Vortices on world sheets of strings and superstrings

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We discuss the dynamics of the world sheet of a string propagating in a non-simply-connected space. A significant role is played in this case by vortex configurations, which give rise to phase transitions on the world sheet. The cases of bosonic strings and superstrings are considered. The connection between the Hagedorn phase transition occurring at finite temperatures and the Berezinskiĭ-Kosterlitz-Thouless transition on the world sheet is discussed.

I. INTRODUCTION

At the present time string theory[1,2] is one of the rapidly developing areas of theoretical physics, including a variety of problems on the boundary of elementary particle physics, field theory and statistical physics. One of the possible goals of these investigations is the construction of a unique theory of fundamental interactions. To realize such a program it is necessary to achieve a much deeper understanding of string properties, to search for additional hidden symmetries, to attempt to recognize what is string field theory—a theory describing the interaction of strings in the same way conventional field theory describes interactions of particles. One of the advantages of the field formulation consists of the fact that it permits a unified way description of the character of the excitations in various classical vacua, to answer the question of which classical vacuum is the true ground state, etc.

All this is missing from contemporary string theory. Its most consistent formulation so far—the theory of the first quantized string—permits the calculation of amplitudes for scattering of strings propagating in a given external space. They are calculated as functional integrals on surfaces, with the boundary of the surface being fixed by the initial and final states. In the case when the latter have definite quantum numbers, i.e., the string finds itself in the state of a "particle", the boundary is transformed into a set of points and the boundary conditions are in effect replaced by the introduction of vertex operators inside the functional integral, so that the amplitude takes on the form

$$A(p_1,\ldots,p_N)\propto\left\langle \prod_{i=1}^{N}\int d^2\xi_i[g(\xi_i)]^{1/2}V_i(P_i,x^\mu(\xi_i))\right\rangle, \tag{1.1}$$

and the averaging is done with the help of the functional integral

$$\int Dg_{ab}(\xi)Dx^\mu(\xi)\exp\left\{-\frac{1}{4\pi\alpha'}\int d^2\xi[g(\xi)]^{1/2}g^{ab}(\xi)\cdot \times\partial_a x^\mu(\xi)\partial_b x^\nu(\xi)\eta_{\mu\nu}\right\},\quad a,b=1,2,\quad \mu,\nu=1,\ldots,D, \tag{1.2}$$

where the integration is over the two-dimensional metric $g_{ab}(\xi)$ and coordinates $x^\mu(\xi)$, describing the embedding of the two-dimensional surface with coordinates ξ^1, ξ^2—the world sheet of the string—in the external D-dimensional flat space, $\eta_{\mu\nu}$ is the standard Minkowski (or Euclidean, after the Wick rotation) metric. One may also consider the case of a space with curvature by replacing $\eta_{\mu\nu}$ by the appropriate metric $g_{\mu\nu}(x)$. The vertex operator $V_i(P_i, x^\mu(\xi))$ corresponds to the emission or absorption of a particle with momentum P and quantum numbers symbolically denoted by the subscript i.

The most important property of this theory is the existence of two-dimensional conformal and reparametrization symmetry in certain cases, for example in flat space for $D = 26$, which permits one to view the two-dimensional metric $g_{ab}(\xi)$ as pure gauge degrees of freedom. The condition of conformal symmetry on the world sheet is equivalent to the classical equations of motion of the string field; all possible vacua of the string theory consist of various two-dimensional conformal theories defined by critical parameters: central charge and the spectrum of anomalous dimensions.[3] In this fashion, in studying two-dimensional models we obtain information on many possible vacua in string theory; in view of the complete absence of an adequate field formalism this is at the present time the only possibility for studying the structure of the vacuum states.

The simplest examples of nontrivial vacua, different from flat space R^D, are flat spaces with one or several compact dimensions: $R^{D-1}\times S^1$ or $R^{D-K}\times S^1_{(1)}\times\ldots\times S^1_{(K)}$, parametrized by the radii of the circles (cycles) S^1 and, in the case of several cycles, by the angles between them. At first sight it may seem that the dimensions of the cycle are arbitrary but it will be shown below that there exist critical, limiting dimensions to the cycles. For radii smaller than limiting the conformal symmetry on the world sheet disappears and such a space is no longer acceptable as the vacuum state. This phenomenon, noted in Refs. 4 and 5, is connected with the Berezinskiĭ-Kosterlitz-Thouless (BKT)[6,7] phase transition on the world sheet, whose essence is the appearance of new dominating field configurations—vortices. Violation of conformal symmetry occurs because for small radii the vortices are in the plasma phase, in which Debye screening is present. The appearance of a new scale parameter—the Debye radius—results in violation of conformal invariance.

The appearance of vortex configurations is possible because the space S^1 is not simply connected, or, in other words, because of the nontriviality of the first homotopic group $\pi_1(S^1) = Z$. It can be shown that also in the case of more general manifolds the existence of vortex excitations, and therefore the possibility of phase transitions, follows from nontriviality of π_1.

It is found that vortices on the world sheet correspond

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directly to certain states of the string. The Hilbert space of the string states is defined by the function embedding the string in the external space $X^\mu(\sigma)$, where σ is the coordinate along the string varying (by convention) between 0 and 2π. for a closed string we have $X^\mu(0) = X^\mu(2\pi)$, i.e., σ is a coordinate on the circle S^1. In this case there exist, for a non-simply-connected manifold M with $\pi_1(M) \neq 0$, sectors in Hilbert space unconnected with each other, and $\pi_1(M)$ enumerates the components of the full Hilbert space:

$$H = \bigoplus_{k\in\pi_1(M)} H_k.$$

States of the string corresponding to sectors with $k \neq 0$, are called soliton or twisted states. The first term is conventionally used in compactification on a torus, for example in the theory of the heterotic string,[8] the second term is used in description of orbifolds,[9] where noncontractible cycles exist round singular points and their radii are not bounded from below. We make use of the first term.

One may raise the question about the form of the vertex operator, responsible for the emission of one or another state of the string. It is found that vertex operators accompanied by vortices on the world sheet correspond to the emission of states of the soliton sector. This is a direct reflection of the fact that both the soliton states and the vortices are a consequence of nontriviality of one and the same group, $\pi_1(M)$.

In the following section we shall discuss the simplest case of compactification on S^1—the two-dimensional XY model. We shall construct the vertex operators of the soliton sector and determine the spectrum of their anomalous dimensions, and consequently the mass spectrum of the soliton sector.

In Sec. III we discuss the description of the system of vortices in the XY model as a two-dimensional Coulomb gas, whose plasma phase is equivalent to the sine-Gordon model.[10] This gives rise to an interesting duality between large and small values of the radius of the space S^1, and also between ordinary and soliton sectors. We shall also discuss questions relating to chiral strings, in particular the gauge anomaly and the resultant restrictions on compactification on a torus.

The role of vortices in the conformal phase, when it is thermodynamically advantageous for the vortices to bind into dipoles, will be discussed in Sec. IV. We will present there a self-consistent calculation of dipole corrections to the correlation functions and critical indices of the free theory. In that approximation all correlation functions coincide with the free ones, but parameters, for example the radii of the cycles, are renormalized.

We shall discuss the effect of the dipole corrections on the interaction with the two-dimensional metric $g_{ab}(\xi)$, in particular the possibility of existence of corrections to the critical dimension of the theory that are exponentially small in the radii of the noncontractible cycles.

In Sec. V we discuss vortex effects in heterotic strings and in superstrings. Here we make use of a supergeneralization of the two-dimensional Coulomb gas model,[11] in which a phase transition also occurs.

Section VI is devoted to the behavior of strings at finite temperatures. As is known, passage to a nonzero temperature T is equivalent to studying the system in a periodic imaginary time, in other words the Euclidean space has the form $R^{D-1}\times S^1$, where the radius of S^1 is $r = (2\pi T)^{-1}$ and there appears in the theory a critical temperature T_c, that coincides[4,5] with the Hagedorn temperature—the limiting temperature in string theory[12,13] above which no canonical ensemble exists.

The corresponding divergence of the free energy is connected with the appearance of additional tachyons in the soliton sector. We shall also discuss certain results of the work of Ref. 14, in particular the character of the phase transition.

