Development of Rayleigh-Taylor and bulk convective instabilities in the dynamics of plasma liners and pinches

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A solution is derived for the problem of the initial, linear stage of the growth of small perturbations in the course of the cylindrically symmetric compression and expansion of a plasma liner and a Z-pinch with a sharp boundary. In these systems, Rayleigh-Taylor instabilities localized near the plasma boundaries are the most dangerous. Bulk convective instabilities develop in addition to these Rayleigh-Taylor instabilities. The various instability modes, including local and global Rayleigh-Taylor modes, which grow in an accelerated plasma with distributed profiles of hydrodynamic variables, are classified. The spectra of the instability growth rates are calculated for plasma liners and Z-pinches. The shape of these spectra reveals an explanation of the stratification and filamentation of the plasma observed experimentally in pinches and liners. The imposition of a longitudinal magnetic field gives rise to a "stability window" in the space of the flow parameters. In this window, the Rayleigh-Taylor modes are suppressed completely by magnetic shear, while the bulk convective modes are suppressed to a significant extent. The longitudinal magnetic field B_{z0} required for an effective stabilization of an imploding plasma liner scales as $B_{z0} = (10\text{–}30\ \mathrm{kG}) I[\mathrm{MA}]/R_0[\mathrm{cm}]$, where R_0 is the initial radius of the liner, and I is the average current through the liner.

1. INTRODUCTION

In contrast with the well-developed theory for the MHD stability of equilibrium plasma configurations, which is presented in the familiar books on plasma physics (e.g., Refs. 1 and 2), many questions remain unresolved in the theory of the stability of the motion of an accelerated plasma. The list of physical problems for which questions of the stability of the plasma dynamics are of governing importance is exceedingly long: ranging from the ionospheric plasma and the interstellar plasma in a gravitational field to dense plasmas in the laboratory accelerated by electromagnetic or other mechanisms in pinches or laser-fusion devices. The stability of the plasma compression process determines the most important characteristics of the compression here: its uniformity and the limits on the concentration of energy. Research in this field is necessary (in particular) if we wish to use an imploding plasma liner as a light source for optical pumping of an active medium on the axis, while optimizing the pumping in terms of the uniformity and luminosity of the plasma, if we are interested in the uniformity and heating of the plasma during the compression, etc.[3,4]

The problem of the stability of accelerated plasma motion can be broken up somewhat arbitrarily into problems concerning two stages of the motion. First, there is the problem of the stability of the fronts of the shock waves which are usually excited in the early stage of the plasma acceleration and the Rayleigh-Taylor instabilities which occur at density gradients or discontinuities which are being accelerated after being created in a plasma by shock waves. Second, there are the problems of importance to the subsequent "well-developed" stage of the motion, in which the plasma flow is subsonic, and its instability is manifested as a competition between bulk-convective and Rayleigh-Taylor instability modes.

In the present paper we analyze the growth of instabilities in what is probably the most important stage: the well-developed stage of subsonic flow during the compression and expansion of the plasma of a Z-pinch and a plasma liner in a longitudinal magnetic field. The problem of the growth of small perturbations is solved in the linear stage of the instabilities in the course of an accelerated motion: the compression and expansion of the plasma liner and a Z-pinch with a sharp boundary, in which the plasma density vanishes at a finite distance from the axis. We analyzed the corresponding problem for a diffuse Z-pinch in Ref. 5. The stability problem is solved in the quasiclassical approximation for the time evolution of the characteristic perturbation growth rate. The Rayleigh-Taylor and bulk convective instability modes are classified. Explicit expressions for the growth rates of these modes are found in several cases. To illustrate the use of the theory and to carry out a qualitative study of the behavior of the various instability modes, we examine the stability of the motion of a hollow plasma liner and a Z-pinch whose unperturbed motion is described by self-similar solutions for a collisionless plasma.[5,6] To compare the results with the predictions of various simple models which have been proposed, we examine the limiting cases of an infinitely thin conducting liner and an incompressible fluid.

In contrast with the situation in a diffuse Z-pinch,[5] in the problems under consideration here the bulk convective modes may be accompanied by the most dangerous instability modes: Rayleigh-Taylor modes, either local or global. The spectra of the instability growth rates calculated for plasma liners and Z-pinches show that a comparatively weak longitudinal magnetic field at the beginning of the compression will give rise to a "stability window": a region in the space of the parameters of the plasma motion in which the Rayleigh-Taylor modes are completely suppressed, and the bulk convective modes are significantly suppressed. In particular, the possibility that a stability window will arise opens up the interesting possibility of achieving a stable compression of a liner or Z-pinch by shaping the current

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pulse to match the compression dynamics. The instability growth-rate spectra derived here also yield a qualitative explanation of the appearance of striations, stratification, and filamentation of plasmas observed in experiments with plasma liners and Z-pinches.

2. CALCULATION OF THE INSTANTANEOUS GROWTH RATE

Let us examine the growth of small hydrodynamic perturbations which arise during electrodynamic acceleration of a plasma, e.g., in the compression of the plasma column of a Z-pinch or in the compression of magnetic flux by a plasma liner. In any case, the unperturbed motion of the plasma in a pulsed system is characterized by some finite time τ. Instabilities which do not manage to grow substantially over a time τ pose no danger. The only instabilities which need by studied are those whose growth rates σ exceed τ^{-1}.

Under the assumption $\sigma\tau \gg 1$ we can define an "instantaneous growth rate" of an instability in a given unperturbed state of the plasma, by choosing as σ the largest of the growth rates characterizing the various perturbation modes at the given instant.[5] If the resulting value of σ satistfies the original assumption, the meaning is that it has been calculated with acceptable accuracy. In the opposite case, the perturbations are not dangerous at the given instant, and the error in the determination of σ is inconsequential. The approach which was taken to the calculation of σ in Ref. 5 is applicable in the well-developed stage of the motion, in which the shock waves and strong sound waves in the plasma have died out, and the motion of the plasma has become subsonic in the proper frame of reference. For the compression of shells of thickness δ and radius R, for example, the characteristic velocity gradient which is established in the shell would then be on the order of u/R, not u/δ, where u is a characteristic velocity. In such case, one can use the quasiclassical approximation which was developed in Ref. 5 to study the perturbations which grow most rapidly. In that approximation, one determines a maximum growth rate $\sigma(t)$, which characterizes the growth of plasma perturbations as a whole, at each time t, while the growth of the perturbations over the finite compression time τ is found as $\exp[\int_0^\tau \sigma(t)dt]$.

The quasiclassical approximation has been used in several studies (e.g., Refs. 7–9) of the development of instabilities. The formal condition for the applicability of this approximation can be written

$$\frac{1}{\sigma}\left|\frac{d\ln\sigma}{dt}\right| \ll 1.$$

This condition clearly holds for arbitrary $\sigma \neq 0$ near the time at which the motion comes to a halt. The accuracy of the quasiclassical approximation was discussed in Ref. 5, where it was shown that this approximation agrees well with the results of numerical calculations for problems of the type considered. It was pointed out in Ref. 5 that the instantaneous value of $\sigma(t)$ can be found as an eigenvalue of a corresponding boundary-value problem. A problem of this type reduces to a calculation of the growth rate of small perturbations for a plasma in an effective gravitational field corresponding to an acceleration

$$\mathbf{g}(\mathbf{r}, t) = -d\mathbf{u}(\mathbf{r}, t)/dt, \tag{1}$$

where $\mathbf{u}(\mathbf{r},t)$ is the velocity of the unperturbed motion. For a cylindrically symmetric radial unperturbed motion of the plasma column of a Z-pinch or liner in a magnetic field, the corresponding "equilibrium" equation is

$$\frac{dp}{dr} + \frac{1}{4\pi}\left[B_z\frac{dB_z}{dr} + \frac{B_\varphi}{r}\frac{d}{dr}(rB_\varphi)\right] = \rho g, \tag{2}$$

where

$$g \equiv -\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r}\right)$$

is minus the local acceleration of the plasma at the given instant, and ρ, p, B_z, and B_φ are the unperturbed values of the density, the pressure, and the magnetic-field components, respectively.

Taking account of the original cylindrical symmetry of the unperturbed motion, we can describe, at each instant, the perturbations manifested as a displacement of the plasma particles from their unperturbed paths by

$$\boldsymbol{\xi}(\mathbf{r}, t) = \boldsymbol{\xi}(r)\exp(\sigma t + im\varphi + ikz),$$

and we can reduce the equations for the perturbed motion to a single equation for the radial component ξ_r of the displacement:

$$\begin{aligned}
&\frac{d}{dr}\left\{\frac{1}{D}(\rho\sigma^2+f^2)\left[\rho\sigma^2\left(\gamma p + \frac{B^2}{4\pi}\right) + \gamma p f^2\right]\frac{1}{r}\frac{d}{dr}(r\xi_r)\right\} \\
&-\left\{\rho\sigma^2+f^2+\frac{B_\varphi}{2\pi}\frac{d}{dr}\frac{B_\varphi}{r} + g\frac{d\rho}{dr} - \frac{1}{D}\left[\rho^2g^2\left(k^2+\frac{m^2}{r^2}\right)\right.\right. \\
&\times(\rho\sigma^2+f^2) - \frac{kB_\varphi}{\pi r}\rho^2\sigma^2 g\left(kB_\varphi - \frac{m}{r}B_z\right) \\
&\left.+\frac{k^2B_\varphi^2}{\pi r^2}\left(\rho\sigma^2\frac{B^2}{4\pi} + \gamma p f^2\right)\right] - r\frac{d}{dr}\left(\frac{kB_\varphi}{2\pi r^2 D}\left(kB_\varphi - \frac{m}{r}B_z\right)\right. \\
&\left.\left.\times\left[\rho\sigma^2\left(\gamma p + \frac{B^2}{4\pi}\right) + \gamma p f^2\right] + \frac{\rho^2\sigma^2 g}{rD}(\rho\sigma^2+f^2)\right)\right\}\xi_r = 0,
\end{aligned} \tag{3}$$

where

$$D = \rho^2\sigma^4 + \left(k^2 + \frac{m^2}{r^2}\right)\left[\rho\sigma^2\left(\gamma p + \frac{B^2}{4\pi}\right) + \gamma p f^2\right]$$

$$f^2 = \frac{1}{4\pi}\left(kB_z + \frac{m}{r}B_\varphi\right)^2, \quad B^2 = B_z^2 + B_\varphi^2.$$

