $\Phi(x)$ represent an arbitrary field and $\varphi(x) = \delta\Phi(x)$ a small increment of the field. We shall assume that $\Phi(x)$ is a classical field and that $\Phi(x)$ is a quantum-mechanical operator. Let us consider the Heisenberg equations for the field φ. These equations can be obtained with the aid of the Hamiltonian H which explicitly contains the background field Φ. Let $H^{(2)}$ be the part of the Hamiltonian H that is quadratic in φ. The Hamiltonian $H^{(2)}$ can be diagonalized at all times t. Let $\{\varphi_N(t,x)\}$ be the set of modes that diagonalize the Hamiltonian $H^{(2)}$ at time t, and let $\{\omega_N\{\Phi\}\}$ be the set of corresponding eigenfrequencies. The frequencies $\omega_N\{\Phi\}$ and modes $\varphi_N(t,\mathbf{x})$ are functional of the field $\Phi(t,\mathbf{x})$. Since the set of functions $\{\varphi_N(t,\mathbf{x})\}$ is complete to all times t, the field $\varphi(x)$ can be expanded as follows:

$$\varphi(x) = \sum_N a_N(t)\,\varphi_N(x), \qquad (x) = (t,\mathbf{x}), \tag{1}$$

where the coefficients $a_N(t)$ are functions of time only. The set of quantities $\{a_N(t)\}$ may be regarded as a set of dynamic coordinate variables. Let π_N be the momentum variable that is the canonical conjugate of a_N, and let us consider the dynamics of the system relative to the moving frame $\{\varphi_N(t,\mathbf{x})\}$. To do this, we consider $\{\pi_N, a_N\}$ as a set of local dynamic variables and express all operators in terms of them.

In the space of the parameters N (which contains momentum space as a direct cofactor), we draw a closed surface $\sigma(\Lambda)$ in the deep ultraviolet region that contains the infrared region and is specified by the following condition. The index N belongs to the surface $\sigma(\Lambda)$ if for the field $\Phi = 0$ we have $(\omega_N^2\{0\} = \Lambda^2 \to \infty)$. We say that the mode $\varphi_N(x)$ and the corresponding degree of freedom (π_N, a_N) lie in the interior of the surface $\sigma(\Lambda)$ if $\omega_N^2\{0\} < \Lambda^2$, and that it refers to the ultraviolet tail if $\omega_N^2\{0\} > \Lambda^2$.

To regularize the theory, we assume that there are no degrees of freedom referring to the ultraviolet tail, and that the bare constant $e^2(\Lambda)$ tends to zero. This means that we are applying to the system an infinite set of second-order constraints in the terminology of Dirac's theory[9]: the degrees of freedom referring to the ultraviolet tail are all assumed to be equal to zero. We assert that this regularization is self-consistent and, without proving this proposition, we shall outline the corresponding physical picture.

Consider the dynamics of the degrees of freedom a_N near the surface $\sigma(\Lambda)$. Since the bare constant is $e^2(\Lambda) \ll 1$, the nonlinearity of these degrees of freedom in the Heisenberg equations is naturally taken into account by perturbation theory (PT).

Purely kinematic considerations show that the effective coupling constant between the degrees of freedom near the surface $\sigma(\Lambda)$ is not renormalized and remains of order $e^2(\Lambda) \ll 1$. The reason for this is that, in this case, the ranges of integration over the intermediate frequencies become greatly reduced, and do not give the usual logarithmic divergences, whatever the duration of the interaction in time.

Let $\Lambda_1 < \Lambda$. We can show in the usual way, using PT, that if we take into account fluctuations in the degrees of freedom between the surfaces $\sigma(\Lambda_1)$ and $\sigma(\Lambda)$, this lead to the renormalization of the constant $e^2(\Lambda) \to e^2(\Lambda_1)$. For the coupling between the degrees of freedom near the surface $\sigma(\Lambda_1)$, we must now use the constant $e^2(\Lambda_1)$ if the integrals over the intermediate frequencies are cut off in PT calculations on the surface $\sigma(\Lambda_1)$.

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It follows that, as we enter the infrared region, the effective coupling constant between the degrees of freedom rises in accordance with the formulas that are well known in asymptotically free theories. This means that the coupling will cease to be weak for some scale $\Lambda_1 \ll \Lambda$. The region adjacent to the boundary of $\sigma(\Lambda)$, in which the couping remains weak, will be referred to as the ultraviolet region. The low-frequency region in the space of the parameters N, which is not included in the ultraviolet region, will be referred to as the infrared region. The coupling between the degrees of freedom and their fluctuations is strong in the infrared region.