II. MOTION OF THE STRING IN A NON-SIMPLY-CONNECTED MANIFOLD

1. The two-dimensional *XY* model: the Berezinskiĭ-Kosterlitz-Thouless phase transition

The simplest example of a non-simply-connected manifold is the circle S^1. The corresponding σ model is also called the XY model. As we already know, $S^1 = R^1/Z$ and the group Z acts on the straight line R^1 by translations: $x \to x + 2\pi n$, $n\in Z$. If the radius of the circle S^1 equals R then the initial action has the form

$$S = \frac{R^2}{4\pi\alpha'}\int d^2\xi\,(\partial_a y)^2,\qquad y\in[0,2\pi),\tag{2.1}$$

where R^2 is just the metric on S^1. After passage to the universal cover R^1 we obtain

$$S_R = \frac{R^2}{4\pi\alpha'}\sum_{i,e=1,2}(x_{i+e}-x_i-aA_{i,e})^2,\qquad aA_{i,e}\in 2\pi Z\tag{2.2}$$

(i is the node number and $\mathbf{e}$ is the lattice vector), or in the continuum limit

$$S = \frac{R^2}{4\pi\alpha'}\int d^2\xi\,(\partial_a x - A_a)^2.\tag{2.3}$$

The quantum theory is defined by the functional integral over the field x and summation over the gauge equivalence classes of the field A_a. The partition function for the XY model is

$$Z = \sum_{\{A_a\}}\int Dx\,\exp\left\{-\frac{R^2}{4\pi\alpha'}\int d^2\xi\,(\partial_a x - A_a)^2\right\}.\tag{2.4}$$

The quantity $\beta = R^2/4\pi\alpha' = 1/T$ is ordinarily called the inverse temperature.[1)]

The gauge equivalence classes are characterized by the strength of the gauge field $F_{ab} = \partial_{[a}A_{b]}$, and in lattice terms

$$\exp(ia^2F_{ab}) = U_{i,a}U_{i+a,b}U^{-1}_{i+a+b,a}U^{-1}_{i+b,a},$$

where $U_{i,e} = \exp(iaA_{i,e})$. The field strength is defined on lattice plaquettes, and the scalar quantity $F = \frac{1}{2}\varepsilon_{ab}F_{ab}$ dual to it is defined at the vertices of the dual lattice, but in the continuum limit this distinction is irrelevant.

In the trivial class one may choose $A_{i,e}$ or $A_a(\xi) = 0$, everywhere and we are dealing with the free theory. The next class has nonzero field strength on only one plaquette, in the continuum limit $F = 2\pi Q\delta(\xi - \xi_0)$, where Q is an integer. In the general case the configuration is specified by a set of integers $Q_1, Q_2, \ldots, Q_i, \ldots$ and coordinates $\xi_1, \xi_2, \ldots, \xi_i, \ldots$ The singular nature of the distribution of the field F is a consequence of the discrete character of the gauge symmetry.

The nontrivial point configurations of the gauge field are vortices, they are what distinguishes the simply free theory from a theory on S^1. On a contour about the singular point—the center of the vortex—the field performs Q full rotations $x \to x + 2\pi Q$. If we make no use of the language of gauge fields A_a, then it is convenient to break up the field into a sum of two components—the classical vortex x_v and the quantum x_q. For the vortex located at the origin of the coordinates,

$$x(\xi)=x_v(\xi)+x_q(\xi)=\vartheta Q+x_q(\xi),$$

where ϑ is the polar angle on the world sheet, $\xi = |\xi|\exp(i\vartheta)$, $\xi = \xi_1 + i\xi_2$.

To determine the action for the vortices in Eq. (2.4) we integrate over the fields $x(\xi)$ in the background of the given distribution of the fields $A_a(\xi)$. This problem is solved trivially as we are dealing with a free theory in the presence of an external source:

$$Z=\sum_{\{A\}}\int Dx\exp\left(-\beta\int d^2\xi[(\partial_a x)^2+2x\partial_a A_a+A_a^2]\right)$$

$$=\sum_{\{A\}}\int Dx\exp\left(-\beta\int d^2\xi(\partial_a x)^2\right)\exp\Big[-\beta\int d^2\xi A_a^2$$

$$+\beta\iint d^2\xi\, d^2\xi'\,\partial_a A_a(\xi)\,\partial_b A_b(\xi')G(\xi,\xi')\Big]$$

$$=\sum_{\{A\}}\exp\Big[\beta\iint d^2\xi\, d^2\xi' F(\xi)G(\xi,\xi')F(\xi')\Big]$$

$$\times\int Dx\exp\Big[-\beta\int d^2\xi(\partial_a x)^2\Big].$$

As a result the effective action for the vortices is

$$S_{eff}=-\frac{\beta}{4\pi}\iint d^2\xi\, d^2\xi' F(\xi)\ln\frac{|\xi-\xi'|^2+a^2}{\mathscr{A}}F(\xi'), \quad (2.5)$$

where

$$G(\xi,\xi')=\frac{1}{4\pi}\ln\frac{|\xi-\xi'|^2+a^2}{\mathscr{A}}$$

is the Green's function for the two-dimensional Laplace operator, a is the ultraviolet cutoff (of the order of the lattice step), $\mathscr{A}$ is the infrared cutoff (area of the system). For the vortex configuration we have

$$F(\xi)=2\pi\sum_{i=1}^{N}Q_i\delta^2(\xi-\xi_i) \quad (2.6)$$

and we obtain, after substituting Eq. (2.6) into Eq. (2.5),

$$S_{eff}(Q_1,\ldots,Q_N)=-\pi\beta\sum_{i,j}Q_iQ_j\ln\frac{|\xi_i-\xi_j|^2+a^2}{\mathscr{A}}$$

$$=\pi\beta\left(\sum_{i=1}^{N}Q_i\right)^2\ln\frac{\mathscr{A}}{a^2}-2\pi\beta\sum_{i,j,i<j}Q_iQ_j\ln\frac{|\xi_i-\xi_j|^2}{a^2}+N\beta\mu. \quad (2.7)$$

The first term in Eq. (2.7) leads to a sharp suppression of configurations with uncompensated charge $\Sigma Q_i \neq 0$. The main contribution to the partition function comes from "neutral" vortex systems. The second term describes interaction between vortices, and the last term describes self-interaction. The chemical potential μ is determined by the asymptotic Green's function $G(\xi,\xi')$, $\xi \to \xi'$ and depends on the specific method of regularization. For the square lattice the exact value is $\mu = 2\pi^2$. It is convenient to view μ as a free parameter of the theory.

In this fashion the system of vortices constitutes a two-dimensional Coulomb gas. Its fundamental property is the existence of the BKT phase transition.[6,7] Below the transition point $\beta > \beta_c$ the vortices are bound with antivortices into dipole molecules; however for $\beta < \beta_c$ these molecules dissociate and the vortex system goes over into the plasma phase. Debye screening gives rise to violation of conformal invariance in the high-temperature ($T = \beta^{-1} > \beta_c^{-1}$) phase and the asymptotic behavior of the correlators changes from a power law to an exponential law.

To find β_c it is convenient to calculate the mean square value of the dipole moment of the molecule.

$$\langle \mathbf{p}^2\rangle=\int d^2\mathbf{p}\,\mathbf{p}^2\exp[-S_{dip}(\mathbf{p})]\left\{\int d^2\mathbf{p}\exp[-S_{dip}(\mathbf{p})]\right\}^{-1}. \quad (2.8)$$

The molecules with $Q = \pm 1$ dissociate first. Their action equals

$$S_{dip}=-2\pi\beta Q_1Q_2\ln[(\mathbf{p}^2+a^2)/a^2], \quad Q_1=-Q_2=\pm 1. \quad (2.9)$$

For $\beta < \beta_c = \pi^{-1}$ the integral in the numerator in Eq. (2.8) diverges, which means freeing of the vortices. Since we have $\beta = R^2/4\pi\alpha'$, the critical value of the radius is

$$R_c=2(\alpha')^{1/2}, \quad \beta_c=\pi^{-1}. \quad (2.10)$$

We shall still return to the properties of the two-dimensional Coulomb gas in Sec. III.

2. String in the $R^{D-1}\times S^1$ space

We now discuss the spectrum of a closed string, when one of the spatial dimensions is compactified to a circle. This problem has been solved in the operator formalism in Ref. 8; our study will be in terms of the Virasoro algebra and we will show that soliton states with winding number L correspond to vortices with charge $Q = L$ of the two-dimensional XY model.

