Mass shift of a classical charge in the presence of conducting boundaries

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It is proposed to calculate the classical part of the mass shift of a uniformly accelerating charge,[1,2] by an alternate method in which the role of the "photon mass" is played by the reciprocal of the distance from the charge to a conducting planar boundary parallel to the line of motion of the charge. The mass shift obtained in this manner turns out to be an elementary function of the product of that distance by the acceleration w_0. The existence of a dispersion relation for the real and imaginary parts of this mass shift is proved. The mass shift for a charge accelerating along the axis of a rectangular waveguide with perfectly conducting walls is found. Results of numerical calculations are given for several ratios of the sides of the rectangular cross section. The effect of the boundary on the mass shift vanishes when the distance between the charge and the walls of the waveguide substantially exceeds c^2/w_0, which is the distance characteristic of the self-field of a uniformily accelerating charge.

1. INTRODUCTION

The mass shift of a classical charged particle moving in a constant uniform electric field was found in Ref. 1 as a special case of the quantum shift and was further studied in Refs. 2–4. It was shown there that the mass shift Δm_a arises as result of a change in the infrastucture of the self-field of a uniformly accelerated (UA) charge and determines the correction to the classical action of the particle, equal to the change in the self-action in the external field as compared to the self-action in the absence of the field (the difference between the indicated quantities will be denoted below by the symbol $|_0^F$):

$$\Delta W_a = -\frac{e^2}{2} \iint d\tau\, d\tau' \dot{x}_\alpha(\tau) \dot{x}_\alpha(\tau') \Delta^c(x(\tau) - x(\tau'); \mu) |_0^F$$

$$= -\Delta m_a \tau. \qquad (1)$$

Here e denotes the charge of the particle, $\dot{x}_\alpha(\tau) = dx_\alpha(\tau)/d\tau$, where τ is the proper time along the world line $x_\alpha(\tau)$, and $\Delta^c(x;\mu)$ is the propagator of a "photon" with mass μ, equal for $\mu = 0$ to[1)]

$$D^c(x) = \Delta^c(x; 0) = (i/4\pi^2)(x^2 + i0)^{-1}. \qquad (2)$$

The expression (1) contains the characteristic infrared (IR) divergence as $\mu \to 0$, preventing the direct use of $D^c(x)$ in place of $\Delta^c(x;\mu)$ in Eq. (1).[1]

In this paper we study the mass shift in the presence, in addition to the external accelerating field, of a perfectly conducting boundary (plane, rectangular waveguide) parallel to the line of motion of the charge. In that case the geometry of the region, occupied by the self-field of the UA charge, is modified not only by the acceleration but also as a result of the conditions at the boundary. This leads to elimination of the IR divergence not with the help of a photon mass but owing to one or several parameters characteristic of the transverse dimensions of the waveguide or of the relative position of the charge and the boundary, and to a dependence of Δm_a on these parameters.

Especially simple formulas are obtained in the case of a planar boundary:

$$\Delta m_a = \frac{\alpha w_0}{2\pi} V_1(Rw_0), \qquad (3)$$

where $w_0 = |e\varepsilon|/m$ is the acceleration of the charge in an electric field of intensity ε, R is twice the distance from the charge to the plane, and $V_1(Rw_0)$ is an elementary function (see Sec. 2 below) with the following asymptotic behavior for $Rw_0 \gg 1$:

$$\mathrm{Im}\, V_1(Rw_0) = -\ln(Rw_0)^2 + 1 + \ldots, \qquad (4)$$

$$\mathrm{Re}\, V_1(Rw_0) = -\pi + \ldots \qquad (5)$$

(the subscript 1 refers to the spin of the self-field of the charge). Comparison with formula (80) of Ref. 1 or (7) of Ref. 2 gives the asymptotic correspondence between the "photon mass" and the distance:

$$\mu \approx \frac{2}{R\gamma}, \quad \gamma = 1.781\ldots, \quad Rw_0 \gg 1, \quad \mu/w_0 \ll 1. \qquad (6)$$

Analogous calculations for the mass shift of a scalar charge, performed on the assumption of Dirichlet boundary conditions for the scalar self-field at the boundary, determine the function $V_0(Rw_0)$ (see below). Here the functions $V_1(Rw_0)$ and $V_0(Rw_0)$ replace the functions $S_1(\mu^2/w_0^2)$ and $S_0(\mu^2/\omega^2)$ of Ref. 3 and have similar to them properties, but are simpler. We have also shown that in our case too the real and imaginary parts of the shift Δm_a are connected by a dispersion relation expressing the causal connection between the reactive change of the self-field of the charge and its radiation, see Ref. 3.