We shall see that fluctuations in the degrees of freedom a_N near the surface $\sigma(\Lambda)$ are small. The effect of fluctuations in the ultraviolet region on the dynamics of these degrees of freedom can be taken into account by PT, and their quantum numbers are then conserved. PT calculations show that the frequencies ω_N^2 and modes $\varphi_N(x)$ near the surface $\sigma(\Lambda)$ depend only on the infrared component of the field $\Phi^i(x)$, which is indicated by the notation $\varphi_N\{\Phi^I\}(x)$ [variations in $\Phi^i(x)$ are proportional to the modes $\varphi_N\{\Phi^i\}(x)$ from the infrared region]. Fluctuations in the field $\Phi^i(x)$ have large amplitudes, but they vary slowly in space-time. Their effect on the dynamics of the ultraviolet degress of freedom a_N is therefore correctly taken into account in the adiabatic approximation. All this leads to the following important result: the occupation numbers n_N for modes $\varphi_N\{\Phi^i\}(x)$ in the deep ultraviolet region are conserved, i.e., they are constants of motion. Of course, this proposition is valid only for states near the vacuum state for which all $n_N = 0$, or only some of them, are different from zero. We shall assume that this restriction is satisfied in the ensuing analysis.

In the infinitely-dimensional affine space of the fields $\Phi(x)$, we can construct the hypersurface $\Sigma(\Lambda)$ in which the sets of modes $\{\varphi_N\{\Phi^i\}(x)\}'$ is a complete tangential set of vectors at each point. (Here and in what follows the prime will indicate that the index N does not assume values that belong to the ultraviolet tail.) This problem is mathematically correct. The hypersurface $\Phi(x)$ is invariant under the group of gauge transformations of the fields $\Sigma(\Lambda)$. Actually, the vector fields that generate the gauge transformations are the zero modes of $H^{(2)}$ and can therefore be expanded in the set $\{\varphi_N\{\Phi^i\}(x)\}'$.

It follows that the configuration space of the system can be bounded up to the hypersurface $\Sigma(\Lambda)$. The dynamics conserves the system on this hypersurface and the system is regularized thereby. Moreover, the following propositions are valid: (1) for the regularized system, the number of deκgrees of freedom is determined and bounded (per unit of volume) and (2) there is no transport of particles, energy, or other quantum numbers across the surface $\sigma(\Lambda)$ for any Λ in the deep ultraviolet region.

Physical fields are also naturally considered to be regularized in the above regularization procedure. By definition, a regularized field depends only on points on the hypersurface $\Sigma(\Lambda)$. In particular, a regularized field $\varphi(x)$ depends on the variables a_N lying in the interior of the surface $\sigma(\Lambda)$ [see Ref. 1]. The commutation relations (CR) for the regularized fields differ from the original CR because the former do not involve the variables $\{\pi_N, a_N\}$ from the ultraviolet tail. The CR for the regularized fields are none other than the Dirac CR (see Ref. 9 and the appendix) that arise when the second-order constraints $a_N \approx 0$ and $\pi_N \approx 0$ are applied to the degrees of freedom $\{\pi_N, a_N\}$, in the ultraviolet tail. The imposition of these constraints is possible because the variables $\{\pi_N, a_N\}$ commute with all the first-order constraints, i.e., the generators of the gauge transformations. Moreover, the commutators $[a_N, H']$ and $[\pi_N, H']$ are proportional to a_N and π_N, which follows from propositions (1) and (2) above, where H' is the Hamiltonian terms of the variables $\{a_N, \pi_N\}$ (see Sec. 2).

We thus arrive at the following picture. Suppose that all the operators are expressed in terms of the regularized fields in the usual way. We derive the Heisenberg equations of motion in an explicit form, using the Dirac CR. The equations of motion obtained in this way have the usual form on the hypersurface $\Sigma(\Lambda)$. Since the regularized equations of motion are Lorentz invariant, we find that Lorentz invariance applies to physical momenta (in the infrared region).

It follows from the foregoing that, in particular, the gauge anomaly does not arise in chiral gauge theory. Because of the importance of this result, we shall derive it explicitly in the language of the Hamiltonian formalism in 4-dimensional Minkowski space.