The physical states of a closed string are characterized by three quantum numbers: the D-momentum P^μ, the number of the excited state N and the soliton quantum number L. The numbers P^μ and N are introduced in the same way as in ordinary space, but the presence of the compact dimension gives rise to quantization of the corresponding component of the momentum:

$$P^0=m/R, \quad m=0,\pm 1,\ldots. \quad (2.11)$$

The second peculiarity consists in the appearance of the number L, corresponding to the number of windings of the string in the coordinate x^0. The spectrum of physical states is described by the equation

$$\alpha' M^2=-4+2(N+\tilde{N})+m^2\alpha'/R^2+L^2R^2/\alpha', \quad N-\tilde{N}=mL. \quad (2.12)$$

Let us show how this equation is obtained. As is well known, the interaction amplitudes in string theory are expressed in terms of the vacuum expectation values of the vertex operators $V_i(P_i,N_i,L_i)$ corresponding to the in- and out-states [see Eqs. (1.1) and (1.2)]:

$$A(in\to out)\propto\left\langle\prod_i\int V_i(P_i,N_i,L_i;x(\xi_i))d^2\xi_i\right\rangle. \quad (2.13)$$

The vertex $V(P, N, L; x(\xi))$ has the form

$$V=\sum_{\{n_i\},\{\tilde{n}_j\}} C^{\mu_i\ldots;\nu_j\ldots}_{n_i,\tilde{n}_j}\prod_i\left(\frac{\partial^{n_i}x^{\mu_i}}{\partial z^{n_i}}\right)$$

$$\times\prod_j\left(\frac{\partial^{\tilde{n}_j}x^{\nu_j}}{\partial\bar{z}^{\tilde{n}_j}}\right)\exp[iPx(z,\bar{z})],$$

$$z,\bar{z}=\xi_1\pm i\xi_2,\quad \sum n_i=N,\quad \sum\tilde{n}_j=\tilde{N}, \qquad (2.14)$$

with the field x^0 containing the classical component x_v: the vortex with charge L is located at the point z. In terms of complex coordinates we have

$$x_v^0(\xi)=\frac{L}{2i}\ln\frac{\xi-z}{\xi-\bar{z}}. \qquad (2.15)$$

For the spectrum of physical states u the tensor coefficients C are determined by the Virasoro conditions (see Ref. 16):

$$L_nV=\tilde{L}_nV=0,\quad n>0, \qquad (2.16a)$$

$$(L_0-1)V=(\tilde{L}_0-1)V=0. \qquad (2.16b)$$

The Virasoro operators L_n and $\tilde{L}_n$ act as follows:

$$L_mV(z)=-\frac{1}{2\pi i\alpha'}\oint_{C_z}d\xi(\xi-z)^{m+1}:\left(\frac{\partial x^\lambda}{\partial\xi}\right)^2:V(z,\bar{z}), \qquad (2.17a)$$

$$\tilde{L}_mV(z)=-\frac{1}{2\pi i\alpha'}\oint_{C_{\bar{z}}}d\bar{\xi}(\bar{\xi}-\bar{z})^{m+1}:\left(\frac{\partial x^\lambda}{\partial\bar{\xi}}\right)^2:V(z,\bar{z}). \qquad (2.17b)$$

The contour C_z encircles the point z and contains only one vertex V.

The physical spectrum is determined by conditions (2.16b), which presuppose that the conformal dimensions Δ and $\bar{\Delta}$ of the fields V are equal to unity. In addition the relation $(L_0-\tilde{L}_0)V=0$ guarantees invariance of the amplitude under rotations in the world sheet.

We shall demonstrate two ways of evaluating the dimensions Δ and $\bar{\Delta}$. The first is based on the evaluation of the correlator

$$\langle V(z)V(z')\rangle\propto 1/(z-z')^{2\Delta}(\bar{z}-\bar{z}')^{2\bar{\Delta}}. \qquad (2.18)$$

It will be used in Sec. IV in summing the dipole corrections to the spectra. For now we apply directly the Virasoro condition (2.16b). Let us act with the operator L_0 on the vertex $V(P^\mu=(m/R,\mathbf{P}),N,\tilde{N},L;z=0)$. To simplify the calculations we assume that only one number in the set $\{n_i\}$ in Eq. (2.14) is different from zero: $n_1=N\neq 0$. In evaluating the integrals it is necessary to consider different pairings of the operators $\partial x^\mu(\xi)$ entering L_0, and the operators $x^\mu(z,\bar{z})$ and $\partial^{n_i}x^\mu/\partial z^{n_i}$ from V:

$$L_0\left(\frac{\partial^nx^\mu}{\partial z^n}\right)\exp[ip^\mu x^\mu(z,\bar{z})]=-\frac{1}{\alpha'}\int\frac{d\xi}{2\pi i}(\xi-z)$$

$$\times\Big[-P_\mu P_\nu\frac{\partial}{\partial\xi}G^{\lambda\nu}\frac{\partial}{\partial\xi}G^{\lambda\mu}V+2iP_\nu\frac{\partial}{\partial\xi}G^{\lambda\nu}\frac{\partial^n}{\partial z^n}\frac{\partial}{\partial\xi}$$

$$\times G^{\lambda\mu}:e^{iPx}:+2iP_\nu\frac{\partial}{\partial\xi}G^{\lambda\nu}:\frac{\partial x^\lambda}{\partial\xi}\frac{\partial^n x^\mu}{\partial z^n}e^{iPx}:$$

$$+2\frac{\partial^n}{\partial z^n}\frac{\partial}{\partial\xi}G^{\lambda\mu}:\frac{\partial x^\lambda}{\partial\xi}e^{iPx}:+:\left(\frac{\partial x^\lambda}{\partial\xi}\right)^2\frac{\partial^nx^\mu}{\partial z^n}e^{iPx}:\Big]. \qquad (2.19)$$

Here

$$G^{\mu\nu}=\langle x^\mu(\xi)x^\nu(z)\rangle=-{}^1/_2\delta^{\mu\nu}\alpha'\ln|\xi-z|^2. \qquad (2.20)$$

However, in contrast to the procedure applicable in the absence of vortices, in the current calculations it is necessary to take into account the presence in the field of the classical component x_v^0, Eq. (2.15). It can be shown that for an arbitrary set $\{n_i\}$, such that $\Sigma n_i=N$ in Eq. (2.14), the result looks like the following:

$$L_0V=(m^2\alpha'/4R^2+\mathbf{P}^2\alpha'/4-mL/2+L^2R^2/4\alpha'+N)V. \qquad (2.21a)$$

The expression for the operator $\tilde{L}_0$ differs by the sign of the mL term:

$$\tilde{L}_0V=(m^2\alpha'/4R^2+\mathbf{P}^2\alpha'/4+mL/2+L^2R^2/4\alpha'+\tilde{N})V, \qquad (2.21b)$$

and in view of the requirement $(L_0-\tilde{L}_0)V=0$ we have

$$N-\tilde{N}=mL, \qquad (2.22)$$

i.e., the total orders of the derivatives ∂_z and $\partial_{\bar{z}}$, which enter V, differ by mL.

The mass spectrum is obtained from the condition (2.16b)

$$\alpha'M^2=\alpha'\mathbf{P}^2=\alpha'm^2/R^2+2mL+L^2R^2/\alpha'+4(N-1)$$

and coincides with Eq. (2.12). This proves that soliton states correspond to vortices on the world sheet.

We note that for the critical value of the radius $R_c=2(\alpha')^{1/2}$, Eq. (2.10), there appear in the spectrum massless states with $L=\pm 1$, $m=N=\tilde{N}=0$. For $R<R_c$ tachyon states appear in the soliton sectors.

III. FIELD DESCRIPTION OF VORTICES IN CLOSED AND CHIRAL STRINGS

The defining role in the vortex system of the XY model is played by vortices with the smallest charge $Q=\pm 1$. Their contribution is described by the partition function of the two-dimensional Coulomb gas:

$$Z=\sum_{n_+=n_-=0}^{\infty}\int\frac{d^2\xi_1/a^2\ldots d^2\xi_{n_+}/a^2}{n_+!}\frac{d^2z_1/a^2\ldots d^2z_{n_-}/a^2}{n_-!}$$

$$\times\exp\Big[-2n\beta\mu+\pi\beta\sum_{i,j=1}^{n_+=n_-}\Big(\ln\frac{|\xi_i-\xi_j|^2+a^2}{a^2}+\ln\frac{|z_i-z_j|^2+a^2}{a^2}$$

$$-\ln\frac{|\xi_i-z_j|^2+a^2}{a^2}\Big)\Big]. \qquad (3.1)$$

Expression (3.1) has a field representation. It can be shown that Eq. (3.1) is equivalent to the partition function for the sine-Gordon model[10]:

$$Z=N_0^{-1}\int D\varphi\exp\Big[-\int d^2\xi\Big(\frac{1}{2}(\partial_a\varphi)^2+\lambda\cos 2\pi(2\beta)^{1/2}\varphi\Big)\Big],$$

$$N_0=\int D\varphi\exp\Big[-\frac{1}{2}\int d^2\xi(\partial_a\varphi)^2\Big],\quad \lambda=2e^{-\beta\mu}/a^2. \qquad (3.2)$$

Expansion of Eq. (3.2) in a power series in λ coincides with Eq. (3.1). Upon making in Eq. (3.2) the change of variable $y = 2\pi(2\beta)^{1/2}\varphi$ we obtain the Lagrangian

$$\mathcal{L} = (\partial_a y)^2/16\pi^2\beta + \lambda \cos y. \tag{3.3}$$

Coleman[10] showed that the quantum theory, Eq. (3.2), exists only for $\beta < \beta_c = \pi^{-1}$, and for $\beta > \beta_c$ the theory has no ground state. The critical value coincides with Eq. (2.10)—the BKT phase transition point, and the absence of a ground state in the model (3.2) corresponds to the dipole phase of the Coulomb gas.