The ratio of specific heats $\gamma(r,t) = (\partial\ln p/\partial\ln\rho)$ is found here, for the unperturbed motion from the actual equation of state. We wish to stress that the derivation of Eq. (3) did not require any assumptions regarding the ideal nature of the unperturbed motion in terms of the equation of state or the absence of dissipation mechanisms. The only assumption was that the perturbations were adiabatic, so their wavelengths must not be too short. The boundary conditions on Eq. (3) follow from the requirement that the total pressure and the normal component of the magnetic field be continuous on the unperturbed plasma surface. We will write these conditions for a plasma column (or hollow liner) in vacuum, assuming that there are no discontinuities, current shells, etc., at the free plasma boundary. The density and thermal pressure of the plasma vanish at the the plasma boundary, and in the case of a hollow plasma liner there is only an axial magnetic field $B_z = B_{zi}$ in the cavity. The boundary condition at the inner boundary of the plasma $[r = R_i(t)]$ is thus

$$R_i\left(\frac{d\xi_r}{dr}\right)_{r=R_i}=\left[\frac{k^2R_i^2+m^2}{kR_i}\frac{I_m(kR_i)}{I_m'(kR_i)}-1\right]\xi_r; \quad (4)$$

If there is no inner plasma boundary, condition (4) is replaced by the condition that the solution be regular on the axis:

$$r\xi_r=0 \quad \text{as} \quad r\to 0. \quad (4a)$$

The boundary condition on the outer boundary of the plasma $[r=R_e(t)]$ is

$$R_e\left(\frac{d\xi_r}{dr}\right)_{r=R_e}=\left[\frac{k^2R_e^2+m^2}{kR_e}\frac{K_m(kR_e)}{K_m'(kR_e)}+\frac{mB_{\varphi e}-kR_eB_{ze}}{mB_{\varphi e}+kR_eB_{ze}}\right]\xi_r, \quad (5)$$

where I_m and K_m are modified Bessel functions, and B_{ze} and $B_{\varphi e}$ are the values of the static axial magnetic field and of the azimuthal magnetic field outside the plasma as $r\to R_e(t)+0$. In those special cases in which Eq. (3) has a singular point at a boundary, e.g., in the case $m=0, B_{ze}=0$, and $r\to R_e(t)$, the corresponding boundary condition is replaced by the requirement that ξ_r be regular at this boundary.[2)]

As in Ref. 5, the maximum value of σ is found as the eigenvalue of boundary-value problem (3)–(5) for which the eigenfunction $\xi_r(r)$ does not vanish on the interval between $r=R_i(t)$ and $r=R_e(t)$ [or between $r=0$ and $r=R_e(t)$.] The boundary conditions (4) and (5) above correspond to a smooth joining of the solutions of Eq. (3) in the plasma and in the vacuum region outside the plasma, where $\rho=0$ and $p=0$. The displacement ξ for the vacuum region is found from the condition that the expression for the perturbed magnetic field $\mathbf{B}^{(1)}$ in terms of ξ in vacuum remain of the same form as inside the plasma[11]: $\mathbf{B}^{(1)}=\mathrm{curl}[\xi\times\mathbf{B}]$. Solutions of Eq. (3) which are regular at $r=0$ and $r\to\infty$ are[12]

$$\xi_r=\begin{cases} I_m'(kr), & 0\leqslant r\leqslant R_i(t),\\ \dfrac{kr}{krB_z+mB_\varphi}K_m'(kr), & r\geqslant R_e(t),\end{cases} \quad (6)$$

where $B_z=B_{ze}=\text{const}$ and $B_\varphi=B_{\varphi e}(R_e/r)$. In the problem of determining the maximum growth rate, a plasma column of finite radius and a plasma liner can thus be thought of as the limiting cases of a diffuse plasma column between $r=0$ and $r=\alpha$ in which ρ and p tend toward zero for $0\leqslant r\leqslant R_i$ and $R_e\leqslant r\leqslant\alpha$. The boundary conditions correspond to the vanishing of $r\xi_r$ at $r=0$ and $r=\alpha$ as $\alpha\to\infty$. We can thus conclude that the results of Ref. 13 remain in force here. According to those results, the spectrum pf positive eigenvalues σ^2 of a given boundary-value problem which correspond to the growth rates of the various radial instability modes for given m and k is discrete and is bounded from above (it may have a condensation point $\sigma^2=0$). Consequently, the maximum eigenvalue σ^2 which we are seeking does indeed correspond to an eigenfunction which has the behavior specified above.[13] This is an isolated eigenvalue, so the problem of calculating it is simplified; for example, one could use the standard regula falsi method. In the degenerate case $f^2\equiv 0$, in which the magnetic field is not perturbed, we cannot work directly from the study of the properties of the spectrum of σ^2 which was carried out in Ref. 13. It is clear, however, that these properties remain the same in this case, since they prevail for any small $f^2>0$.

For a detailed study of the important limiting cases of short-wavelength perturbations and a thin cylindrical shell we can ignore effects of the cylindrical geometry. For such limiting cases we have the problem of calculating the instability growth rates for a plasma slap which is oriented parallel to the xy plane and which lies between $z=-\alpha$ and $z=+\alpha$ in an effective gravitational field, which is directed along the z axis. The magnetic fields $\mathbf{B}=\mathbf{B}_-$ at $z<-\alpha$ and $\mathbf{B}=\mathbf{B}_+$ at $z>\alpha$ are assumed to be parallel to the plane of the slab. We assume that the x and y dependence of the perturbation is of the form $\exp(ik_xx+ik_yy)$. Working directly [or from (3), taking the limit $1/r\to 0$ and making the substitutions $\varphi\to x$, $z\to y$, $r\to z$, $k\to k_y$, and $m/r\to k_x$, we can easily derive "equilibrium" equations analogous to (2) and (3):

$$\frac{dp}{dz}+\frac{1}{4\pi}\left(B_x\frac{dB_x}{dz}+B_y\frac{dB_y}{dz}\right)=\rho g. \quad (2a)$$

We also find an equation for ξ_z:

$$\frac{d}{dz}\left\{\frac{1}{D}(\rho\sigma^2+f^2)\left[\rho\sigma^2\left(\gamma p+\frac{B^2}{4\pi}\right)+\gamma pf^2\right]\frac{d\xi_z}{dz}\right\}$$
$$-\left\{\rho\sigma^2+f^2+g\frac{d\rho}{dz}-\frac{1}{D}k^2\rho^2g^2(\rho\sigma^2+f^2)\right.$$
$$\left.-\frac{d}{dz}\left[\frac{\rho^2\sigma^2g}{D}(\rho\sigma^2+f^2)\right]\right\}\xi_z=0, \quad (3a)$$

where

$$D=\rho^2\sigma^4+k^2\left[\rho\sigma^2\left(\gamma p+\frac{B^2}{4\pi}\right)+\gamma pf^2\right],$$

$$f^2=\frac{1}{4\pi}(k_xB_x+k_yB_y)^2, \quad k^2=k_x^2+k_y^2, \quad B^2=B_x^2+B_y^2.$$

In the general case, the boundary conditions are

$$d\xi_z/dz\pm k\xi_z=0, \quad z=\pm\alpha, \quad (4b)$$

where $k>0$. In those special cases in which Eq. (3a) has a singular point at $z=-\alpha$ or $z=\alpha$, the corresponding boundary condition is replaced by the requirement that ξ_z be regular at the corresponding boundary. If the plasma fills the upper half-space ($z>-\alpha$) or the lower one ($z<\alpha$), we are left with only one of the boundary conditions in (4b)—that which corresponds to the minus sign or the plus sign, respectively.

As above, the maximum eigenvalue σ^2 corresponds to an eigenfunction which does not vanish in the interval between $z=-\alpha$ and $z=\alpha$ (or in the half-space occupied by the plasma). Everything which we said above regarding the spectral properties of boundary-value problem (3)–(5) also remains in force.

3. RAYLEIGH-TAYLOR INSTABILITY MODES

3.1. Spectrum of growth rates of an instability caused by a discontinuity in hydrodynamic variables

In the literature, the "Rayleigh-Taylor instability" is usually understood as the well-known instability of the surface of a "heavy liquid," usually assumed to be incompressible, which is supported by a "light liquid" in an effective gravitational field. Playing the role of this light liquid might be, for example, a magnetic field which accelerates or confines a conducting plasma. The classical expression for the growth rate of the Rayleigh-Taylor instability, which is the limiting case of a convective or "interchange" instability in the latter case,[8] is

$$\sigma^2 = |g|k; \tag{7}$$

we are assuming $k > 0$ everywhere. We will now show how this result can be derived from general equation (3a).

In Eq. (3a) we take the limit of an incompressible fluid, $\gamma \to \infty$. Introducing the new variable ζ be means of the relation

$$d/d\zeta = (\rho\sigma^2 + f^2)\, d/dz, \tag{8}$$

we can put (3a) in the form

$$d^2\xi_z/d\zeta^2 - [k^2(\rho\sigma^2+f^2)^2 + gk^2 d\rho/d\zeta]\xi_z = 0. \tag{9}$$

We assume that the hydrodynamic variables vary in a slab whose thickness δ is small in comparison with the perturbation wavelength; i.e., we assume $k\delta \ll 1$. The variation in g within the slab is small. We can then set

$$d\rho/d\zeta = (\rho_2 - \rho_1)\delta(\zeta), \tag{10}$$

where ρ_1 and ρ_2 are the values of the density on the opposite sides of the slab. Under the assumption that the hydrodynamic variables vary slowly over a length scale $1/k$ outside the slab, we choose quasiclassical solutions of (9) outside the slab which vanish with distance from the slab:

$$\xi_z \propto \exp\left[\pm k \int (\rho\sigma^2 + f^2)\, d\zeta\right], \tag{11}$$

where the plus and minus signs correspond to the solutions on the opposite sides of the slab. We can integrate (9) over a thin slab and find an expression for the discontinuity in $d\xi_z/d\zeta$ by means of (11). We find an expression for the result of the integration of the second term in (9) by means of (10). We then find that the condition $\sigma^2 > 0$ can be satisfied only under the condition $g(\rho_2 - \rho_1) < 0$ and that in this case we have

$$\sigma^2 = k\frac{|g(\rho_2-\rho_1)|}{\rho_2+\rho_1} - \frac{1}{4\pi}\frac{(\mathbf{kB}_1)^2 + (\mathbf{kB}_2)^2}{\rho_2+\rho_1}, \tag{12}$$

where the vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ correspond to the magnetic field on the opposite sides of the slab. Expression (12) is well known.[1] In the case $\rho_1 = 0$, $\mathbf{k}\cdot\mathbf{B}_1 = \mathbf{k}\cdot\mathbf{B}_2 = 0$, it becomes (7).