Since fermion fluctuations give rise to the gauge anomaly, let us consider fluctuations in the quantum-mechanical Weyl field $\varphi(x)$ in an external gauge field $A_\mu(x)$. The Hamiltonian is

$$H_F = H_F^{(2)} = -\int d^3x\,(\varphi^+ i\tau^i \nabla_i \varphi + i\varphi^+ A_0 \varphi), \tag{2}$$

where τ^i are the Pauli matrices and $i = 1, 2, 3$ and $\nabla_\mu = \partial/\partial x^\mu + A_\mu$. The Heisenberg equations $i\dot{\varphi} = [\varphi, H]$ and the commutation relations

$$[\varphi^+(t, \mathbf{x}), \varphi(t, \mathbf{y})] = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{3}$$

lead to the Weyl equation

$$i\dot{\varphi} = h\varphi, \tag{4}$$

where $h(t) = -i\tau^i\nabla_i - iA_0$ is a hermitian Weyl operator that depends on the gauge field (and, consequently, on time).

Let $\{\varphi_N(t,\mathbf{x})\}$ be the complete orthonormal set of solutions of the Weyl equation. This means that all the functions φ_N satisfy (4) and, at the time t_0, the set $\{\varphi_N(t_0,\mathbf{x})\}$ is a complete orthonormal set of eigenfunctions of the operator $h(t_0)$:

$$h(t_0)\varphi_N(t_0, \mathbf{x}) = \omega_N(t_0)\varphi_N(t_0, \mathbf{x}). \tag{5}$$

Since the operator $h(t)$ is hermitian, it follows from (4) and (5) that

$$\sum_N \varphi_N(t, \mathbf{x})\varphi^+(t, \mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{6}$$

at any time t. Because of the completeness condition (6), any fields $\varphi(x)$ and $\varphi^+(x)$ can be expanded in the sets of functions $\{\varphi_N(x)\}$ and $\{\varphi_N{}^+(x)\}$ with coefficients $\{a_N(t)\}$ and $\{a_N^+(t)\}$, respectively:

$$\varphi(x) = \sum_N a_N\varphi_N(x), \qquad \varphi^+(x) = \sum_N a_N{}^+\varphi_N{}^+(x).$$

The set of Grassmann quantities $\{a_N^+, a_N\}$ can be regarded as a complete set of fermion degrees of freedom of the system. The commutation relations (3) assume the following form in terms of variables $\{a_N^+, a_N\}$:

$$[a_N{}^+, a_N]_+ = \delta_{MN}. \tag{7}$$

The Weyl equations (4) then signify that

$$i\dot{a}_N = [a_N, H_F'] = 0, \quad i\dot{a}_N{}^+ = [a_N{}^+, H_F'] = 0. \tag{8}$$

The following second-order constraints can then be imposed in (8): $a_N \approx 0$, $a_N^+ \approx 0$, if $|\omega_N(t_0)| > \Lambda \to \infty$. This procedure leads to regularization. Instead of the CR given by (3), we must now use the Dirac bracket. There is no need to reproduce the explicit formulas because the entire situation reduces to an obvious redefinition of symbols [see Appendix, equation (A8)]. This leads to the following equations of motion:

$$\begin{aligned} &\dot{a}_N = 0, \ \dot{a}_N{}^+ = 0, \ |\omega_N(t_0)| < \Lambda, \\ &a_N = 0, \ a_N{}^+ = 0, \ |\omega_N(t_0)| > \Lambda, \end{aligned} \tag{9}$$

which show that the Weyl equations given by (4) are valid for the regularized fields

$$\varphi(x) = \sum_N{}' a_N \varphi_N(x), \qquad \varphi^+(x) = \sum_N{}' a_N{}^+ \varphi_N{}^+(x). \tag{10}$$

The prime and the summation symbol indicate that the sums are evaluated over the indices N for which $|\omega_N(t_0)| < \Lambda$. All the operators can now be expressed in terms of the regularized fields (10), and their dynamics can be investigated using the Weyl equations given by (4). Hence, for the regularized fermion current

$$J^{\mu a} = (\varphi^+ t^a \varphi, \ \varphi^+ \tau^i t^a \varphi),$$

we have $\nabla_\mu J^\mu = 0$ where t^a are the generators of the Lie algebra of the gauge group. This equation shows that the gauge invariance of the chiral theory remains invalid.[1)]

We now note the qualitative difference between Feynman and dynamic quantization. Dynamic quantization can be applied to asymptotically free theories for which there is a region in momentum space, which we call the ultraviolet region, in which quantum-mechanical fluctuations in the field are small at all times.

The coupling is thus seen to be weak at infinity in momentum space. On the other hand, at infinity in x-space, the coupling is not assumed to be weak and, in general, it cannot be weak. The imposition of the condition $\Phi(x) \to 0$ for $x \to \infty$ is therefore incorrect in this case.