It is clear that the Lagrangian (3.3) is invariant under the shift $y \to y + \pi$ and y may be interpreted as an angle variable. If we ignore the $\lambda \cos y$ term describing Debye screening and violating conformal symmetry, then we obtain a Lagrangian of the same type as the original one but with new values of the parameters β and R:

$$\tilde\beta = 1/16\pi^2\beta, \quad \tilde R = \alpha'/R. \tag{3.4}$$

The duality (3.4) is also observed in the spectrum (2.12), invariant under the replacement $R \to \tilde R$ and $m \rightleftarrows L$ (see Ref. 17 and references therein). This is nothing but duality of the Kramers-Wannier type for the XY model, where we have chosen the initial field $x(\xi)$ as the order parameter, while the field $y(\xi)$ arising in the field description of the vortices is the disorder parameter. We note that it is defined as it should be on the dual lattice, which is where the coordinates of the vortices are defined. In this fashion there exist in the theory three regions

$$\begin{aligned} &\text{I. } R > R_c = 2(\alpha')^{1/2}, \\ &\text{II. } R_c > R > \alpha'/R_c = {}^1/_2(\alpha')^{1/2}, \\ &\text{III. } R < {}^1/_2(\alpha')^{1/2}. \end{aligned} \tag{3.5}$$

Under the transformation (3.4) regions I and III go into each other, while region II goes into itself. However in comparing regions I and III it is necessary to take into account the term $\lambda \cos y$ in Eq. (3.3). It can be interpreted as an external tachyonic field, with the string propagating in its background. In this manner we have duality between the theory with $R < \frac{1}{2}(\alpha')^{1/2}$ and the theory with $\tilde R = \alpha' R^{-1} > R_c$, but in the presence of the condensate of the external tachyonic field.

The critical parameters of open strings with a compact dimension are the same as for closed strings, aside for some minor technical complications connected with boundary conditions. The situation is different if we are interested in chiral (heterotic) strings. In that case it is necessary to consider the chiral sector of the theory (2.4) and assume, when evaluating the effective action of the type of Eq. (2.5), that the components $A_+ = A_0 + A_1$ or $A_- = A_0 - A_1$ of the gauge field are independent. Making use of standard conic variables $\xi_0 \pm \xi_1$ we easily see that Eq. (2.5) can be represented in the form (for Minkowski signature)

$$L_{eff} = \frac{\beta}{4}(\partial_- A_+ - \partial_+ A_-)\frac{1}{\partial_+\partial_-}(\partial_- A_+ - \partial_+ A_-)$$

and, setting $A_- = 0$, we obtain for the chiral theory the gauge-noninvariant action

$$S_{eff} = \frac{\beta}{4}\int d^2\xi\, \partial_- A_+ \frac{1}{\partial_+\partial_-}\partial_- A_+. \tag{3.6}$$

Under the gauge transformation (2.8) we have

$$\delta A_+ = 2\partial_+\varepsilon, \quad \varepsilon(\xi) = 2\pi n(\xi); \quad n \in Z,$$

$$\begin{aligned} \delta S_{eff} &= \beta \int d^2\xi\, \varepsilon(\xi)\,\partial_- A_+ \\ &= \frac{\beta}{2}\int d^2\xi\, \varepsilon(\xi)\left[\frac{1}{2}\varepsilon_{ab}F_{ab}(\xi) + \partial_a A_a(\xi)\right]. \end{aligned} \tag{3.7}$$

In spite of the fact that the theory may be gauge-noninvariant, since only $\exp(i\delta S)$ is relevant for quantization we obtain unbroken gauge symmetry if for all $\varepsilon(\xi)$ and $A_a(\xi)$ we have $\delta S = 2\pi N$. Taking into account that $\varepsilon(\xi)/2\pi$ and $A_a(\xi)/2\pi$ are integers we obtain for β the condition

$$\beta = \pi^{-1}N, \quad N \in Z_+. \tag{3.8}$$

Upon substituting in Eq. (3.8) the value $\beta = R^2/4\pi\alpha'$ we obtain $R^2 = 4\alpha' N$, where the radius R is the parameter of the ordinary, closed string. Functional integration over the fields x is carried out for the full left-right theory, and the chiral sector is extracted only in the final formulas. The field $x(\xi)$ can be decomposed into a formal sum of two fields:

$$x(\xi) = x_+(\xi_+) + x_-(\xi_-),$$

and if the fields $x \pm (\xi \pm)$ are compactified on the circle S^1 of radius R_k then the full field is found to be compactified on a circle of radius $R = 2R_k$. Therefore $R_k^2 = \alpha' N$ and such values were obtained for heterotic strings previously. The relation $R = 2R_k$ can be interpreted to mean that in chiral strings only even numbers of vortices are considered on the world sheet of the corresponding left-right theory.

Up to this point we have confined ourselves to one cycle S^1. In the heterotic string it is necessary to consider the more general case of compactification on tori

$$T^d = \underbrace{S^1 \times \ldots \times S^1}_{d}.$$

Now the action equals

$$S = \beta \int d^2\xi\, a_{ij}(\partial_a x^i + A_a^i)(\partial_a x^j + A_a^j), \tag{3.9}$$

where a_{ij} is the matrix of the lattice Γ^d, such that $T^d = R^d/\Gamma^d$.

It is easy to show that the effective action for the vortices in the chiral sector of such a theory is obtained by the obvious generalization of Eq. (3.6):

$$S_{eff} = \frac{\beta}{4} a_{ij} \int d^2\xi\, d^2\xi'\, \partial_- A_+^i(\xi)\frac{1}{\partial_+\partial_-}\partial_- A_+^j(\xi'), \tag{3.10}$$

while the condition $\delta S_{eff} = 0 \pmod{2\pi}$ results now not only in quantization of β [see Eq. (3.8)], but also in a new condition on the matrix a_{ij}:

$$a_{ij} n_i m_j \in Z, \quad \forall n_i, m_j \in Z, \tag{3.11}$$

i.e., the lattice Γ^d should be integral. In this way gauge symmetry naturally allows compactification of chiral strings on integral tori only.

IV. INFLUENCE OF VORTICES ON THE SPECTRUM OF PHYSICAL STATES OF THE STRING

1. Vortex contribution to the correlation functions of the *XY* model

We show that taking into account corrections due to dipole vortices leads to renormalization of the constant β, corresponding to the compact dimension. We study the influence of dipole vortex configurations on the asymptotic behavior of the correlation functions for $\beta > \beta_c$. In this region the theory is in the conformal phase and the vortices combine with antivortices into molecules. The partition function of the dipole gas differs somewhat from expression (3.1) for the free plasma. It is given by

$$Z=\sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{i=1}^{N}\frac{d^2x_i\,d^2\mathbf{p}_i}{a^4}\exp\Big\{-\beta\Big[\sum_{i\neq j}U_{ij}+\sum_{i=1}^{N}(U_i(\mathbf{p}_i)+2\mu)\Big]\Big\}. \quad (4.1)$$

Here $U_i(\mathbf{p}_i)$ is the binding energy of the vortex molecule with dipole moment $\mathbf{p}_i$ (we shall assume that the charges of the vortices equal ± 1), and $U_{ij}=U(x_i-x_j,\mathbf{p}_i,\mathbf{p}_j)$ is the interaction energy of the ith and jth dipole. The total chemical potential of the vortex and antivortex, forming the molecule, equals 2μ.