A corresponding expression for the spectrum of growth rates of a "density-step" instability can be found from (3a) in the case of a compressible medium also. For this purpose we should assume that the slab thickness δ is small in comparison with the length scale of the variation in the quantities outside the slab, $L = c_f^2/g$, where $c_f = [(\gamma p/\rho) + (B^2/4\pi\rho)]^{1/2}$ is the velocity of a fast magnetosonic wave accross the magnetic field. In the case $\mathbf{k}\cdot\mathbf{B}_1 = \mathbf{k}\cdot\mathbf{B}_2 = 0$, and for perturbation wavelengths λ satisfying the condition

$$\delta \ll \lambda \ll L, \tag{13}$$

we can take the limit $k \to \infty$ in Eq. (3a). An expression like (10) remains in force for the density discontinuity. We can then put (3a) in the form

$$\frac{d}{dz}\left(\frac{\rho\sigma^2}{k^2}\frac{d\xi_z}{dz}\right) - \left[\rho\sigma^2 + g\frac{d\rho}{dz} - \frac{\rho^2 g^2}{\gamma p + B^2/4\pi}\right]\xi_z = 0. \tag{14}$$

By virtue of inequality (13), the last term in square brackets is small in comparison with the first, and we can put (14) in the form of (9) with $f^2 = 0$. As above, we then find the Rayleigh-Taylor spectrum:

$$\sigma^2 = k|g(\rho_2 - \rho_1)|/(\rho_2 + \rho_1). \tag{15}$$

3.2. Global modes of the Rayleigh-Taylor instability

In a more general case, we would be dealing with distributed profiles of the density, pressure, and current in the plasma, so it is worthwhile to explicitly analyze situations in which the spectrum of the most rapidly growing instability modes of the system is of the form in (7). We refer to instabilities with a spectrum of this sort as "Rayleigh-Taylor instabilities in the general sense."

We will show that instabilities of this type occur in the following cases. First, they occur if the unperturbed motion satisfies certain special conditions. Rayleigh-Taylor instability (7) then occurs globally, over the entire range of wave numbers k and m (or of k_x and k_y.) The perturbation profiles are independent of the profiles of the hydrodynamic variables of the unperturbed motion at any instant. Second, for large wave numbers, Rayleigh-Taylor spectrum (7) may be realized asymptotically. The corresponding modes would be localized near the surface at which the instability occurs, and the profiles of the perturbations would again be asymptotically independent of the unperturbed profiles near the boundary.

A global Rayleigh-Taylor mode is realized under the condition $\mathbf{k}\cdot\mathbf{B} = 0$, under which the perturbations do not bend the magnetic field lines. In this case the boundary conditions reduce to the requirement that the thermal pressure be continuous at the perturbed plasma surface. The boundary conditions are satisfied automatically if we have $\nabla\xi = 0$ throughout the volume and at the surface of the plasma, i.e., if the pressure is completely unperturbed in each part of the plasma. In the latter case one can show[14] that we also have $[\nabla\boldsymbol{\xi}] = 0$, i.e., that the displacement can be written in the form $\xi = \nabla\Phi$, where Φ is a harmonic function: $\nabla^2\Phi = 0$. In this case, the profiles of ξ are obviously independent of the profiles of the hydrodynamic variables of the unperturbed motion.

Let us consider a very simple example in plane geometry. Since the terms containing f^2 vanish under the condition $\mathbf{k}\cdot\mathbf{B} = 0$ in Eq. (3a), we can show that perturbations of the type $\xi_z = \exp(\pm kz)$ satisfy Eq. (3a) identically for $\sigma^2 = \pm gk$, regardless of the profiles of the unperturbed hydrodynamic variables, provided that the condition $g(z) = \text{const}$ holds. To demonstrate this point, it is sufficient to substitute $d\xi_z/dz = \pm k\xi_z$ into (3a) and to make use of the following expression, which holds in this case:

$$D = \rho\sigma^2[\rho\sigma^2 + k^2(\gamma p + B^2/4\pi)]. \tag{16}$$

If $g < 0$, and if the plasma fills the upper half-space, $z > 0$, a perturbation mode which is bounded everywhere at $z > 0$ is $\xi_z = \exp(-kz)$, which corresponds to growth rate (7). The perturbation decays exponentially with distance from the unstable surface ($z=0$). Since an exponential eigenfunction satisfies the boundary conditions, and since it vanishes nowhere, the corresponding eigenvalue is, in accordance with the discussion in Sec. 2, the growth rate of the most rapidly growing instability. This point was demonstrated especially for the Rayleigh-Taylor instability in Ref. 15.

If the plasma fills the lower half-space, $z<0$, there is no instability. A global perturbation mode $\xi_z = \exp(kz)$, which corresponds to the dispersion relation $\sigma^2 = -|g|k$ with purely imaginary eigenvalues $\sigma = \pm i\omega$, is a surface (Rayleigh) wave. For a plasma slab which is supported or accelerated in a magnetic field whose direction does not change inside or near the slab, so that we have $\mathbf{k}\cdot\mathbf{B}=0$ everywhere, we will see the development of a global Rayleigh-Taylor instability and a simultaneous propagation of surface waves. The global nature of these modes, whose spectrum does not depend on the profiles of the unperturbed variables, was established for the case of an incompressible fluid in Ref. 16 and for an arbitrary equation of state of the unperturbed medium in Ref. 17.

In the case of spherical geometry[14] or cylindrical geometry,[18] an global Rayleigh-Taylor mode exists in the case of a linear acceleration profile in the plasma volume:

$$g(r)/r=\text{const}. \tag{17}$$

Profiles of this type are typical of, for example, uniform compression of a plasma, including self-similar compression (see Refs. 5, 6, and 14 and Secs. 5 and 6 below). In spherical geometry these modes exist for arbitrary values of the wave numbers l and m; we should set $k=(l+1)/R$ in (7). For cylindrical geometry it can be shown that global Rayleigh-Taylor modes exist only for $\mathbf{B}_\varphi = 0$, $k=0$, $m\neq 0$, since the dispersion relation analogous to (7) in this case is

$$\sigma^4 = g^2(k^2+m^2/r^2). \tag{18}$$

In other words, this relation can be satisfied throughout the plasma volume under condition (17) only if we set $k=0$. [We are not considering here the special case of a profile of g for which relation (18) can be satisfied identically even in the case $k\neq 0$.] Solutions for perturbation profiles ξ_r which satisfy Eq. (3) identically under these conditions are of a power-law form: $\xi_r = r^{-1\pm m}$. Furthermore, the eigenfunctions do not vanish at $r>0$; i.e., the corresponding eigenvalues for which we have $\sigma^2>0$ correspond to the growth rates of the most rapidly growing modes.

These instability modes are probably the ones of greatest importance during the compression of a magnetic field B_z by a cylindrical liner, either a plasma liner or a metal liner, which is converging on an axis. We recall that the behavior of the global modes which we are discussing here does not depend on the equation of state of the unperturbed medium.

3.3. Local modes of the Rayleigh-Taylor instability

Another possible manisfestation of the Rayleigh-Taylor instability stems from the development of local perturbations near a plasma surface at large wave numbers. In the short-wavelength limit, we can restrict the analysis to plane geometry. We assume here that at the surface at which the plasma density ρ vanishes a perturbation does not bend the magnetic field lines; i.e., we have $\mathbf{k}\cdot\mathbf{B}=0$ and $f^2=0$. We assume that the shear characterizing the unperturbed magnetic field is inconsequential on the other surface, i.e., that f^2 falls off more rapidly than ρ toward the boundary. We might say that in this case the plasma boundary can be assumed to be abrupt regardless of the density profile near it. Near the boundary, Eq. (3a) takes the following form in this case:

$$\frac{d}{dz}\left(\rho\frac{d\xi_z}{dz}\right)-k^2\left(\rho-\frac{|g|}{\sigma^2}\frac{d\rho}{dz}\right)\xi_z=0. \tag{19}$$

This equation is typical of an incompressible fluid (the limit $\gamma\to\infty$). Here we assume $g=\text{const}(z)<0$ near the surface, that the plasma fills the half-space $z>0$, and that $\rho(0)=0$. The boundary conditions in this case reduce to the requirement that ξ_z be regular at $z\geqslant 0$. It is easy to see that for any profile $\rho(z)$ a solution of Eq. (19) which satisfies the boundary conditions is

$$\xi_z=\exp(-kz) \tag{20}$$

and that the corresponding eigenvalue is indeed given by expression (7). The exponential decay of the solutions into the interior of the plasma with a length scale $1/k$ justifies the approximations which were made in the derivation of Eq. (19), since in the short-wavelength limit we can indeed assume that the value of g is constant over this length scale, and we can ignore terms of higher order in ρ in comparison with those which we have retained in (19).