On the contrary, the condition $\Phi(x) \to 0$ for $x \to \infty$ is imposed in the case of Feynman quantization, which is equivalent to the hypothesis that the physical vacuum is qualitatively no different from free vacuum with the coupling turned off; the coupling can be turned on adiabatically. On the other hand, there is no region in momentum space in which the field fluctuations are small. Actually, suppose that a relatively hard quantum is created at time t_0. Its coupling to the field will increase in the course of time, and will eventually become strong.

2. REGULARIZATION OF THE CHIRAL SCHWINGER MODEL

Let us now apply the ideas formulated above to the simplest of the chiral theories, namely, the chiral Schwinger model.

We consider in two-dimensional Minkowski space a system with action

$$S = \int d^2x \left[-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\varphi^+ (\nabla_0 + \nabla_1) \varphi \right], \tag{11}$$

where $\nabla_\mu = \partial / \partial x^\mu - iA_\mu$, A_μ is an abelian gauge field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and φ is the one-component complex Grassman field. We use the letters x, y to represent the spatial coordinate x^1. The Hamiltonian and the angular momentum operator are obtained in the usual way:

$$H = \int dx \left[\frac{e^2}{2} E^2 - i\varphi^+ \nabla_1 \varphi - A_0 (\partial_1 E + \varphi^+ \varphi) \right], \tag{12}$$

$$P = \int dx (-i\varphi^+ \nabla_1 \varphi), \tag{13}$$

where A_0 plays the part of the Lagrange multiplier for $\chi = \partial_1 E + \varphi^+ \varphi$. Nonzero simultaneous commutation relations take the form

$$[\varphi^+(t, x), \varphi(t, y)]_+ = \delta(x-y), \tag{14a}$$

$$[A_1(t, x), E(t, y)] = i\delta(x-y). \tag{14b}$$

We shall use the representation in which the operators A_1 and φ^+ are diagonal and their effect reduces to multiplication, and the conjugate operators become $E = -i(\delta / \delta A_1)$ and $\varphi = \delta / \delta \varphi^+$. The Hamiltonian (12) and the commutation relations (14) lead to the Heisenberg equations

$$i(\nabla_0 + \nabla_1)\varphi = 0, \tag{15}$$

$$\dot{A}_1 = e^2 E + \partial_1 A_0, \tag{16}$$

$$\dot{E} = \varphi^+ \varphi. \tag{17}$$

In addition to the equations of motion (15)–(17), we must use the constraints

$$\chi = \partial_1 E + \varphi^+ \varphi \approx 0. \tag{18}$$

These constraints show that the gauge field has no dynamic degrees of freedom of its own. It only provides the Coulomb interaction between the Fermi quanta. Hence the Fermi field contains all the dynamic degrees of freedom, and (15) is the key equation.

The Weyl equation (15) can be solved exactly. Let $P(t, x)$ be the phase factor determined by the equation

$$(\nabla_0 + \nabla_1) P(t, x) = 0. \tag{19}$$

The initial condition at $t = t_0$ is

$$P(t_0, x) = \exp\left[i \int_\infty^x A_1(t_0, y)\, dy \right]. \tag{20}$$

It is readily verified that the functions

$$\varphi_k(t, x) = P(t, x) \exp[-ik(t-x)], \quad -\infty < k < +\infty \tag{21}$$

satisfy the Weyl equation (15). Since

$$P(t, x) P^+(t, x) = P^+(t, x) P(t, x) = 1,$$

the set of functions given by (21) is complete and orthonormal at all times:

$$\int \frac{dk}{2\pi} \varphi_k(t,x)\varphi_k^+(t,y)=\delta(x-y). \quad (22)$$

The Fermi fields can therefore be expanded in terms of the functions (21) for all values of time t:

$$\varphi(t,x)=\int \frac{dk}{2\pi}\varphi_k(t,x)a_k,$$
$$\varphi^+(t,x)=\int \frac{dk}{2\pi}a_k^+\varphi_k^+(t,x). \quad (23)$$

The set of quantities $\{a_k^+, a_k\}$ is a single-valued ($\varphi=0\rightarrow a_k=0$) set of Grassmann degrees of freedom. It follows from the Weyl equation (15) that these variables satisfy the equations of motion

$$\dot{a}_k^+=0, \ \dot{a}_k=0. \quad (24)$$

Consider the commutation relations at time $t=t_0$. According to (20) and (21), the functions $\varphi_k(t_0,x)$ are functionals of the field A_1. Since the commutator $[\varphi(t_0,x), A_1(t_0,x)]$ is zero, it follows that