The main contribution to the renormalization of the scale dimensions of the fields is due only to terms proportional to powers of $\langle\mathbf{p}^2\rangle$. Higher dipole corrections have the form $\langle\mathbf{p}^{2k}\rangle/z^{2k-2}$, and being of short range are of less interest.

First of all we study the contribution of the free dipole gas to the correlator (x is the angle variable on the circle)

$$D(\xi,\eta)=\langle\exp[imx(\xi)]\exp[-imx(\eta)]\rangle. \quad (4.2)$$

It turns out that taking into account the interaction between the dipoles has no effect on the result, i.e., we shall obtain an answer that is exact in the dipole approximation.

To perform the calculations we need to know the field of the vortex dipole. Although in going round a single vortex the field acquires the increment $x\to x+2\pi$, the dipole field at large distances is determined unambiguously as the phase accumulations of vortex and antivortex cancel each other. According to Eq. (2.15), the field produced at the point ξ by a dipole $\mathbf{p}$ located at the point z_0, is given by

$$x_{\mathbf{p}}(\xi,z_0)=\frac{1}{2i}\left[\ln\frac{\xi-z_0-\mathbf{p}/2}{\bar{\xi}-\bar{z}_0-\bar{\mathbf{p}}/2}-\ln\frac{\xi-z_0+\mathbf{p}/2}{\bar{\xi}-\bar{z}_0+\bar{\mathbf{p}}/2}\right]. \quad (4.3)$$

Expansion in a series in $|\mathbf{p}|/|\xi-z_0|\ll 1$ gives

$$x_{\mathbf{p}}(\xi,z_0)=[\bar{\mathbf{p}}(\xi-z_0)-\mathbf{p}(\bar{\xi}-\bar{z}_0)]/2i|\xi-z_0|^2. \quad (4.4)$$

The field at the point ξ due to N dipoles is given by the superposition of the individual fields. In Euclidean coordinates z_a, $z=z_1+iz_2$, we have

$$x_N(\xi)=\sum_{i=1}^{N}\frac{\mathbf{p}_{ai}\varepsilon_{ab}(\xi-z_i)_b}{|\xi-z_i|^2}. \quad (4.5)$$

The correlator of interest to us equals

$$\langle\exp[imx(\xi)]\exp[-imx(\eta)]\rangle=\sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{i=1}^{N}\frac{d^2x_i\,d^2\mathbf{p}_i}{a^4}$$
$$\times\exp\{im[x(\xi)-x(\eta)]\}\exp\Big\{-\beta\Big[\sum_{i=1}^{N}U(\mathbf{p}_i)+2\mu\Big]\Big\}$$
$$\times\Big\{\sum_{N=0}^{\infty}\frac{1}{N!}\int\prod_{i=1}^{N}\frac{d^2x_i\,d^2\mathbf{p}_i}{a^4}\exp\Big(-\beta\Big[\sum_{i=1}^{N}U(\mathbf{p}_i)+2\mu\Big]\Big)\Big\}^{-1}. \quad (4.6)$$

If terms containing higher moments $\langle\mathbf{p}^4\rangle$, $\langle\mathbf{p}^6\rangle$,..., are ignored then Wick's theorem holds and therefore

$$\langle\exp\{im[x(\xi)-x(\eta)]\}\rangle=\exp\{\bar{N}m^2[\langle x_{\mathbf{p}}(\xi)x_{\mathbf{p}}(\eta)\rangle-\langle x_{\mathbf{p}}^2\rangle]\}, \quad (4.7)$$

where $\bar{N}$ is the average dipole density,

$$\bar{N}=-\frac{1}{2\beta A}\frac{\partial}{\partial\mu}Z=\int\frac{d^2\mathbf{p}}{a^4}\exp[-\beta U(\mathbf{p})-2\beta\mu]. \quad (4.8)$$

The right side of Eq. (4.7) contains the field $x_{\mathbf{p}}$, produced by one dipole and averaged over its positions. The one-dipole correction may be represented graphically, see Fig. 1. The single line stands for the field of the dipole, Eq. (4.4), the dot stands for the dipole molecule. The correlator is

$$\langle x_{\mathbf{p}}(\xi)x_{\mathbf{p}}(\eta)\rangle-\langle x_{\mathbf{p}}^2\rangle=\int d^2z\,d^2\mathbf{p}[x_{\mathbf{p}}(\eta)x_{\mathbf{p}}(\xi)-x_{\mathbf{p}}^2(\eta)]$$
$$\times\exp(-4\pi\beta\ln|\mathbf{p}|)\Big[\int d^2\mathbf{p}\exp(-4\pi\beta\ln|\mathbf{p}|)\Big]^{-1} \quad (4.9)$$

Integration over $d^2\mathbf{p}$ is elementary:

$$\int d^2\mathbf{p}\,\mathbf{p}_a\mathbf{p}_b e^{-U(\mathbf{p})}={}^1/_2\delta_{ab}\langle\mathbf{p}^2\rangle, \quad (4.10)$$

and we obtain

$$\langle x_{\mathbf{p}}(\xi)x_{\mathbf{p}}(\eta)-x_{\mathbf{p}}^2(\eta)\rangle$$
$$=\frac{1}{2}\langle\mathbf{p}^2\rangle\int d^2z\left[\frac{(\eta-z)_a(\xi-z)_a}{|\eta-z|^2|\xi-z|^2}-\frac{1}{|z|^2}\right]. \quad (4.11)$$

The integral diverges logarithmically for $z\to 0$. We regularize it in the same way as the analogous divergence in the Green's function for the Laplace operator, i.e.,

$$\langle x_{\mathbf{p}}(\xi)x_{\mathbf{p}}(\eta)-x_{\mathbf{p}}^2(\eta)\rangle=-\frac{1}{2}\langle\mathbf{p}^2\rangle 2\pi\ln\frac{|\eta-\xi|^2+a^2}{a^2}. \quad (4.12)$$

The last integration is easily performed using the symbolic notation

$$\int d^2z\frac{(\xi-z)_a(\eta-z)_b}{|\xi-z|^2|\eta-z|^2}=4\pi^2\int d^2z\Big[\partial_a\frac{1}{\partial^2}(\eta-z)\Big]$$
$$\times\Big[\partial_a\frac{1}{\partial^2}(\xi-z)\Big]=-4\pi^2\int d^2z\frac{1}{\partial^2}\partial^2\frac{1}{\partial^2}=-4\pi^2\frac{1}{\partial^2}(\xi-\eta), \quad (4.13)$$

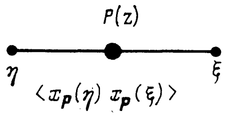

FIG. 1.

where

$$-\frac{1}{\partial^2}(\xi-\eta)=G(\xi,\eta)=-\frac{1}{2\pi}\ln\frac{(\eta-\xi)^2+a^2}{a^2}. \quad (4.14)$$

Let us give the expression for the full correlator with dipole corrections. The contributions of the dipoles Δ_v and the quantum fluctuations Δ^0 to the scale dimension are independent, so we have

$$\langle \exp\{im[x(\eta)-x(\xi)]\}\rangle=1/(\xi-\eta)(2\Delta^0+2\Delta_v)m^2. \quad (4.15)$$

The dimension Δ^0 is the same as for the free field:

$$\Delta^0=1/8\pi\beta, \quad (4.16)$$

while Eqs. (4.7) and (4.12) give for the vortex component

$$\Delta_v={}^1/_2\pi\bar{N}\langle \mathbf{p}^2\rangle=\chi/4\beta. \quad (4.17)$$

The quantity χ characterizes the polarizability of the dipole gas

$$\chi=2\pi\beta\bar{N}\langle \mathbf{p}^2\rangle=2\pi\bar{N}\langle\partial\mathbf{p}\rangle/\partial\mathbf{E}, \quad (4.18)$$

E is the electric field intensity. According to Eqs. (2.8) and (2.9) we have

$$\chi=\pi\beta e^{-2\beta\mu}/(\beta-\beta_c)(2\pi\beta-1). \quad (4.19)$$

It thus turns out that the one-vortex renormalization of the scale dimension can be described by one numerical parameter

$$\varepsilon=1+2\pi\chi \quad (4.20)$$

—the dielectric permeability of the Coulomb gas:

$$\Delta=\Delta^0\varepsilon. \quad (4.21)$$

In statistical physics this is achieved by temperature renormalization $T\to\varepsilon T$.