As we have already mentioned, the Rayleigh-Taylor mode which grows most rapidly has a feature which distinguishes it from other modes. Solution (20) differs from the other solutions of the same boundary-value problem in that it does not depend on the unperturbed profiles and is divergence-free; i.e., it locally reproduces the properties of global Rayleigh-Taylor modes. We can illustrate the situation in the simple example of a power-law profile of the unperturbed plasma density near the boundary: $\rho\propto z^s$, where $s=1,2,\ldots$. In this case Eq. (19) reduces to an equation for the confluent hypergeometric function.[19] The spectrum corresponding to the boundary conditions is

$$\sigma_n^2=|g|ks/(s+2n), \quad n=0,1,2,\ldots; \tag{21}$$

The corresponding eigenfunctions are expressed in terms of generalized Laguerre polynomials:

$$\xi_{z,n}=\exp(-kz)L_n^{(s-1)}(2kz). \tag{22}$$

We see that all eigenvalues and eigenfunctions other than those which correspond to the $n=0$ local Rayleigh-Taylor mode depend explicitly on the parameter s, which characterizes the density profile near the boundary. Consequently, under the restrictions on the magnetic shear near the plasma boundary which we listed above, this boundary behaves exactly as a sharp boundary would, regardless of the nature of the unperturbed profiles near it. In other words, it behaves in the manner of a density step, generating a classical Rayleigh-Taylor spectrum of instability growth rates, (7).

In the short-wavelength limit with which we are concerned here the spectrum of local instability modes can be studied asymptotically at $n\gg 1$ in the quasiclassical approximation. We put Eq. (19) in the form

$$\frac{d^2\xi_z}{d\zeta^2}+q^2(\zeta)\xi_z=0, \tag{23}$$

where ζ is defined by (8), and

$$q^2=k^2\rho^2\left[-1+\frac{|g|}{\sigma^2\rho^2}\frac{d\rho}{d\zeta}\right]. \qquad (24)$$

The eigenvalues σ^2 for Eq. (23) with the boundary conditions specified above can be found from the Bohr-Sommerfield quantization condition,[20] which is asymptotically exact at $n\gg 1$:

$$\int q\,d\zeta=k\int dz\left[-1+\frac{|g|}{\sigma^2}\frac{d\ln\rho}{dz}\right]^{1/2}=\pi(n+1/2), \qquad (25)$$

where $n = 0, 1, 2, \ldots$. The integration over z in (25) is restricted to the "classically accessible" region, in which we have $q^2>0$. For example, for the power-law density profiles which we examined above, we find the following spectrum from (25):

$$\sigma_n^2=|g|ks/(2n+1). \qquad (26)$$

As expected, this spectrum is asymptotically the same as (21) at $n\gg s$.

3.4. Effect of shear on the spectrum of local Rayleigh-Taylor modes

We turn now to the question of just when it is legitimate to ignore the shear of the unperturbed magnetic field near the plasma boundary, i.e., to ignore the increase in the stabilizing term f^2 with distance into the plasma. By assumption, this term is zero at the plasma surface. It can be seen from (12) that if the magnetic field changes direction abruptly at the plasma surface there will be no growth of sufficiently short-wavelength instabilities. We can show that local Rayleigh-Taylor modes can indeed be suppressed when there is a shear, regardless of the profile of the unperturbed density near the boundary. This circumstance gives rise to a "stability window" in the space of parameters of the accelerated motion of the plasma. Within this window, the basid MHD instability modes are suppressed.

We assume $k = k_x$, and we assume that at $z = 0$ we have $B_x = 0$. We can then write

$$f^2=k^2B_x^2/4\pi. \qquad (27)$$

Integrating the equilibrium equation (2a), we find

$$\frac{B_x^2}{8\pi}+\frac{B_y^2-B_y^2(0)}{8\pi}+p=\int_0^z\rho g\,dz'. \qquad (28)$$

It follows that in the general case in which all terms on the left side of (28) are of the same order of magnitude the quantity B_x^2 falls off more rapidly than ρ toward the plasma boundary. For example, if we have $\rho\propto z^s$ near the boundary then we have $B_x^2\propto z^{s+1}$, i.e., the value of f^2 at a point sufficiently close to the boundary is small in comparison with $\rho\sigma^2$. We made use of this relation in the derivation of Eq. (19) in Subsection 3.3. Expression (27) for f^2 contains the large factor k^2, so the effect of this form should be examined in more detail. For simplicity we consider a power-law density profile near the boundary:

$$\rho=\rho_0(z/L)^s. \qquad (29)$$

According to (28) we then have

$$f^2=\frac{k^2B_\parallel^2}{4\pi}\left(\frac{z}{L}\right)^{s+1}, \qquad (30)$$

where L is a length scale of the variations along the z direction, and ρ_0 and $B_\parallel^2$ are characteristic values of the plasma density and of $(\mathbf{k}\cdot\mathbf{B})^2/k^2$. Retaining the term f^2 in (3a) in the short-wavelength limit, we again find Eq. (9). For an approximate determination of the eigenvalues σ^2, we again (as in Subsection 3.3) use the Bohr-Sommerfeld quantization condition in the form

$$k\int dz\left[-1+\frac{|g|d\rho/dz}{\rho\sigma^2+f^2}\right]^{1/2}=\pi(n+1/2). \qquad (31)$$

Substituting (29) and (30) into (31), and integrating, we find

$$[K(Q)-E(Q)]/\chi Q=\pi(n+1/2), \qquad (32)$$

where

$$\chi^2=\frac{2}{s}\frac{B_\parallel^2}{8\pi\rho_0|g|L},\qquad Q^2=\left[1+\left(\frac{\sigma_2}{2gks\chi}\right)^2\right]^{-1},$$

and $K(Q)$ and $E(Q)$ are complete elliptic intervals. The dimensionless paramter χ characterizes the effect of the shear on the local Rayleigh-Taylor modes. It is easy to see that in the limit $\chi\to 0$ spectrum (32) is the same as expression (26) found earlier. On the contrary, the limit $\chi\to\infty$ corresponds to a suppression of the Rayleigh-Taylor modes by the shear. In this case we find

$$\frac{\sigma^2}{|g|k}=2^{7/2}s\chi\exp\left[-\frac{\pi}{2}\left(n+\frac{1}{2}\right)\chi\right]. \qquad (33)$$

It can be seen from (33) that the growth rates of modes with $n\gg 1$ are exponentially small at large values of χ. If we assume that (33) gives us a rough estimate of the growth rate of the fundamental ($n = 0$) mode, we see that the latter is also exponentially small. We recall that according to the discussion in Sec. 2 this mode is again the dominant one for the given values of the wave numbers ($k_x\neq 0$, $k_y = 0$).

What is the physical meaning of the parameter χ? For a thin shell inside which a magnetic field B_z is being accelerated by the pressure of an azimuthal magnetic field B_φ we have

$$|g|\approx|B_\varphi^2-B_z^2|/8\pi\rho_0L. \qquad (34)$$

where L represents the thickness of the shell in this case. During the inward acceleration of the liner, under the condition $B_\varphi^2>B_z^2$, the Rayleigh-Taylor mode localized on the outer surface of the plasma is dominant for constrictions (perturbations with $m = 0$) according to the discussion above, and in this case we have $B_\parallel = B_z$. During the stopping stage with $B_z^2>B_\varphi^2$, perturbations with $m\gg 1$, which have the same Rayleigh-Taylor spectrum, are predominant, and in this case we have $B_\parallel\equiv B_\varphi$. According to (34), and under the assumption $s\sim 1$, we have

$$\chi^2\sim\min(B_z^2,B_\varphi^2)/|B_\varphi^2-B_z^2|. \qquad (35)$$

We see that if B_z and B_φ are close enough in magnitude the value of χ will be large, and the growth rate of the local Rayleigh-Taylor mode exponentially small, despite the finite acceleration (g) of the plasma shell. This effect is specific to a plasma shell of finite thickness and cannot be derived in the approximation of an infinitely thin shell (Sec. 4).

Figure 1 shows spectra of the instability growth rates of a plasma liner of finite thickness (the ratio of the liner thickness δ to its average radius R is 1) found through a numerical solution of boundary-value problem (3)–(5). The liner is accelerated inward; the corresponding unperturbed solution is given below in Sec. 5. Here we have $s = 2$, and the curves have been plotted for various values of χ and B_z at a fixed value of the current through the liner. In this case B_z in (35) is the field inside the liner, and B_φ is the azimuthal field at the outer boundary of the liner. We see that, in agreement with (33), at sufficiently large values of χ, reached by virtue of an increase in B_z, the growth rate falls off more rapidly than $|g|$, while retaining the characteristic Rayleigh-Taylor shape $\sigma \propto k^{1/2}$. This result means that over the time required to reach a fixed velocity increment $\Delta u = |g|\Delta t$ the development of the Rayleigh-Taylor instability, characterized by the value of $\sigma(t)\Delta t$, can be kept arbitrarily small. It follows that it is possible to use shear to suppress the most dangerous instability (local Rayleigh-Taylor) mode.

4. SPECTRA OF THE INSTABILITY GROWTH RATES OF AN INFINITELY THIN LINER

Among the various problems involving calculation of instantaneous instability growth rates for the compression of magnetic flux in a hollow cylindrical liner by a longitudinal current flowing through the liner, we focus here on the limiting case of an infinitely thin, ideally conducting liner. This is the simplest problem in this category for which an exact analytic solution can be found. It is convenient to work directly from the equations of motion of a thin liner in these calculations.