$$[a_k, A_1(t_0,x)]=0, \ [a_k, \varphi_p(t_0,x)]=0. \quad (25)$$

From this and from (14), (22), and (23) we obtain the commutation relations

$$[a_k, a_p^+]_+=2\pi\delta(k-p), \ [a_k, a_p]_+=0, \quad (26)$$

which remains valid at all times t because (24) are satisfied. From the CR

$$[\varphi(t_0,x), \chi_\lambda]=\lambda(x)\varphi(t_0,x),$$
$$[P(t_0,x), \chi_\lambda]=\lambda(x)P(t_0,x),$$

where $\chi_\lambda=\int dx\lambda\chi$, we obtain

$$[a_k, \chi_\lambda]=0, \ [a_k^+, \chi_\lambda]=0. \quad (27)$$

The set of functions (21) is therefore an analog of the set of modes $\{\varphi_N\{\Phi^i\}\}$ introduced above. The equations of motion (24) show that, to regularize the theory, it is natural to impose the second-order constraints

$$a_k^+\approx0, \ a_k\approx0, \ |k|>\Lambda\rightarrow\infty. \quad (28)$$

It follows from (27) that all second-order constraints commute with the first-order constraints (18). This means that the imposition of the constraints (28) is formally correct.

The canonical transformation can be used to pass from the variables $(A_1, E, \varphi^+, \varphi)$ and the Hamiltonian H to the variables $(A_1, \mathscr{E}, a_k^+, a_k)$ and the Hamiltonian H'. We note that $\mathscr{E}-E=\eta\neq0$ where η contains the dependence of the fermion degrees of freedom. We do not need the explicit form of H' and of the equations of motion in terms of the new variables,[2] except for (24):

$$[a_k^+, H']=0, \ [a_k, H']=0. \quad (29)$$

The Dirac CR for the fundamental fields $(A_1, E, \varphi^+, \varphi)$ at time t_0 are constructed in the Appendix using the above formulas. There commutation relations, and all the relationships and equations that follow from them, can be extended to any time t as follows. Let us suppose that, at some instant of time t, the Dirac CR have been determined and have the properties defined by (A5)–(A7). They are then also determined at an infinitesimally close time $t+dt$. Actually, $A(t+dt)=A(t)+idt[H(t), A(t)]^*$. Hence, the CR $[A(t+dt), B(t+dt)]^*$ are determined in terms of the Dirac CR at time t. Using the properties defined by (A5)–(A7), we can readily verify that the Dirac CR must also satisfy these properties at time $t+dt$. It is readily shown that

$$[A, B]^*(t+dt)=[A(t+dt), B(t+dt)]^*. \quad (30)$$

This follows from (A7) if we substitute $C=H$. Equation (30) shows that all the relationships between the operators that include the Dirac CR remain valid at all times. This means that all the commutation relations and the equations obtained with the aid of the Dirac commutation relations (A8)–(A12) and, in particular, (31)–(33), remain valid at all times t.

We shall now use commutation relations (A8)–(A12) to carry our certain calculations. We find that

$$[\varphi(x), \chi(y)]^*=\delta(x-y)\varphi(x), \quad (31)$$
$$[\partial_1 E(x), \chi(y)]^*=0.$$

It follows from (31), (A8), and (A9) that

$$[\chi(x), \chi(y)]^*=0, \ [\chi(x), H]^*=0. \quad (32)$$

We must now evaluate $(i\dot{\varphi}-[\varphi, H]^*)$, using (A8), (A10), and (A31), and project the result on to the functions φ_k where $|k|<\Lambda$. It is readily seen that, in classical mechanics, this projection does not include the contribution due to the term

$$\left\{\int dy\frac{e^2}{2}(E(y))^2, \varphi(x)\right\}^*.$$

In quantum mechanics, the situation is somewhat more complicted. From the definitions of the functions φ_k and from (A10), we have

$$[\varphi(x), E(y)]^*=-\int_{|p|>\Lambda}\frac{dp}{2\pi}\varphi_p(x)\int_y^\infty dz\,\varphi_p^+(z)\varphi(z).$$

Using this equation and also (A8) and (31), we obtain the projection of the Heisenberg equation for the field φ onto the function $\varphi_k, |k|<\Lambda$:

$$i\int dx\,\varphi_k^+(x)(\nabla_0+\nabla_1)\varphi(x)=-\frac{e^2}{2}\int dx\,\varphi_k^+(x)\int dy\,E(y)$$
$$\times\int_{|p|>\Lambda}\frac{dp}{2\pi}\varphi_p(x)\int_y^\infty dz(\varphi_p^+(z)\varphi(z))=-\frac{e^2}{2}\int_{|p|>\Lambda}\frac{dp}{2\pi}\int dy$$
$$\times\int_y^\infty dx\int_y^\infty dz(\varphi_k^+(x)\varphi_p(x))(\varphi_p^+(z)\varphi(z)),$$

since $\varphi_p(x)$ depends only on the field $A_1(z)$ for $z<x$. If we now simplify this equation, we obtain

$$i\dot{a}_k=-\frac{e^2}{2\pi}\frac{\Lambda}{\Lambda^2-k^2}a_k, \quad |k|<\Lambda.$$