2. Multidipole corrections to the correlation functions of the *XY* model

We show now how to take into account the influence on the scale dimension of interactions between dipoles. It turns out that it is possible to sum up the corrections to the correlators in all orders of $\langle \mathbf{p}^2\rangle$, and the contributions of the higher moments ($\langle \mathbf{p}^4\rangle$ etc.) decrease rapidly and have no effect on the asymptotic behavior of the correlators at large distances.

The main assumption in our calculations is that the dipole approximation for the interaction of the vortex molecules is applicable down to short distances between them. This is justified for the vortex interaction potential, Eq. (2.18), but it is entirely possible that this result has a universal character.

We consider the correlation function

$$G_{ab}(\xi,\eta)=\langle\partial_a x(\xi)\partial_b x(\eta)\rangle, \quad (4.22)$$

where the averaging is performed in the standard manner with respect to the partition function, Eq. (4.1).

The gradient of the field due to a dipole $\mathbf{p}_c$ located at the point z is given by [see Eq. (4.14)]

$$\partial_a x(\xi)=\varepsilon_{ab}\partial_b\partial_c\frac{2\pi}{\partial^2}(\xi-z)\mathbf{p}_c. \quad (4.23)$$

This expression, exact in the dipole approximation, differs by $2\pi\varepsilon_{ab}\mathbf{p}_b\delta(\xi-z)$ from what would be obtained by direct differentiation of Eq. (4.5), valid for $|\xi-z|\gg|\mathbf{p}|$.

The interaction energy of two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ has the form

$$\frac{1}{8\pi^2}U(\mathbf{p}_1,\mathbf{p}_2,z_1-z_2)=-\mathbf{p}_{1a}\mathbf{p}_{2b}\frac{\partial}{\partial z_{1a}}\frac{\partial}{\partial z_{2b}}\frac{1}{\partial^2}(z_1-z_2)$$
$$=\left[\frac{\delta_{ab}z_{12}{}^2-2z_{12a}z_{12b}}{(z_{12}{}^2+a^2)^2}+\frac{a^2\delta_{ab}}{2\pi(z_{12}{}^2+a^2)}\right]\mathbf{p}_{1a}\mathbf{p}_{2b}, \qquad z_{12a}=z_{1a}-z_{2a}. \quad (4.24)$$

To find the n-dipole correction we expand $\exp(-\beta\Sigma U_{ij})$ in Eq. (4.1) in a series in U_{ij}. After averaging over the dipole moments (see below) it becomes apparent that a significant role is played only by terms of the form

$$G_{ab}^{(n)}=\int\left[\prod^n\frac{d^2\mathbf{p}_i\,d^2z_i}{a^4}\right.$$
$$\left.\times\exp[-\beta U(\mathbf{p}_i-2\beta\mu]\right]\partial_a x(\xi-z_1)\partial_b x(\eta-z_n)\prod_{i=1}^{n-1}(-\beta U_{i,i+1}). \quad (4.25)$$

They correspond to the line graphs, Fig. 2, where the double line symbolizes the dipole-dipole interaction, Eq. (4.24).

Further calculations proceed as follows. First, since the dipole moment of a single molecule is not large:

$$\langle \mathbf{p}^2\rangle=a^2/2\pi(\beta-\beta_c)\ll a^2, \quad (4.26)$$

one may integrate independently over the dipole moments $\mathbf{p}_i$ with the help of Eq. (4.10). Thereafter one should integrate over the coordinates of the molecules. We show how this is done using the ith dipole as an example. The contraction

$$\int d^2z_i\left[\partial_{i-1,a}\partial_{i,c}\frac{8\pi^2\beta}{\partial^2}(z_i-z_{i-1})\right]\frac{\bar{N}\langle\mathbf{p}^2\rangle}{2}\left[\partial_{i+1,b}\partial_{i,c}\right.$$
$$\left.\times\frac{8\pi^2\beta}{\partial^2}(z_{i+1}-z_i)\right] \quad (4.27)$$

is most simply evaluated by parts. The operators $\partial_c{}^2$ and $1/\partial^2$ cancel each other, and one of the double lines (Fig. 2) collapses into a dot leaving the factor $-4\pi^2\langle\mathbf{p}^2\rangle\bar{N}\beta$. This means that the line graphs represent a geometric progression whose sum equals

$$G_{ab}=G_{ab}^{(0)}+\frac{1}{2\beta}\sum_{k=1}^{\infty}(-4\pi^2\beta\langle\mathbf{p}^2\rangle\bar{N})^k\varepsilon_{am}\varepsilon_{bn}\frac{\partial}{\partial\xi_m}\frac{\partial}{\partial\eta_n}\frac{1}{\partial^2}(\xi-\eta), \quad (4.28)$$

where for noncoincident ξ and η one may replace $\varepsilon_{am}\varepsilon_{bn}$ by $-\delta_{an}\delta_{bm}$ since $\varepsilon_{am}\varepsilon_{bn}=\delta_{ab}\delta_{mn}-\delta_{an}\delta_{bm}$. The first term equals

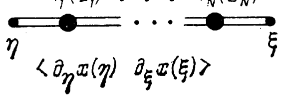

FIG. 2.

$$G_{ab}^{(0)}=-\frac{\partial}{\partial\xi_a}\frac{\partial}{\partial\eta_b}\frac{1}{2\beta}\frac{1}{\partial^2}(\xi-\eta), \quad (4.29)$$

after which we finally obtain

$$G_{ab}(\xi,\eta)=-\frac{1}{1+4\pi^2\beta\langle\mathbf{p}^2\rangle\bar{N}}\frac{1}{4\pi\beta}\frac{\partial}{\partial\xi_a}\frac{\partial}{\partial\eta_b}\ln|\xi-\eta|. \quad (4.30)$$

Turning to the equation for the renormalization of the scale dimension of fields e^{ix}, Eq. (4.21), we may rewrite Eq. (4.30) as

$$G_{ab}(\xi,\eta)=\frac{1}{\varepsilon}G_{ab}^{(0)}(\xi,\eta)=\frac{\Delta^0}{\Delta}G_{ab}^{(0)}(\xi,\eta). \quad (4.31)$$

The beauty of this expression suggests that it may be universal and model-independent.

In precisely the same way we may calculate dipole corrections to the Coulomb interaction of vortices, Eqs. (2.14) and (2.18), with the result

$$S=\frac{S^{(0)}}{\varepsilon}=-\frac{2\pi\beta Q_1Q_2}{\varepsilon}\ln\frac{(z_1-z_2)^2+a^2}{a^2}. \quad (4.32)$$

Thus taking the dipoles into account also reduces here to the renormalization of the temperature $T\to\varepsilon T$. In conclusion we call attention to the fact that we have made heavy use of the requirement $\langle\mathbf{p}^2\rangle\ll a^2$. This allowed us to first integrate over $d^2\mathbf{p}$, and then "contract" the propagators, which would be problematical for $\langle\mathbf{p}^2\rangle\sim a^2$, for example on the lattice or in our regularization scheme for $\beta<\frac{3}{2}\beta_c$. In that case the main contribution would come from configurations in which the distances between dipoles and between vortices in a dipole are of the same order. Starting from our results one may suppose that in that case one can no longer speak of dipoles, and higher corrections are determined by neutral multivortex clusters.

The last remark refers to the vanishing of multidipole corrections to the correlators $\langle x(\xi)x(\eta)\rangle$. One can verify that all higher ($N\geqslant2$) line graphs contain the operators $\partial_a\varepsilon_{ab}\partial_b=0$, so that the first dipole correction is also the last.