The equation of motion of a perturbed element $R(t)d\varphi dz$ of the liner surface is[21]

$$\frac{\partial}{\partial t}\left[\varkappa(t)R(t)\left(\mathbf{e}_r\frac{dR}{dt}+\frac{\partial\xi}{\partial t}\right)\right]=-\mathbf{n}\,dS\,(p^{(0)}+p^{(1)}),\qquad(36)$$

where $R(t)$ is the liner radius, $\mu \equiv 2\pi\varkappa(t)R(t) = \text{const}$ is the mass per unit length of the liner, ξ is the displacement of the point of the liner under consideration with respect to its unperturbed position $\mathbf{r} = \mathbf{e}_r R(t)$, $\mathbf{e}_r$ is a unit vector in the radial direction, $\mathbf{n}dS$ is an element of the perturbed liner surface,

$$p^{(0)}=\frac{B_\varphi^2-B_z^2}{8\pi},\qquad p^{(1)}=\frac{B_\varphi B_\varphi^{(1)}}{4\pi}+\frac{B_\varphi^2}{4\pi}\frac{\xi_r}{R(t)}-\frac{B_zB_z^{(1)}}{4\pi},\qquad(37)$$

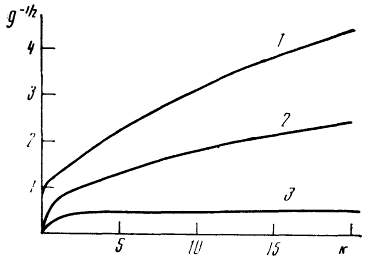

FIG. 1. Spectra of the growth rates of the Rayleigh-Taylor ($m = 0$) instability mode of a hollow plasma liner of finite thickness ($\delta/R = 1$) for various values of the parameter χ: 1) $\chi = 0.01$; 2) $\chi = 1$; 3) $\chi = 2.33$.

and $B_\varphi^{(1)}$ and $B_z^{(1)}$ are the perturbations of the azimuthal magnetic field outside the liner and at the axial field inside it. These perturbations can be calculated in the same way as in Sec. 2. Choosing a perturbation in the form $\xi = \xi(t) \times\exp(im\varphi + ikz)$, and setting $\partial\xi/\partial t = \sigma\xi$ in the spirit of this approach, we find the following dispersion relation for σ:

$$\sigma^4+\left\{-\frac{g}{R}\left[1+\frac{2m^2K_m(kR)}{kRK_m'(kR)}\right]+\frac{B_z^2}{2\mu}\left[\frac{kRI_m(kR)}{I_m'(kR)}-\frac{m^2K_m(kR)}{kRK_m'(kR)}-1\right]\right\}\sigma^2-\left(k^2+\frac{m^2}{R^2}\right)g^2=0,\qquad(38)$$

$$g=(B_\varphi^2-B_z^2)/8\pi\varkappa=-\ddot{R}(t),\qquad(39)$$

and $\ddot{R}(t)$ is the unperturbed acceleration of the liner.

Since the free term in dispersion relation (38) is negative for all values of m and k which are not zero simultaneously, there exists one root of Eq. (38) which corresponds to a finite instability growth rate. In the limit $m = 0$, $k \to 0$ the instability growth rate remains finite if $B_\varphi^2 > 3B_z^2$. Equation (38) describes a joint manifestation of MHD-instabilities of a conducting shell in a magnetic field, e.g., the instability with respect to constrictions and the Rayleigh-Taylor instabilities due to acceleration. With $B_z = 0$ and $k \to \infty$ we find classical Rayleigh-Taylor spectrum (7) from (38). The growth rates of the instabilities of a shell which is compressing (by inertia) a longitudinal magnetic field B_z inside itself in the absence of an external field B_φ also forms Rayleigh-Taylor spectrum (7) in the limit $k \to 0$, $m \to \infty$. Here $m/R(t)$ serves as a wave number.

Let us take a more detailed look at the important particular case of one of the fundamental instabillty modes—the $m = 0$ constriction mode in the limit of large wave numbers, $k \to \infty$. In this case, relation (38) takes the form

$$\sigma^4-2\chi^2gk\sigma^2-g^2k^2=0,\qquad(40)$$

where we have used the notation $\chi^2 = B_z^2/|B_\varphi^2 - B_z^2|$ in accordance with (35). At small values of χ, Eq. (40) becomes (7). For large values of χ we have

$$\sigma^2/|g|k=1/2\chi^2=|g|/2g_{max},\qquad(41)$$

where $g_{\max} = B_\varphi^2/8\pi\varkappa$.

There is an important difference between the behavior of the Rayleigh-Taylor mode in the limit $\chi \to \infty$ for a plasma slab of finite thickness, (33), and that in the limit of an infinitely thin liner, (41). In the latter case the growth rate falls off with increasing χ, just as g does. Accordingly, a perturbation growth $\sigma\Delta t = \Delta u(k/2g_{\max})^{1/2}$ will correspond to a fixed velocity increment $\Delta u = |g|\Delta t$; this perturbation growth is smaller by a factor of only $\sqrt{2}$ than the corresponding value found for $B_z = 0$, $|g| = g_{\max}$, i.e., in the absence of "stabilization" by a longitudinal magnetic field. Ths mechanism for the suppression of Rayleigh-Taylor instability modes by shear which we discussed in the preceding section of the paper operates only in a slab of finite thickness δ. In other words, as the liner becomes thinner a longitudinal magnetic field fails to stabilize perturbations of progressively shorter length, and a Rayleigh-Taylor instability sets in more rapidly. A finite liner thickness is thus a stabilizing factor in the presence of a longitudinal magnetic field.

Figure 2 shows the instability growth-rate spectra for an infinitely thin liner which correspond to a joint solution of Eqs. (38) and (39). It was assumed that the liner is compressed from an initial state of rest, $R(0) = R_0$, $\dot{R}(0) = 0$, during the flow of a constant current through the liner. In the process, the liner compresses an initial magnetic field B_{z0} which it encloses. Hence

$$B_\varphi(t)=B_{\varphi 0}\alpha^{-1}(t), \quad B_z(t)=B_{z0}\alpha^{-2}(t), \tag{42}$$

where $\alpha(t) \equiv R(t)/R_0$ is the solution of Eq. (39) with (42). A characteristic parameter of the problem is the ratio of the pressures of the longitudinal and azimuthal magnetic fields. The initial value of this ratio is

$$b=B_{z0}^2/B_{\varphi 0}^2. \tag{43}$$

As the compression proceeds, it increases in proportion to α^{-2}, according to (42). Figure 2 is plotted for the value $b = 0.1$. The unit of time here is the quantity

$$t_0=(4\pi \varkappa_0 R_0)^{1/2}/B_{\varphi 0}, \tag{44}$$

where $\varkappa_0$ is the initial value of the mass per unit surface area of the liner.

Figure 2a shows the spectrum of growth rates at the beginning of the compression for the case $\alpha = 1$, $\ddot{\alpha} = -0.9$. For all wave numbers $m = 0, 1, 2, \ldots$ the spectrum has the characteristic Rayleigh-Taylor shape. The growth rates increase in a square root fashion with increasing wave number, as was pointed out above. Figure 2b corresponds to a time near the equilibrium position, at which $\ddot{\alpha}$ is close to zero; here we have $\alpha = 0.32$, $\ddot{\alpha} = -0.075$. For moderate values of the wave numbers m and k, the growth rates are small, although they increase asymptotically as the root of the wave number, as before. Finally, Fig. 2c shows the growth-rate spectrum during the stopping of a liner by an axial magnetic field: $\alpha = 0.164$, $\ddot{\alpha} = 16.5 > 0$. The spectrum retains the characteristic Rayleigh-Taylor root asymptotic behavior, but modes with large values of m of course become predominant in this case.

5. SPECTRA OF GROWTH RATES OF RAYLEIGH-TAYLOR AND BULK CONVECTIVE INSTABILITIES OF A LINER OF FINITE THICKNESS

Let us track the qualitative changes in the instability growth-rate spectrum during the motion of a plasma liner of finite thickness, which is compressing an axial magnetic field via flow of a strong longitudinal current through the liner.[22,23] To illustrate the method developed above, we will take a more detailed look at self-similar solutions of the problem of the implosion of a plasma liner in a magnetic field, for which analytic expressions are known for the profiles of the hydrodynamic variables which figure in (3). Self-similar solutions with uniform deformation[6,24] describe specifically the well-developed stage of the compression proceeds, to which the approach which we are using here is applicable (Sec. 2.). For such solutions the hydrodynamic variables depend in the following way on the self-similar coordinate $\eta = r/R(t)$ and on the time:

$$u_r(r,t)=R_0\dot{\alpha}(t)\eta, \quad \rho(r,t)=\rho_0\alpha(t)^{-2}N(\eta),$$
$$p(r,t)=p_0\alpha(t)^{-2\gamma}P(\eta),$$
$$B_\varphi(r,t)=B_{\varphi 0}\alpha(t)^{-1}H_\varphi(\eta), \quad B_z(r,t)=B_{z0}\alpha(t)^{-2}H_z(\eta), \tag{45}$$

where $R(t)$ is the time-dependent characteristic radius of the plasma cylinder (e.g., the mean radius of a hollow plasma cylinder or the external radius of a solid Z-pinch);

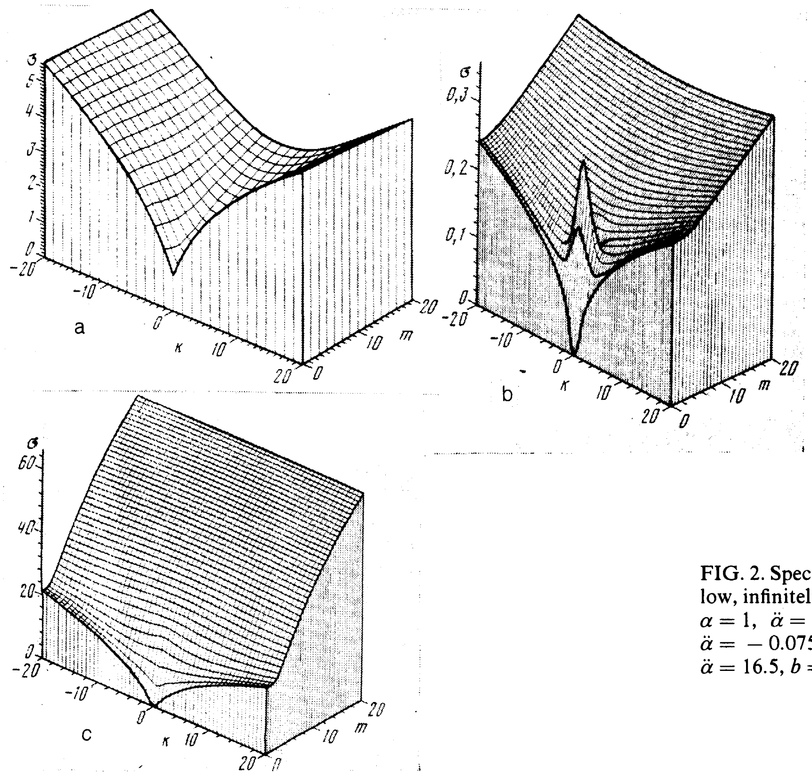

FIG. 2. Spectra of the instability growth rates of a hollow, infinitely thin, ideally conducting liner. a—$t = 0$, $\alpha = 1$, $\ddot{\alpha} = -0.9$, $b = 0.1$; b—$t = 1.18$; $\alpha = 0.32$, $\ddot{\alpha} = -0.075$, $b = 0.1$; c—$t = 1.36$, $\alpha = 0.164$, $\ddot{\alpha} = 16.5$, $b = 0.1$.