Hence it is clear that the commutation relations (26) are reproduced dynamically, and in the limit as $\Lambda\rightarrow\infty$ we have the equations of motion given by (24). Consequently, the Heisenberg equations for the regularized fields

$$\varphi(t,x)=\int\limits_{|k|<\Lambda}\frac{dk}{2\pi}\varphi_k(t,x)a_k,$$

$$\varphi^+(t,x)=\int\limits_{|k|<\Lambda}\frac{dk}{2\pi}a_k^+\varphi_k^+(t,x)$$

have the previous form (15) in the limit as $\Lambda\to\infty$. The rapid accumulation of phase factors in the operators a_k and a_k^+ for $|k|\sim\Lambda$ has no significance because these factors mutually cancel out if we confine our attention to finitely excited states. Hence, the regularized Heisenberg equations $i\dot{A}=[A,H]^*$ are obtained from the formal equations (15)–(18) merely by crossing out the variables a_k^+ and a_k for $|k|>\Lambda$. All the quantities must be expressed in terms of the regularized Fermi fields. Consequently, when the quantities $\partial_\mu J^\mu$ are evaluated $J^\mu=(\varphi^+\varphi, \varphi^+\varphi$ is the fermion current), we can use (15) directly (the separation of the Fermi fields is not required at this point because we are using the regularized fields). Hence we find that

$$\partial_\mu J^\mu=0 \tag{33}$$

Equations (32) and (33) show that the theory remains gauge invariant and the generators of the gauge transformations χ remain as second-order constraints.

Similar calculations applied to the projection of $(i\dot{\varphi}-[\varphi,H]^*)$ onto the function of φ_k, where $|k|>\Lambda$, lead to an identify. These calculations are based on (16) and the constraints given by (28), in addition to the commutation relations (A8)–(A12).

We note that (16) follows directly from commutation relation (A9). Equation (17) is a consequence of (18) and (33), established independently.

Equation (33) can be obtained by the method developed in Ref. 1. Formula (2.22) in Ref. 1 gives the general expression for the gauge anomaly which, in our case, assumes the form

$$\partial_\mu J^\mu(t,x)=-i\int\limits_{|k|<\Lambda}\frac{dk}{2\pi}(\varphi_k^+\varphi_k|_{t+\varepsilon/2,x}-\varphi_k^+\varphi_k|_{t,x})=0,\qquad \varepsilon\to 0.$$

3. THE GROUND STATE AND ELEMENTARY EXCITATIONS

By definition, the ground state $|0\rangle$ satisfies the following conditions:

$$a_k^+|0\rangle=0,\quad -\Lambda<k<0, \tag{34a}$$

$$a_k|0\rangle=0,\quad 0<k<\Lambda, \tag{34b}$$

$$\chi(x)\,|0\rangle=0. \tag{35}$$

Clearly, the state

$$|0\rangle=\prod_{-\Lambda<k<0}a_k^+ \tag{36}$$

satisfies (34)–(35). Consider the gauge invariant operator

$$d_k\{\psi\}=\int dl\,\psi(l)\int dx\exp(-ikx)\varphi^+(x+l)$$
$$\times\exp\Big[i\int\limits_{x-l}^{x+l}A_1(y)\,dy\Big]\varphi(x-l), \tag{37}$$

that creates or annihilates a state with a particular value of momentum (13), equal to k. We also find that

$$[d_k\{\psi\},H]=kd_k\{\psi\}+e^2d_k\{\psi l\}-e^2$$
$$\times\int dl\,\psi(l)\int dx\exp(-ikx)\varphi^+(x+l)$$
$$\times\exp\Big[i\int\limits_{x-l}^{x+l}A_1(y)\,dy\Big]\varphi(x-l)\int\limits_{x-l}^{x+l}dz\,E(z). \tag{38}$$

It is readily seen that operators such as (37) create normalized states with a particular energy, but only when $\psi(l)\sim\delta(l)$. Actually, when this is not the case, the second and third terms on the right hand side of (38) cannot be proportional to the operator (37). The reason for this is that, according to (35),

$$E(x)=-\int\limits_{-\infty}^{x}dy\,\varphi^+(y)\varphi(y).$$

Hence it is clear that the field $E(x)$ is not a spatially homogenous quantity. The second term on the right hand side of (38) is therefore found to vary rapidly as the "particle" (37) travels along the x axis.