To evaluate the mass shift for a charge moving in a rectangular waveguide we made use of the results of Ref. 5, where, in particular, the Green function (GF) for the electromagnetic field in a rectangular parallelepiped is obtained. The GF for the waveguide is obtained by letting one of the edges of the parallelepiped go to infinity. The mass shift in the waveguide is represented as a double sum over image charges whose terms are elementary functions, which is convenient for numerical calculations. The mass shift $\mathrm{Re}\Delta m$ obtained in formula (31) differs from $\mathrm{Re}\,\Delta m_a$ by a quantity due only to the effects of the boundaries (see below) and exhibits correct asymptotic behavior in the external field: In

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a weak field it coincides with half of the Coulomb interaction energy between the charge and the image charges, while in a strong field it coincides with the mass shift $-\alpha w_0/2$ of a UA charge in a space without boundary.[1] In the intermediate region, $w_0^{-1} \sim a_1$ (where a_1 is the length of one of the sides of the cross section of the waveguide), the shift Re Δm depends on the position of the charge and does not reduce to a simple sum of the Coulomb shift and the shift $-\alpha w_0/2$.

2. MASS SHIFT IN THE PRESENCE OF A CONDUCTING PLANE

The change produced in the self-action of a classical charge by the presence of the plane is taken into account by a modification of the causal propagator, Eq. (2) (see, e.g., Ref. 6, p. 105 of Russian translation):

$$D_B^c(x,x') = \frac{i}{4\pi^2}\left[\frac{1}{(x-x')^2+i0} - \frac{1}{(x-\tilde{x}')^2+i0}\right], \tag{7}$$

which ensures that the Dirichlet boundary conditions are satisfied for the self-field of the charge; here $\tilde{x}'_\alpha$ is the "mirror image" of the vector $\tilde{x}'_\alpha$. We assume that the conducting boundary coincides with the (x_2,x_3) plane and the motion takes place along a straight line parallel to the axis x_3, so that the $\tilde{x}'_\alpha$ in Eq. (7) is connected with $\tilde{x}'_\alpha = (\mathbf{x}', ix'_0)$ as follows:

$$\tilde{x}_\alpha' = (-x_1', x_2', x_3', ix_0').$$

Then for the UA charge we find upon settting $x = x(\tau)$, $x' = x(\tau')$, see Ref. 2, that

$$(x-x')^2 = -2w_0^{-2}[\operatorname{ch} w_0(\tau-\tau')-1],\quad \dot{x}_\alpha\dot{x}_\alpha' = -\operatorname{ch} w_0(\tau-\tau'),$$
$$(x-\tilde{x}')^2 = R^2 - 2w_0^{-2}[\operatorname{ch} w_0(\tau-\tau')-1], \tag{8}$$

where $R = 2x_1$.

In the presence of boundaries formula (1) should contain $D_B^c(x,x')$ instead of $\Delta^c(x-x';\mu)$. If, moreover, the procedure $|_0^F$ is understood as $\Big|_{0,0}^{F,B} = \Big|_{0,B}^{F,B} + \Big|_{0,0}^{0,B}$, then the right-hand side of Eq. (1) determines the total shift $\Delta m = \Delta m_a + \Delta m_C$, where Δm_a is the shift that vanishes together with the acceleration (field), while Δm_C is the shift due to the Coulomb interaction of the particle with the image charge. It is obvious that to obtain Δm_C one should use formula (1) with Δ^c replaces by $D_B^c - D^c$ [see Eq. (7)], and the factor $\dot{x}_\mu(\tau)\,\dot{x}_\mu(\tau')$ should be replaced by its limit for $\varepsilon = 0$, equal to (-1), with the symbol $|_0^F$ omitted:

$$\Delta W_C = \frac{-i\alpha}{2\pi}\tau\int_{-\infty}^{\infty}\frac{dy}{y^2-R^2-i0} = \frac{\alpha}{2R}\tau,\qquad y=\tau-\tau', \tag{9}$$

$$\Delta m_C = -\alpha/2R. \tag{10}$$

It is relevant that Δm_C is equal to half the Coulomb interaction energy of two equal and opposite charges at a distance of $R = 2|x_1|$ from each other. The reason is that the work performed in moving the charge away from the boundary should be equal to

$$A = -\Delta m_C = \int_{x_1}^{\infty}\frac{\alpha}{(2x_1)^2}dx_1$$

(no work is performed to move the image charge).

Keeping the notation Δm_a for the mass shift due to acceleration, see Eq. (1), we have in the case of the plane

$$\Delta W_a = \frac{e^2}{2}\iint d\tau\, d\tau'\, \dot{x}_\alpha\dot{x}_\alpha' D_B^c(x,x')\Big|_{0,B}^{F,B} = -\frac{\alpha w_0}{2\pi}V_1(Rw_0)\,\tau, \tag{11}$$

$$V_1(Rw_0) = \pi\left(-\operatorname{cth}\theta + \frac{1}{2\operatorname{sh}(\theta/2)}\right) - i(\theta\operatorname{cth}\theta - 1), \tag{12}$$

$$\operatorname{ch}\theta = 1 + R^2w_0^2/2,\quad \theta \geqslant 0, \tag{13}$$

so that the exact expressions for Re Δm_a and Im Δm_a take on the form

$$\operatorname{Re}\Delta m_a = \frac{\alpha w_0}{2}\left(-\operatorname{cth}\theta + \frac{1}{2\operatorname{sh}(\theta/2)}\right), \tag{14}$$

$$\operatorname{Im}\Delta m_a = \frac{\alpha w_0}{2\pi}(-\theta\operatorname{cth}\theta + 1). \tag{15}$$