ρ_0, p_0, $B_{\varphi 0}$, and B_{z0} are normalization constants; and $N(\eta)$, $P(\eta)$, $H_\varphi(\eta)$, and $H_z(\eta)$ are dimensionless functions (representatives) which describe the self-similar spatial profiles. The $\alpha(t)$ dependence is determined by the equation of motion[6,25]

$$t_0^2\, d^2\alpha/dt^2+\alpha^{-1}-\beta\alpha^{1-2\gamma}-b\alpha^{-3}=0, \tag{46}$$

where the parameters $\beta=4\pi p_0/B_{\varphi 0}^2$ and $b=B_{z0}^2/B_{\varphi 0}^2$ characterize the relative roles of the kinetic pressure of the plasma and the pressure of the axial magnetic field, the unit of time is $t_0=(4\pi\rho_0)^{1/2}R_0/B_{\varphi 0}$, and R_0 and $\alpha(t)$ are defined as in Sec. 4. The initial conditions are $\alpha(0)=1$, $\dot\alpha(0)=0$.

With $\beta+b=1$, Eq. (46) describes a steady-state equilibrium of the plasma: $\alpha(t)=1=\text{const}$. For $0<\beta+b<1$ we find periodic solutions which describe radial oscillations of the liner around its equilibrium position with a period on the order of t_0 (Ref. 24). For $\beta+b\ll 1$ the degree of compression of the liner in the course of these oscillations is significant. Finally, the case $\beta=b=0$ corresponds to an unbounded compression of the liner in the absence of a counterpressure (a collapse) over a finite time $\tau=(\pi/2)^{1/2}t_0$.

To avoid complicating the analysis below with a discussion of the tangential shock at the liner surface,[25] we restrict the analysis to self-similar solutions which describe the compression of a liner with a negligibly low plasma pressure: $\beta=0$. The dynamics of such a liner is determined completely by its inertia and by the balance between the magnetic-pressure forces exerted on the liner from without and from within, as in the simple model in Sec. 4. We will discuss the stabilizing effect of a finite plasma pressure in Sec. 6 below. In this case we choose a density profile with a smooth decay toward the inner and outer boundaries of the liner:

$$N(\eta)=15(\eta_2-\eta)^2(\eta-\eta_1)^2/\delta^5\bar{R}, \tag{47}$$

where $\delta=\eta_2-\eta_1$, $\bar{R}=(\eta_1+\eta_2)/2$, and η_1 and η_2 are the inner and outer radii of the liner. This profile corresponds[6] to the following profiles of the magnetic-field components:

$$\begin{aligned}H_\varphi(\eta)=\frac{1}{\eta}\Big\{\frac{(\eta-\eta_1)^3}{28\delta^5\bar{R}}\,&[105(\eta-\eta_1)^5+120(3\eta_1-2\delta)(\eta-\eta_1)^4\\&+140(3\eta_1^2-6\eta_1\delta+\delta^2)(\eta-\eta_1)^3\\&+168(\eta_1^3-6\eta_1^2\delta+3\eta_1\delta^2)(\eta-\eta_1)^2\\&+210(3\eta_1^2\delta^2-2\delta\eta_1^3)(\eta-\eta_1)+280\delta^2\eta_1^3]\Big\}^{1/2},\end{aligned} \tag{48}$$

$$\begin{aligned}H_z(\eta)=\Big\{1-\frac{(\eta-\eta_1)^3}{2\delta^5\bar{R}}\,&[10(\eta-\eta_1)^3+12(\eta_1-2\delta)(\eta-\eta_1)^2\\&+15\delta(\delta-2\eta_1)(\eta-\eta_1)+20\delta^2\eta_1]\Big\}^{1/2}.\end{aligned} \tag{49}$$

In the case under consideration here, both surface and bulk convective instability modes can occur. From the stability standpoint, the most dangerous combinations of flow parameters are the vanishing of the plasma density ($\rho=0$) and of f^2, the latter describing the bending of the field lines by a perturbation:

$$kB_z+(m/r)B_\varphi=0. \tag{50}$$

In accordance with the discussion in Sec. 3, surface Rayleigh-Taylor modes occur in just the case in which these two quantities vanish simultaneously. For the plasma liner under consideration here, this situation may be realized at the outer surface of the liner as it is being accelerated inward by the pressure of an azimuthal magnetic field for the $m=0$ mode, and during the stopping of the liner by the axial magnetic field for the $k=0$ mode. In these cases, the local Rayleigh-Taylor modes described in Subsection 3.3, with the growth-rate spectra given by (7), develop as $k\to\infty$ and $m\to\infty$, respectively. As was mentioned in Subsection 3.3, the corresponding perturbations become localized over a distance on the order of $1/k$ or R/m near the unstable liner surface. This situation is illustrated by Fig. 3, which shows perturbation profiles $\xi_r(r)$ for modes with $kR_0=5$ and 10, $m=0$, at the initial time with $b=0.01$. Although according to (7) the growth rates of these instabilities are not limited at large wave numbers, these instabilities can be suppressed by shear with an appropriate combination of values of B_z and B_φ.

Other instability modes are naturally called "bulk convective modes." Bulk instabilities develop more rapidly or more slowly, depending on whether they band the magnetic field lines. Since their growth rates are higher for those wave-vector directions for which Eq. (50) can be satisfied—with our choice of the signs of B_φ, B_z, $m\geqslant 0$ this situation corresponds to negative values of k—the spectra of the growth rates are asymmetric under the substitution $k\to -k$ in the case $B_z\neq 0$. Note the difference from the case of a liner of zero thickness (Fig. 2): The bulk instability modes are not concentrated near the surface. At large wave numbers $|kR_0|$, $m\gg 1$, they localize near the point $r=r^*$, at which condition (50) holds for the given k and m. The situation is illustrated by Fig. 3, which shows profiles of perturbations of bulk modes with $m=-kR_0=5$ and $m=-kR_0=10$. The growth rates of the bulk convective modes differ from those of the Rayleigh-Taylor modes in that they are bounded at large wave numbers. In particular, in the problem at hand we find saturation for these modes as $k\to\infty$ and/or $m\to\infty$.

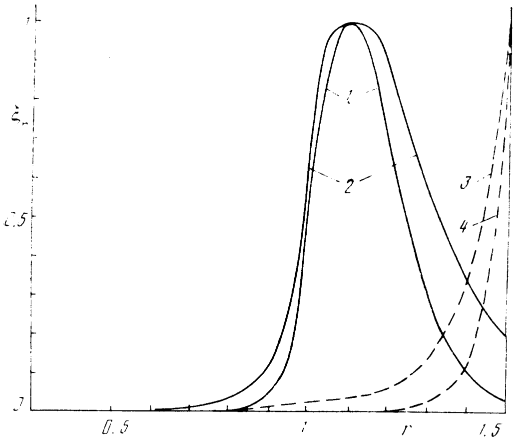

FIG. 3. Perturbation profiles $\xi_r(r)$ corresponding to eigenfunctions of various perturbation modes of a hollow cylindrical liner ($\delta/R=1$) for $t=0$ and $b=0.01$. The solid lines correspond to bulk convective modes: 1) $m=kR_0=5$; 2) $m=-kR_0=10$. The dashed lines correspond to Rayleigh-Taylor modes: 3) $m=0$, $kR_0=5$; 4) $m=0$, $kR_0=0$.

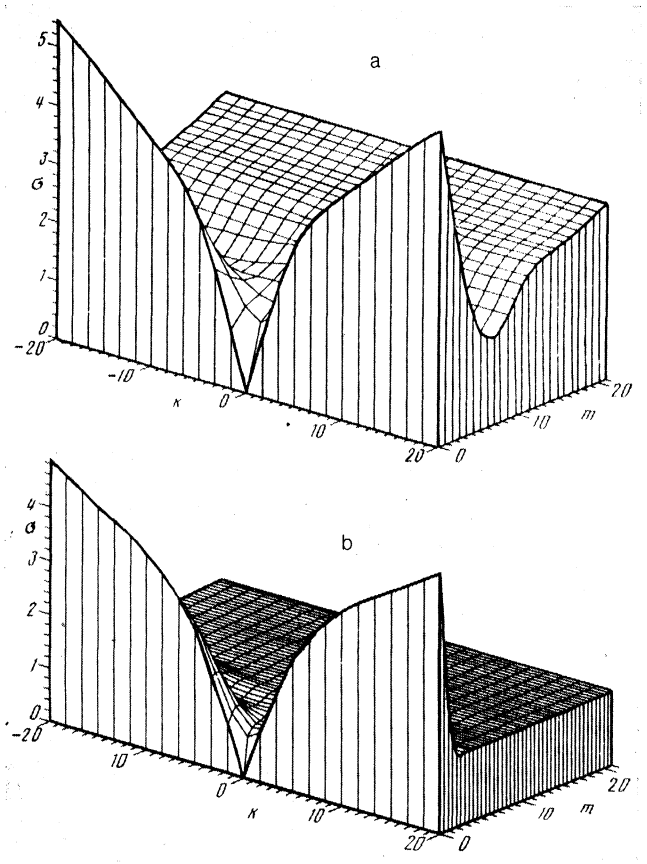

FIG. 4. Growth-rate spectra of the instabilities of a hollow plasma liner of finite thickness ($\delta/R=1$) at $t=0$ for $\alpha=1$. a) $\ddot{\alpha}=-1$, $b=0$; b) $\ddot{\alpha}=-0.9$; $b=0.1$.