Consequently, the particle creation and annihilation operators with momenta $k>0$ have the form

$$c_k=\int dx\exp(-ikx)\varphi^+(x)\varphi(x),\qquad k>0,$$
$$c_k^+=\int dx\exp(ikx)\varphi^+(x)\varphi(x),\qquad k>0. \tag{39}$$

It follows from (34) that

$$c_k|0\rangle=0,$$

$$c_k^+|0\rangle=\exp(ikt)\int\limits_0^k\frac{dp}{2\pi}a_{k-p}^+a_{-p}|0\rangle.$$

The commutation relations $[c_k^+,H]^*=[c_k^+,P]^*=-kc_k^+$ then show that the excited state $|k\rangle=c_k^+|0\rangle$ has momentum k and energy $\omega(k)=k$. Hence Lorentz invariance holds if we pass to the limit as $\Lambda\to\infty$.

If we confine our attention to finitely excited states, we have

$$[c_k,c_p^+]^*=k\delta(k-p), \tag{40}$$

where it is assumed that (34) is valid in the deep ultraviolet region.

We now state a final obvious proposition: the theory is unitary because, according to (35), we have $\langle 0|0\rangle=1$.

4. CONCLUSIONS

Regularization of the model defined by (11) by means of the Dirac CR is thus seen to provide us with the possibility of a quantization in which all the necessary properties of the theory remain valid, i.e., unitarily and gauge and Lorentz invariance. As a result, we obtain a spatially odd theory of free massless neutral Bose particles with positive momenta. We note that Lorentz invariance and violation of spatial parity ensure that the quanta of excitations in the one-dimensional theory have no mass.

The example of the chiral Schwinger model thus shows that field theory can be regularized by the imposition of second-order constraints on particular degrees of freedom in the ultraviolet region. This regularization is often preferable to traditional methods because it enables us to retain the necessary first-order constraints (for example, gauge invariance). This confirms the hypothesis of dynamic anomaly-free quantization, which we used previously to investigate the nonabelian Weyl theory in Minkowski 4-space.[1]

We note in conclusion that the nonabelian chiral Schwinger model or the chiral 't Hooft model can be quantized by the above method. The result is a spatially odd theory of free colorless massless baryons and mesons with the spectrum $\omega(k) = k$, $k>0$.

APPENDIX

We not introduce the Dirac commutation relations for the operators of the Schwinger model (12), and apply the second-order constraints (28).

We recall the expression for the Dirac brackets in classical theory (see Ref. 9), and then generalize them to the quantum field in the case of the chiral Schwinger model.

Let $\varkappa_s$, $s = 1,\ldots, S$ be a complete set of second-order constraints in the classical Hamiltonian theory, and let $\{\ldots,\ldots\}$ denote the Poisson bracket. We use $c_{ss'}$ to represent the inverse of the matrix $\{\varkappa_s, \varkappa_{s'}\}$. The Dirac brackets for A and B are then defined by

$$\{A, B\}^*=\{A, B\}-\{A, \varkappa_s\}c_{ss'}\{\varkappa_{s'}, B\}. \tag{A1}$$

This expression has all the necessary properties of Poisson brackets. The equations of motion obtained by using the Poisson and Dirac brackets are identical in the weak sense, and the constraints $\varkappa_s$ can be set equal to zero even prior to the evaluation of the Dirac brackets.

We now turn to the model (12). We first evaluate the Dirac CR for the fundamental fields $A_1(x)$, $E(x)$, $\varphi^+(x)$, and $\varphi(x)$ at time t_0. We need the parity function α, defined for homogeneous operators with values in the group Z_2. By definition, $\alpha(A_1) = \alpha(E) = 0$, $\alpha(\varphi^+) = \alpha(\varphi) = 1$. If the function α is defined for the operators A and B, then $\alpha(AB) = \alpha(A) + \alpha(B) \pmod 2$. The commutator of the homogeneous operators A and B is defined by

$$[A, B]=AB-BA(-1)^{\alpha(A)\alpha(B)}. \tag{A2}$$

Let $a_i(k)$ denote a_k^+ for $i = 1$ and a_k for $i = 2$. According to (26), the analogs of the matrix $\{\varkappa_s, \varkappa_{s'}\}$ and its inverse $c_{ss'}$ takes the form

$$2\pi\delta(k-p)\tau_{ij}^1,\ |k|>\Lambda,\ |p|>\Lambda, \tag{A3}$$

where τ_{ij}' is the first Pauli matrix. Let $[\ldots,\ldots]^*$ represent the Dirac CR. For the fundamental homogeneous fields, we have by analogy with the Dirac brackets (A1)

$$[A,B]^*=[A,B]-\frac{1}{2}\int\limits_{|k|>\Lambda}\frac{dk}{2\pi}([A,a_i(k)]\tau_{ij}^1[a_j(k),B]$$
$$-[B,a_i(k)]\tau_{ij}^1[a_j(k),A](-1)^{\alpha(A)\alpha(B)}). \tag{A4}$$