3. Multidipole corrections to the spectrum of states

Let us investigate how the spectrum of physical states of the closed string is affected by the interaction between the dipoles. It turns out that it results in the replacement of the radius R in Eq. (2.21) by the effective quantity

$$R_{eff}^2=R^2/\varepsilon. \quad (4.33)$$

The vertex operators of the physical fields are determined by Eq. (2.14). For brevity, as in Sec. II, we confine ourselves to the simplest term, in which only $n_1=N\neq0$. We demand that characteristic dimensions Δ, $\bar{\Delta}$ of the field $V(m/R,P,N,\tilde{N},L)$ equal unity. This condition, Eq. (2.16b), determines the mass spectrum. According to Eq. (2.18) it is sufficient to study the pair correlator of the field V with the field $V(-m/R,-P,N,\tilde{N},-L)$. In view of Wick's theorem it is expressible through the sum of products of all possible pair expectation values:

$$\left\langle:\left[\frac{\partial^N}{\partial\xi^N}x^\mu(\xi)\right]\exp[iPx(\xi)]::\exp[-iPx(\eta)]\frac{\partial^N}{\partial\eta^N}x^\nu(\eta):\right\rangle$$
$$=\left\langle:\exp[iPx(\xi)]::\exp[-iPx(\eta)]:\right\rangle\{\partial_\xi{}^N\partial_\eta{}^N\langle x^\mu(\xi)x^\nu(\eta)\rangle$$
$$+P_\lambda P_\sigma[\partial_\xi{}^N\langle x^\mu(\xi)x^\lambda(\eta)\rangle][\partial_\eta{}^N\langle x^\sigma(\xi)x^\nu(\eta)\rangle]\}$$
$$\times\left\langle\exp\left(\frac{-L^2R^2}{\alpha'^2}\ln|\xi-\eta|\right)\right\rangle. \quad (4.34)$$

The last factor arises from taking into account the Coulomb interaction of vortices with charges $\pm L$ at points ξ and η, corresponding to our solitons. Taking the dipoles into account results in screening, Eq. (4.32). In the first factor the field x contains the vortex component x_v, Eq. (2.15), besides which the dipoles renormalize the dimension of the field e^{iPx} itself, Eq. (4.21). Therefore

$$\langle:\exp[iPx(\xi)]::\exp[-iPx(\eta)]:\rangle$$
$$\propto|\xi-\eta|^{-P^2\alpha'-\varepsilon m^2\alpha'/R^2}\left(\frac{\xi-\eta}{\bar{\xi}-\bar{\eta}}\right)^{mL}. \quad (4.35)$$

As regards the second factor, its dimension is determined simply by the total order of the derivatives. In this way we find, taking Eqs. (4.35) and (4.32) into account, that

$$\Delta=P^2\alpha'/4+m^2\alpha'\varepsilon/4R^2-mL/2+L^2R^2/4\alpha'\varepsilon+N, \quad (4.36a)$$

and analogously

$$\bar{\Delta}=P^2\alpha'/4+m^2\alpha'\varepsilon/4R^2+mL/2+L^2R^2/4\alpha'\varepsilon+\tilde{N}. \quad (4.36b)$$

Upon equating Δ and $\bar{\Delta}$ to unity we indeed find the spectrum, Eq. (2.21), with the radius R replaced by R_{eff}, Eq. (4.33). We have thus proved that taking into account the dipole corrections to the conformal phase results in renormalization of the inverse temperature β of the compact dimension. It is true that we cannot say that this is all that happens, because the renormalization of the correlators

$$\langle xx\rangle\to\varepsilon\langle xx\rangle,\quad\langle\partial_\xi x,x\rangle\to\langle\partial_\xi x,x\rangle,$$
$$\langle\partial_\xi x,\partial_\eta x\rangle\to\frac{\langle\partial_\xi x,\partial_\eta x\rangle}{\varepsilon} \quad (4.37)$$

is different from that of the free theory. In addition it follows from Eq. (4.31) that it is necessary to redefine the term $\frac{1}{2}{:}\partial x^0\partial x^0{:}\to\frac{1}{2}\varepsilon{:}\partial x^0\partial x^0{:}$ in the energy-momentum tensor, i.e., the behavior of the cyclic degrees of freedom, differs from the others.

V. VORTICES IN FERMIONIC STRINGS

Our goal is the description of the motion of the superstring in a non-simply-connected space and the determination of the parameters of the phase transition. For simplicity we confine ourselves, as in the bosonic case, to the space $R^d\times S^1$. The action of the free superstring in the superconformal gauge has the form

$$S=\frac{1}{4\pi\alpha'}\int d^2\xi[\partial_a x^\mu\partial_a x^\mu+i\psi^\mu\gamma_a\partial_a\psi^\mu], \quad (5.1)$$

with the supertransformation law

$$\delta x^\mu=\varepsilon\psi^\mu,\quad\delta\psi^\mu=-i\gamma^a\varepsilon\partial_a x^\mu. \quad (5.2)$$

In considering the coordinate x, corresponding to S^1, and going to its covering, we should have introduced in Eq. (5.1) the gauge superfield, since Eq. (5.2) does not permit the introduction of the boson vector field $A_a(\xi)$ by itself, if supersymmetry on the world sheet is to be preserved. In the general case this program is quite tedious, but one may separately supersymmetrize the vortex contribution, which is nothing but Eq. (2.5)—the action of the Coulomb gas. A detailed description of the super-Coulomb gas is given in Ref. 11; here we shall only state some essential information.

The model is constructed in the superspace (ξ_a, θ), where θ is a two-dimensional spinor whose supertransformation law has the form

$$\xi_a' = \xi_a + i\varepsilon^+\gamma_a\theta, \quad \theta' = \theta + \varepsilon. \tag{5.3}$$

The partition function is a direct generalization of Eq. (3.1), where in place of every integration over $d^2\xi$ one should write the measure in superspace $\frac{1}{2}d^2\xi d\theta^+\gamma_5 d\theta$, and the logarithmic interaction in ordinary space is replaced by the same kind of interaction in superspace: $\ln(R^2(i,j)+a^2)/a^2$, where the distance between the points (ξ_a^i, θ^i) and (ξ_a^j, θ^j) is defined by the expression

$$R_a(i,j) = \xi_a^i - \xi_a^j + i\bar\theta^i\gamma_a\theta^j \tag{5.4}$$

and is invariant under the transformations (5.3). Finally

$$Z_{ferm} = \sum_{n_+=n_-=0}^{\infty} \lambda^{n_++n_-} \int \frac{d^2\xi_1 \ldots d^2\xi_{n_+} d\theta_1^+\gamma_5 d\theta_1 \ldots d\theta_{n_+}^+\gamma_5 d\theta_{n_+}}{2^{n_+}n_{n+}!}$$
$$\times \int \frac{d^2z_1 \ldots d^2z_{n_-} d\omega_1^+\gamma_5 d\omega_1 \ldots d\omega_{n_-}^+\gamma_5 d\omega_{n_-}}{2^{n_-}n_-!}$$
$$\times \exp\left[\pi\beta \sum_{i,j=1}^{n_+=n_-} q_i q_j \ln \frac{R^2(i,j)+a^2}{a^2}\right], \tag{5.5}$$

where $q_i = \pm 1$ are the charges of the vortices and antivortices. After integration over the Grassmann variables we obtain (for details see Ref. 11)

$$Z_{ferm} = \sum_{p=0}^{\infty} \frac{\lambda^{2p}}{(p!)^2} \int d^2\xi_1 \ldots d^2\xi_p d^2z_1 \ldots d^2z_p \, \Omega_{2p}(\xi_i, z_j)$$
$$\times \exp\left\{\pi\beta\left[\sum_{i,j=1}^{p} \ln\frac{|\xi_i-\xi_j|^2+a^2}{a^2} + \ln\frac{|z_{ij}-z_j|^2+a^2}{a^2}\right.\right.$$
$$\left.\left. - \ln\frac{|\xi_i - z_j|^2+a^2}{a^2}\right]\right\}, \tag{5.6}$$

$$\Omega_{2p}(\xi_i, z_j) = \left|\sum_M (-1)^M \frac{1}{t_{m_1m_2}} \cdots \frac{1}{t_{m_{2p-1}m_{2p}}}\right|^2, \quad t_{ij} = t_i - t_j, \tag{5.7}$$

where t_i denote the holomorphic variables ξ_i and z_j and the sum in Eq. (5.7) is over all possible permutations of M points. $(-1)^M$ is the parity of the permutation. The expression (5.6) differs from the partition function of the bosonic case by the factor (5.7), resulting in a different value of β_c. Let us consider the mean square value of the dipole moment of the vortex-antivortex pair, where the divergence corresponds to pair dissociation and BKT-transition:

$$\langle p^2\rangle = \int d^2\xi\, d^2\eta\, |\xi-\eta|^2 \exp\left(-2\pi\beta \ln\frac{|\xi-\eta|^2+a^2}{a^2}\right)\Omega_2(\xi,\eta)$$
$$\times\left[\int d^2\xi\, d^2\eta \exp\left(-2\pi\beta\ln\frac{|\xi-\eta|^2+a^2}{a^2}\right)\Omega_2(\xi,\eta)\right]^{-1}$$
$$\sim a^2\frac{\pi\beta}{2\pi\beta-1}, \quad \beta_c{}^f = \frac{1}{2}\pi^{-1} = \frac{1}{2}\beta_c, \tag{5.8}$$

i.e., the critical value for $\beta_c{}^f$, and therefore for $R_c{}^2$, is two times smaller than in the bosonic case.