Corresponding estimates of the asymptotic values of the growth rate were derived in Ref. 5. As $k\to\infty$, with finite m, for example, we have

$$\sigma_{m,\infty}=\frac{1}{\alpha(t)t_0}\max_r\left[\frac{B_z^2}{B^2}\frac{d\ln B_z}{d\ln r}+\frac{B_\varphi^2}{B^2}\left(\frac{d\ln B_\varphi}{d\ln r}+1\right)\right]^{1/2}. \qquad (51)$$

When the finite magnitude of the thermal pressure of the plasma is taken into account, the asymptotic saturation of the growth rate gives way to a maximum of the growth rate at a large value of m, followed by a decay to zero as $m\to\infty$. The bulk convective instability modes, just as the surface modes, can be stabilized by shear. The necessary condition on the shear for this stabilization is of the form of the Suydam criterion, generalized to incorporate the acceleration of the plasma. It follows from this discussion that as long as the local Rayleigh-Taylor modes are not suppressed they will grow the fastest.

Figures 4–6 show the growth-rate spectra calculated by numerical solution of boundary-value problem (3)–(5) for an unperturbed motion described by the self-similar solution (46)–(49) with the parameter values $\eta_1=0.5$, $\eta_2=1.5$, and $\delta=\overline{R}$.

Figures 4a and 4b, compare the growth-rate spectra plotted for the time $t=0$ with $\alpha=1$ under conditions differing in that there is a comparatively weak magnetic field in Fig. 4b: $b=0$ for Fig. 4a, and $b=0.1$ for Fig. 4b. It can be seen from Figs. 4a and 4b that a magnetic field of this strength has essentially no effect on the growth rate of the local Rayleigh-Taylor instability mode. The growth rates of the bulk modes in Fig. 4b, on the other hand, are considerably smaller, although the dimensionless acceleration of the plasma is essentially the same in the two cases: $\ddot{\alpha}=-1$ for Fig. 4a and $\ddot{\alpha}=-0.9$ for Fig. 4b. This behavior of the growth rates agrees with (51), where the first term in square brackets is negative.

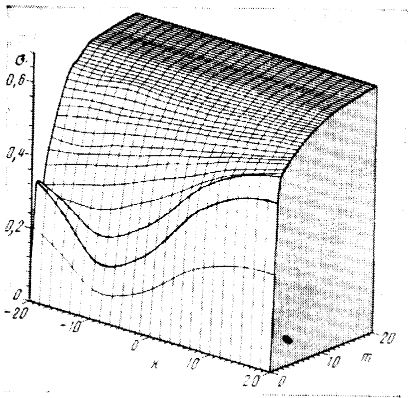

FIG. 5. Growth-rate spectra of the instabilities of a hollow plasma liner of finite thickness ($\delta/R=1$) at $t=1.225$, $\alpha=0.103$, $\ddot{\alpha}=-0.64$, and $b=0.01$.

Figure 5 shows the growth-rate spectrum calculated for $b=0.01$ at the time with $\alpha=0.103$ and $\ddot{\alpha}=-0.64$. The dimensionless acceleration $|\ddot{\alpha}|$ is only 30% lower than the initial value ($\ddot{\alpha}=-0.99$), but in this case the axial magnetic field is strong enough that the local Rayleigh-Taylor instability mode can be completely suppressed by shear according to the discussion in Subsection 3.4. In addition, there is a substantial decrease in the growth rates of the bulk modes. In this case the parameter χ has a value of only 4 [see (35)].

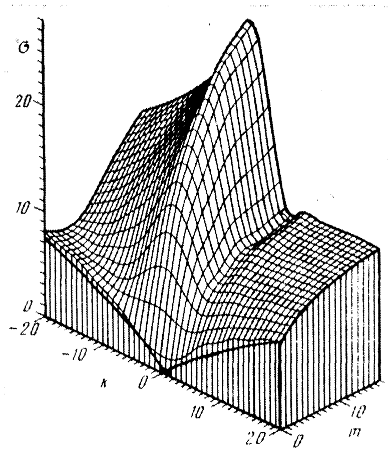

FIG. 6. Growth-rate spectra of the instabilities of a hollow plasma liner of finite thickness ($\delta/R=1$) at $t=1.36$, $\alpha=0.164$, $\ddot{\alpha}=16.5$, and $b=0.1$.

Figure 5 thus demonstrates the existence of a stability window in the space of the parameters of the motion, in which the local Rayleigh-Taylor instability mode, which is the primary mode for the initial stage of the acceleration, is completely suppressed, and the growth of the bulk convective modes is significantly slower. We wish to stress that these effects are not a consequence of a decrease in the acceleration, which remains on the order of its initial value.

Figure 6 is plotted for the time at which we have $\alpha = 0.164$, $\ddot{\alpha} = 16.5$ and corresponds to the case in which the liner is slowed by the axial magnetic field, with $b = 0.1$. Here again, a local Rayleigh-Taylor instability mode appears, in this case at the inner surface of the liner: $k = 0$, $m \to \infty$. The asymmetry of the spectral regions with $k > 0$ and $k < 0$ is particularly clearly pronounced in Fig. 6. The growth rates of the bulk instability modes reach saturation at large values of k and finite values of m, in accordance with (51).

The saturation of the bulk convective modes at large wave numbers should have the consequence that initial inhomogeneities localized along the φ or z direction may grow exponentially as a whole, without spreading out.[13] They would thereby lead to the appearance of the bands or striations which are observed in the structure of a liner with a magnetic field.

6. GROWTH-RATE SPECTRA OF RAYLEIGH-TAYLOR AND BULK CONVECTIVE INSTABILITIES OF A *Z*-PINCH WITH A SHARP BOUNDARY

We now consider the growth-rate spectra of the instabilities which occur in the plasma of a Z-pinch with a sharp boundary. We are interested in the effect of a finite plasma pressure on the growth rates for instabilities during the implosion of a pinch plasma accelerated toward the axis by the pressure of the azimuthal magnetic field of an axial current flowing through the pinch.

We can illustrate the shape of this spectrum in the case of the unperturbed motion of the plasma of a Z-pinch, which is described by self-similar solution (45), (46), where

$$N(\eta)=3(1-\eta^2)^2, \tag{52}$$

$$H_\varphi(\eta)={}^1/_2\eta(3\eta^4-8\eta^2+6)^{1/2}, \tag{53}$$

$$H_z(\eta)=(1-\eta^2)^{3/2}, \tag{54}$$

$$P(\eta)={}^1/_2(1-\eta^2)^3. \tag{55}$$

The primary distinction between the nature of the instability development in a solid Z-pinch and that in a hollow liner stems from the difference between density profiles (47) and (52). In this case the following condition holds over the entire plasma volume:

$$d\rho/dr \leqslant 0. \tag{56}$$

In the stage of the inward acceleration of the plasma, with $g > 0$, the term $g(d\rho/dr)$ in Eq. (3) is therefore negative, i.e., destabilizing. Consequently, the growth-rate spectra of the instabilities must be qualitatively similar to the corresponding spectra calculated for a hollow liner (cf., for example, Fig. 4, a and b, and Fig. 7, a and b). After the equilibrium position has been crossed, on the other hand, and the plasma is slowed by the thermal and/or magnetic counterpressure, g goes negative, and the corresponding term becomes stabilizing. The stabilizing effect is so strong that the growth rates of all of the instability modes, both surfaces and bulk, vanish (this result was established in Ref. 5 for the case of bulk modes). For a liner, on the other hand, this does not occur in the corresponding slowing stage, since in it we have $g(d\rho/dr) < 0$ near the inner surface, where the Rayleigh-Taylor and bulk instability modes develop. Consequently, Fig. 6 has no analog here.

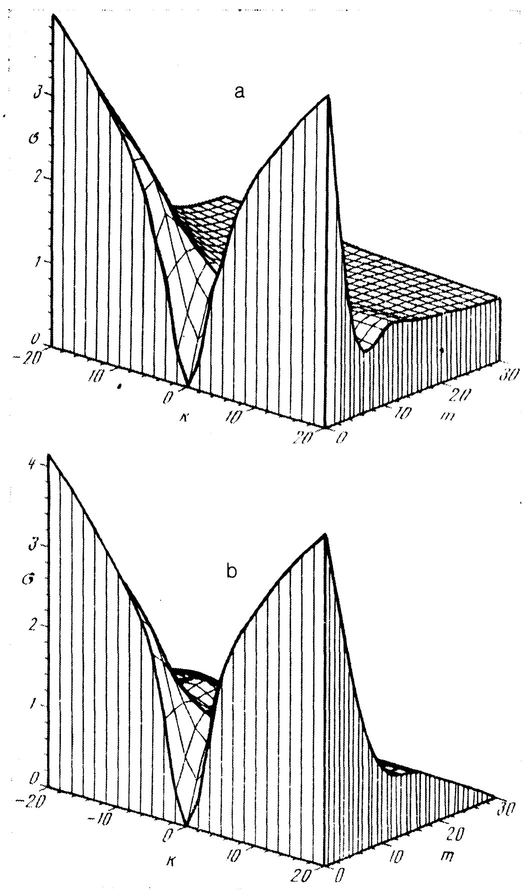

FIG. 7. Growth-rate spectra of the instabilities of a compacted Z-pinch at $t = 0$, $\alpha = 1$, $\ddot{\alpha} = -0.89$. a—$\beta = 0.1$, $b = 0.01$; b—$\beta = 0.01$, $b = 0.1$. For clarity, the scale has been enlarged along the m axis.

It is clear from this discussion that it is sufficient to examine the growth-rate spectra at the beginning of the compression for various combinations of the parameters β and b, in (46), which characterize the relative roles played by the thermal and magnetic counterpressures (for $\beta + b < 1$). Corresponding growth-rate spectra are shown in Figs. 7a and 7b for two combinations of β and b at a fixed value of the acceleration: (a) $\beta = 0.1$, $b = 0.01$; (b) $\beta = 0.01$ $b = 0.1$. Comparing parts a and b of Fig. 7, we see that the growth rates of the local Rayleigh-Taylor modes, which are dominant in both cases, are essentially the same here. The growth rates of the bulk convective instability modes in case a, in which the thermal counterpressure is dominant, and in case b, in which the magnetic counterpressure is dominant, are close in value. A finite thermal pressure, like a longitudinal magnetic field, is thus a factor which stabilizes compression. Note that the finite thermal pressure has its greatest effect on the change in the spectra at large values of m. While the value of the growth rate reaches saturation in the limit $m \to \infty$ in the case $\beta = 0$, at a finite value $\beta \neq 0$ the growth rate goes through a maximum and then vanishes at $m \sim \beta^{-1/2}$ (Ref. 5). In agreement with this estimate, under the conditions in Fig. 7a the only nonvanishing growth rates are those of the bulk convective instability modes with $m < 6$, while under the conditions in Fig. 7b the range of m is considerably wider. This result means that a more significant filamentation should be observed experimentally in the absence of a longitudinal magnetic field, in the case in which the growth rate depends only weakly on k and has a clearly expressed maximum at $m > 1$.