It is clear that

$$[A, B]^*=-[B, A]^*(-1)^{\alpha(A)\alpha(B)},$$
$$[xA+yB, C]^*=x[A, C]^*+y[B, C]^*. \tag{A5}$$

Let A, B, and C be homogeneous operators. By definition

$$[A, BC]^*=[A, B]^*C+B[A, C]^*(-1)^{\alpha(B)\alpha(A)}. \tag{A6}$$

Formulas (A4)–(A6) provide an inductive definition of the Dirac CR for any functionals that depend on the fundamental fields. We shall show that the Jacobi identity is valid for the fundamental fields and takes the form

$$[A, [B, C]^*]^*(-1)^{\alpha(A)\alpha(C)}+[B, [C, A]^*]^*(-1)^{\alpha(B)\alpha(A)}$$
$$+[C, [A, B]^*]^*(-1)^{\alpha(C)\alpha(B)}=0. \tag{A7}$$

Let any three operators out of the four A, B, C, D satisfy the Jacobi identity (A7). Using (A5) and (A6), we can readily verify that, in this case, the operator triplet A, B, and CD will also satisfy (A7). Hence, by induction, we conclude that any functionals of fundamental fields satisfy the Jacobi identity (A7).

The Dirac CR (A4)–(A6) are thus seen to have all the necessary properties.

Let us now evaluate the nonzero Dirac CR for the fundamental fields in accordance with (A4) [$\partial_1 E(x)$ replaces $E(x)$ in some of the CR].

In these evaluations, we use the commutation relations (14), (15)–(27), and also the definition (21). It follows from (27) that $[\partial_1 E(x), a_k] = [a_k, \varphi^+(x)\varphi(x)] = \varphi_k^+(x)\varphi(x)$. Therefore we find that

$$[\varphi(x),\varphi^+(x)]^*=\int\limits_{-\Lambda}^{\Lambda}\frac{dk}{2\pi}\varphi_k(x)\varphi_k^+(y), \tag{A8}$$

$$[A_1(x), E(y)]^*=i\delta(x-y), \tag{A9}$$

$$[\partial_1E(x),\varphi(y)]^*=-\int\limits_{|p|>\Lambda}\frac{dp}{2\pi}\varphi_p(y)(\varphi_p^+(x)\varphi(x)), \tag{A10}$$

$$[\partial_1E(x),\varphi^+(y)]^*=\int\limits_{|p|>\Lambda}\frac{dp}{2\pi}(\varphi^+(x)\varphi_p(x))\varphi_p^+(y), \tag{A11}$$

$$[\partial_1E(x),\partial_1E(y)]^*=\frac{1}{2}\int\limits_{|p|>\Lambda}\frac{dp}{2\pi}\{[(\varphi_p^+(y)\varphi(y))(\varphi^+(x)\varphi_p(x))$$
$$+(\varphi^+(y)\varphi_p(y))(\varphi_p^+(x)\varphi(x))]-[x\leftrightarrow y]\}. \tag{A12}$$

Direct verification shows that the Dirac commutation relations (A8)–(A12) satisfy the Jacobi identity (A7).

We note that, in terms of the variables $(A_1, \mathscr{E}, a_k^+, a_k)$, the Dirac commutation relations take a much simpler form. Nonzero Dirac communication relations have the form of (25) for $|k|<\Lambda$ and $|p|<\Lambda$, and

$$[A_1(x), \mathscr{E}(y)]^*=i\delta(x-y).$$

Hence all the properties of the Dirac commutation relations (A5)–(A7) in terms of these variables are obvious.

[1]A similar approach to the study of gauge anomalies was developed by Gribov.[10]

[2)] We use q and Q to denote the sets of coordinates (A_1, φ^+) and (A_1, a_k^+), and let U be a unitary matrix such that $Q = U^+ qU$. Then we have $i\dot{Q} = [Q, H']$, where $H' = U^+HU + i\dot{U}^+U$.

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