There exists a field description of supervortices, analogous to Eq. (3.2), with the Lagrangian in this case being

$$\mathscr{L} = \frac{1}{2}(\partial_a\Phi)^2 + \frac{1}{2}\bar\psi i\gamma_a\partial_a\psi - \lambda\bar\psi\psi\cos[2\pi(2\beta)^{1/2}\Phi]. \tag{5.9}$$

The equivalence between the partition functions corresponding to Eqs. (5.9) and (5.6) is proved by term-by-term comparison of the series in powers of λ, just as in the bosonic case. The factor $\Omega_{2p}(\xi_i, z_j)$ [see Eq. (5.7)] arises from evaluation of thee average of the product $\bar\Psi\Psi$.

The Lagrangian, Eq. (5.9), differs from the supersymmetric Lagrangian in the sine-Gordon model by the absence of the term $(\lambda^2/4\pi^2\beta)\sin^2[2\pi(2\beta)\frac{1}{2}\Phi]$, which is of higher order in λ and immaterial in the limit of small λ.

The critical value $\beta_c{}^f = \frac{1}{2}\pi^{-1}$ can be obtained from Eq. (5.9) by, for example, bosonization $\bar\Psi\Psi \to \cos[2\pi(2\beta)\frac{1}{2}\Phi]$, $i\bar\Psi\delta_a\partial_a\Psi \to (\partial_a\Phi)^2$. Comparing the Lagrangian

$$\mathscr{L} = (\partial_a\Phi)^2 + \lambda'\cos[4\pi(2\beta)^{1/2}\Phi]$$

with Eq. (3.3) we arrive at the value $\beta_c{}^f = \frac{1}{2}\pi^{-1}$. As in the bosonic case there is the duality (3.4), and the action (5.9) describes the superstring in the tachyonic field condensate. Comparing $\beta_c{}^f = \frac{1}{2}\pi^{-1}$ with the spectrum of closed superstrings,

$$\alpha' M^2/2 = N + \tilde N - 1 + {}^1/_2 m^2\alpha'/R^2 + {}^1/_2 Q^2R^2/\alpha', \quad N - \tilde N = mQ, \tag{5.10}$$

we see that for $R^2 < R_c{}^2 = 2\alpha'$ a tachyon appears in the soliton sector with $N = \tilde N = m = 0, |Q| = 1$. Moreover, the GSO-projection condition must be taken into account for superstrings (see Ref. 1). This usually excludes the state with $N = \tilde N = 0$. Then there are no tachyons in the theory and the sum over spin structures results in the disappearance of vortex effects. However, if the spatial bosons and fermions have different boundary conditions on noncontractible cycles then the values $N = \tilde N = 0$ are allowed and vortex effects reappear. Examples are considered by Rohm in Ref. 18. The same happens in the theories at finite temperature, where the noncontractible cycle may be interpreted as imaginary time.

VI. STRING AT FINITE TEMPERATURES

Properties in thermodynamic equilibrium are conventionally described by means of the canonical Gibbs ensemble, whose main characteristic is the partition function

$$Z = \mathrm{Tr}\exp(-\beta\hat H) = \sum_n \exp(-\beta E_n), \tag{6.1}$$

with $\beta = T^{-1}$ the inverse temperature. All energy levels of the system are included in the sum over n. For a string ensemble one must sum in Eq. (6.1) over all types of string states, and there appear sums of the form

$$\sum_m d(m) \exp(-\beta m),$$

where $d(m)$ is the number of different states with the same mass m. For all types of strings (besides unphysical ones with $N = 2$ and $N = 4$ supersymmetry on the world sheet) the spectrum $d(m)$ has the form[12]

$$d(m) \sim m^{-a} \exp(\beta_H m), \tag{6.2}$$

where a and β_H are constants depending on the type of string. The exponentially growing spectrum (6.2) results in the existence of a critical temperature $T_H = \beta_H^{-1}$, above which the partition function, Eq. (6.1), does not exist due to the divergence as $m \to \infty$. The quantity a determines the behavior of the thermodynamic characteristics as $\beta \to \beta_H$. It is known (see Ref. 13 and the earlier work in Ref. 19), that for $a > (d+1)/2$ the thermodynamic characteristics do not diverge for $\beta \to \beta_H$. In that region one has large energy-density fluctuations and the very description with the help of the canonical ensemble is inapplicable—the microcanonical ensemble is needed. For $a < (d+1)/2$, on the contrary, the thermodynamic characteristics diverge as $\beta \to \beta_H$, i.e., β_H cannot be reached in principle. In that case T_H is a limiting temperature.

The parameters a and β_H were evaluated some time ago. Leaving out the details we note the results for the bosonic (B), super (SST) and heterotic (HS) strings (for details see, e.g., Ref. 1):

$$a_{open} = (d-1)/2, \quad a_{closed} = d, \quad d_B = 26, \quad d_{SST} = d_{HS} = 10 \tag{6.3}$$

The intriguing fact that arbitrary closed strings can be heated up to β_H, while the open ones cannot, without doubt requires further study. It may be that the existence in closed strings of two sectors—left and right—is of importance here. In some sense this corresponds to a hidden symmetry of our space—we observe only $x_L + x_R$, but it is important that they both exist. The quantity β_H turns out to be the same for open and closed strings:

$$\begin{aligned} \beta_{H,B} &= 2\pi[{}^1/_6 (d_B-2)\alpha']^{1/2} = 4\pi(\alpha')^{1/2}, \\ \beta_{H,SST} &= \pi[(d_{SST}-2)\alpha']^{1/2} = 2\pi(2\alpha')^{1/2}, \\ \beta_{H,HS} &= {}^1/_2(\beta_B + \beta_{SST}) = \pi(2+2^{1/2})(\alpha')^{1/2}. \end{aligned} \tag{6.4}$$

We now recall that the description of the system at nonzero temperature T is equivalent to a description in periodic imaginary time, i.e., in the space $R^{d-1} \times S^1$, where for the radius R we have

$$2\pi R = T^{-1} = \beta. \tag{6.5}$$

It is easy to see from Eqs. (6.4) and (6.5) that the critical temperatures correspond to the radii

$$R_B = 2(\alpha')^{1/2}, \quad R_{SST} = (2\alpha')^{1/2}, \quad R_{HS} = (1+2^{-1/2})(\alpha')^{1/2}, \tag{6.6}$$

reflecting, as we know already, the BKT phase transition on the world sheet and the appearance of tachyonic states in the soliton sector. The latter again explains the nature of the divergence of the partition function of the strings at finite temperature as the natural result of the appearance of new tachyonic modes in the spectrum.

To conclude this Section we would like to discuss the assumption made in Ref. 14 on the nature of the phase transition. It was noted that the soliton modes necessarily interact with the dilaton, and the effective potential describing the soliton field Φ and Φ^* and the dilaton σ has the form [Eq. (3.13) of Ref. 14]

$$V(\Phi, \Phi^*, \sigma) = (-4/\alpha' + 1/4\pi^2\alpha'^2 T^2)\Phi^*\Phi + g\sigma\Phi^*\Phi, \tag{6.7}$$

where T is the temperature and g is some constant.

The conclusion was drawn from the presence of the term $\sigma\Phi^*\Phi$ in Eq. (6.7) that a first-order phase transition occurs for $\beta > \beta_H = 4\pi(\alpha')^{1/2}$ due to the appearance of the appropriate vacuum expectation value of the field σ.

However, the shift in the field σ may be interpreted as a shift of the quantity $R^2 \sim 1/T^2$—the radius of S^1. Indeed, the dilaton vertex operator satisfies $V_\sigma \propto \partial_a x^\mu \partial_a x^\mu$, so for $\sigma \neq 0$ there appears an addition to the action proportional to $\sigma(\partial_a x^\mu)^2$, giving rise to the redefinition $R^2 \to R^2 + \text{const}\cdot\sigma$. It is precisely this feature that is reflected in Eq. (6.7), where one should keep in mind that R^2 is a massless field too (the metric is a graviton condensate). In fact only a redefinition of fields has taken place, after which a shift in the dilaton vacuum expectation value reduces simply to a redefinition of the string coupling constant.[20] Therefore, within the framework of the external field formalism we may not shift σ once the radius R (or the temperature T) has been fixed. Of course, the question of precisely what fixes R, remains open.

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[1)] We note that Eq. (2.4) is essentially the Villain[6,15] model, but it seems to us necessary to underline the gauge character of this model, which is important in the case of chiral strings (see Sec. III).

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