As in the case of a hollow plasma liner, there is a stability window here. In accordance with the discussion above, this window is wider than in the case of a liner, since it includes the entire slowing stage and the subsequent expansion of the plasma, with $g < 0$. A solid Z-pinch in a longitudinal magnetic field is thus generally more stable than a plasma liner.

7. GROWTH OF PERTURBATIONS DURING PLASMA COMPRESSION

For each perturbation component (m,k) at each time, we can find the maximum growth rate $\sigma_{m,k}$ by solving the boundary-value problem (3)–(5). This approach is equilvalent to writing the perturbation in the form

$$\xi_{m,k}(r,t) = \tilde{\xi}_{m,k}(r,t) \exp\left[\int_0^t \sigma_{m,k}(t')\,dt' \right], \tag{57}$$

which is characteristic of the semiclassical approximation. The time evolution $\tilde{\xi}_{m,k}(r,t)$ is assumed here to be slow in comparison with the variation of the exponential factor. Using the first semiclassical approximation, we ignore the time dependence of the coefficient of the exponential function in (57). This simplification makes it possible to apply this method to time-dependent problems in which the variables in the equations for the perturbations cannot be separated. A direct calculation[5] demonstrates the high accuracy of the first semiclassical approximation in problems of this class. In this approximation, the growth of the perturbations over a finite time interval t is estimated to be

$$\Gamma_{m,k}(t) = \exp\left[\int_0^t \sigma_{m,k}(t')\,dt' \right]. \tag{58}$$

As we saw above, the growth-rate spectrum of the instabilities of a plasma liner at the beginning of the compression contains rapidly growing Rayleigh-Taylor modes, which are localized near the outer boundary of the liner and whose growth rates are not bounded as $k \to \infty$. It also contains bulk convective modes, whose growth rates are bounded in this limit (Fig. 4, a and b). The growth of each of the perturbation components (m,k) during the compression is characterized by expression (58), in which the integral is evaluated from the beginning of the compression to the time at which the stability window is reached.

Turning-on a longitudinal magnetic field results in a stabilization by shear of the most dangerous Rayleigh-Taylor modes during the compression and in a corresponding decrease of the maximum values of $\Gamma_{m,k}$. It thus bcomes possible to reach higher degrees of radial compression without a disruption of the cylindrical symmetry. This stabilization of the compression of a plasma liner by a longitudinal magnet field has in fact been observed in some recent experiments,[23,24] in which degrees of radial compression $\alpha^{-1} = 15$–22 of a hollow plasma liner were reached; these are record high values for devices of this type. Note that, in contrast with the known results of the theory of the stability of steady-state plasma configurations, the dynamics of the compression is effectively stabilized by a comparatively weak longitudinal magnetic field ($b \lesssim 10^{-2}$). This effect is attributed to a decrease in the integral contribution to $\Gamma_{m,k}$ in the later, most important, stages of the compression, as a result of the attainment of the stability window. A detailed numerical calculation of the stabilizing magnetic fields (to optimize the compression process) is beyond the scope of the present study. We simply note that for the experimental conditions of Refs. 23 and 24 the short-wavelength modes with $kR_0 \sim 20$ are the most dangerous, so the conclusion (of Ref. 5) that the compression is stabilized by a longitudinal field in the interval

$$B_{z0} = (10\text{–}30\ \text{kG})\bar{I}\,[\text{MA}]/R_0[\text{cm}] \tag{59}$$

applies to these modes ($\bar{I}$ is the characteristic current flowing through the liner). That conclusion was reached as the result of a calculation for $kR_0 = 30$.

The theory indicates an interesting opportunity for stabilizing the entire compression process. Here it would be necessary to organize the compression process in such a way that the plasma parameters remained inside the stability window at all times, i.e., in such a way that the stability parameter χ (Subsection 3.4) was quite large. This situation could be arranged by shaping the current pulse flowing through the liner in such a way that the pressure of the azimuthal magnetic field which compresses the liner has the same time evolution as the pressure of the longitudinal magnetic field frozen in the liner. This situation corresponds to a current growth $I(t) \propto R(t)^{-1}$. We will not discuss the possibilities of a technical implementation of this regime here; we simply note that in most experiments the rapid disruption of cylindrical symmetry occurs specifically when the current growth comes to a stop.

8. CONCLUSION

We have derived a theory for the onset of instabilities during the accelerated motion of plasma liners and pinches with a sharp boundary for the initial, linear, stage of the evolution of the instabilities. We have classified the bulk and Rayleigh-Taylor instabilities of an accelerated plasma for Z-pinches and liners. We have calculated the instability spectra. The results show, in particular, that initial perturbations which are localized along the z or φ direction have a tendency to grow as a whole. This tendency probably explains the experimentally observed stratification and filamentation of the plasma. We have shown that a Rayleigh-Taylor surface instability mode is the most important and grows most rapidly, but can be suppressed by a longitudinal magnetic field (by shear). In a finite range of parameter values of the accelerated motion of the plasma, this instability can be suppressed completely, while the growth rates of the bulk convective instability modes can be reduced substantially.

A finite thickness of the plasma liner is another stabilizing factor. To achieve the greatest degree of radial compression and to optimize the energy concentration process, one could vary both the thickness of the liner and the strength of the initial longitudinal magnetic field. A temporal shaping of the current pulse flowing through the liner also makes possible in principle to organize the liner compression process in such a way that the parameters of the plasma being compressed do not go outside the stability window during the stage of the current rise. By taking this approach one could expect to surpass the degrees of radial compression which are presently the record high values, 20–22 (Ref. 23).

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[1)] Jaycor, San Diego, California.

[2)] For certain density profiles near a free boundary where there is a singular point, both solutions of Eq. (3) are bounded at this point. As was shown in Ref. 10, one chooses then specifically the regular solution, i.e., that solution whose derivative is bounded at the singular point.

[1]V. E. Golant, A. P. Zhilinskiĭ, and S. A. Sakharov, *Osnovy fiziki plazmy*, Atomizdat, Moscow, 1977 (*Fundamentals of Plasma Physics*, Wiley, New York, 1980).

[2]N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics*, McGraw-Hill, New York, 1972 (Russ. Transl. Atomizdat, Moscow, 1977).

[3]F. S. Felber, F. J. Wessel, N. C. Wild, *et al.*, J. Appl. Phys. **64**, 3831 (1988).

[4]N. R. Pereira and J. Davies, "X-rays from Z-pinches," Preprint NRL. September 15, 1987.

[5]A. B. Bud'ko, A. L. Velikovich, A. I. Kleev, *et al.*, Zh. Eksp. Teor. Fiz. **95**, 496 (1989) [Sov. Phys. JETP **68**, 279 (1989)].

[6]M. A. Liberman and A. L. Velikovich, Nucl. Fusion **26**, 709 (1986).

[7]S. I. Anisimov and Ya. B. Zel'dovich, Pis'ma Zh. Tekh. Fiz. **3**, 1081 (1977) [Sov. Tech. Phys. Lett. **3**, 445 (1977)].

[8]F. S. Felber and N. Rostoker, Phys. Fluids **24**, 1049 (1981).

[9]F. Hattori, H. Takabe, and K. Mima, Phys. Fluids **29**, 1719 (1986).

[10]N. A. Inogamov, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. *i* Gaza No. 1, 158 (1984).

[11]W. A. Newcomb, Phys. Fluids **4**, 391 (1961).

[12]B. B. Kadomtsev, in: *Voprosy teorii plazmy, Vyp. 2* (ed. M. A. Leontovich), Atomizdat, Moscow, 1963 (*Reviews of Plasma Physics, Vol. 2*, Consultants Bureau, New York, 1966).

[13]J. P. Goedbloed and P. H. Sakanaka, Phys. Fluids **17**, 908 (1974).

[14]D. L. Book and I. B. Bernstein, J. Plasma Phys. **23**, 521 (1980).

[15]N. A. Inogamov, Dokl. Akad. Nauk SSSR **278**, 57 (1984) [Sov. Phys. Dokl. **29**, 714 (1984)].

[16]K. O. Mikaelian, Phys. Rev. Lett. **48**, 1365 (1982).

[17]H. A. Inogamov, Zh. Prikl. Mekh. Tekh. Fiz. No. 5, 110 (1985).

[18]N. A. Inogamov, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. *i* Gaza No. 5, 145 (1985).

[19]N. A. Inogamov, Izv. Akad. Nauk SSSR, Mekhan. Zhidk. *i* Gaza No. 6, 92 (1985).

[20]L. D. Landau and E. M. Lifshitz, *Kvantovaya mekhanika*, Nauka, Moscow, 1974 (*Quantum Mechanics: Non-Relativistic Theory*, Pergamon, New York, 1977).

[21]E. G. Harris, Phys. Fluids **5**, 1057 (1962).

[22]F. S. Felber, F. J. Wessel, N. C. Wild, *et al.*, Laser and Particle Beams **5**, 699 (1987).

[23]F. S. Felber, M. M. Malley, F. J. Wessel, *et al.*, Phys. Fluids **31**, 2053 (1988).

[24]F. S. Felber, Phys. Fluids **25**, 643 (1982).

[25]M. A. Liberman, A. L. Velikovich, and F. S. Felber, in: *Megagauss Technology and Pulsed Power Applications* (Eds. C. M. Fowler, R. S. Caird, and D. J. Ericson), Plenum Press, New York, 1987, p. 107.

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