The interval $\Delta\tau = \tau - \tau' = w_0^{-1}\theta$ is the characteristic interval of proper time between the emission of the photon at the instant τ and its absorption (after reflection from the mirror) at the instant τ', see Fig. 1. This follows from the fact that Eq. (13) may be rewritten in the form [see Eq. (8)]

$$R^2 + (x_3(\tau) - x_3(\tau'))^2 = (x_0(\tau) - x_0(\tau'))^2. \tag{13'}$$

In the case of a scalar charge we obtain by following Ref. 3, replacing $\dot{x}_\alpha x'_\alpha \to 1$ in Eq. (11), and making use of Eq. (8)

$$\Delta m_a = \frac{\alpha w_0}{2\pi}V_0(Rw_0),$$
$$V_0(Rw_0) = i(-1+\theta/\operatorname{sh}\theta) + \pi(1/\operatorname{sh}\theta - 1/2\operatorname{sh}(\theta/2)). \tag{16}$$

Here θ is the same as in Eq. (13).

For $R \gg w_0^{-1}$ formulas (4) and (5) are easily obtained from Eq. (12); the mass shift Δm_a vanishes with vanishing field ($w_0 = 0$).

Graphs of the functions V_1 and V_0 are shown in Fig. 2. For convenience in making comparisons with their analogs—the functions S_1 and S_0 of Ref. 3—we measure the abscissa in units of $2/w_0\gamma$, corresponding to $\lambda^{1/2} = \mu/w_0$ in Ref. 3.

One's attention is called to not only the qualitative but also the quantitative agreement between the functions $V_1(Rw_0)$, $V_0(Rw_0)$ determining the mass shifts, on the one

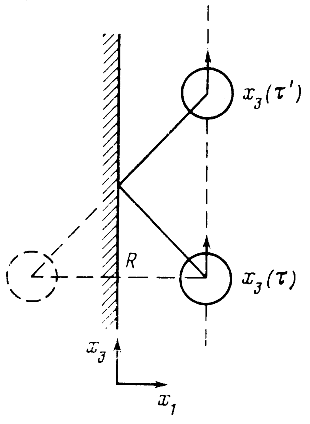


FIG. 1.

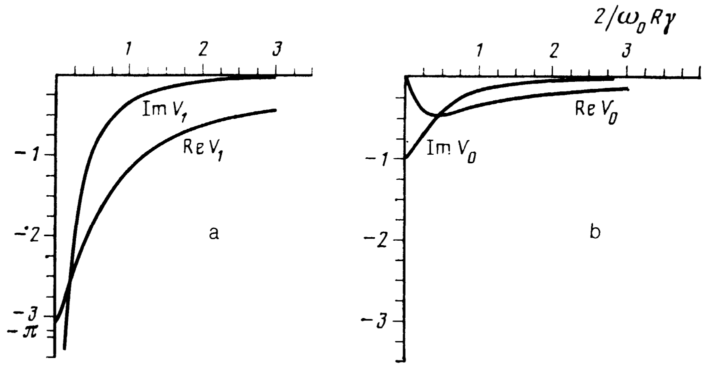


FIG. 2.

hand, and $S_1(\lambda)$, $S_0(\lambda)$ on the other. Besides the agreement of all the limiting values ofthe functions V_1, V_0 and S_1, S_0 (for $R\omega_0 \gg 1$, $Rw_0 \ll 1$ and correspondingly $\lambda \ll 1$, $\lambda \gg 1$), we have the characteristic ratio

$$\lim_{Rw_0\to 0} \frac{\operatorname{Re} V_1}{\operatorname{Re} V_0} = \lim_{\lambda\to\infty} \frac{\operatorname{Re} S_1}{\operatorname{Re} S_0} = 3, \tag{17}$$

whose value was related in Ref. 3 to the differences between the spin degrees of freedom of massive vector and scalar fields.[2)] The minimum point of the function Re V_0 and the intersection points of the graphs of all four functions are also close to the corresponding points for the functions Re S_0, ImS_0, Re S_1, Im S_1. The relation

$$\lim_{\lambda\to\infty} (\operatorname{Im} S_1/\operatorname{Im} S_0) = 1 \tag{18}$$

is not true for V_1 and V_0:

$$\lim_{Rw_0\to 0} \frac{\operatorname{Im} V_1}{\operatorname{Im} V_0} = 2, \tag{19}$$

and the asymptotes of Im $V_1(Rw_0)$ and Im $V_0(Rw_0)$ near zero have themselves a power-law character in contrast to the essential singularity of the functions Im $S_1(\lambda)$ and Im $S_0(\lambda)$ at infinity.

The agreement in the behavior of the functions V_1, V_0 and S_1, S_0 for $Rw_0 \gg 1$ and $\lambda \ll 1$ in fact expressed by the correspondence in Eq. (6). In actuality, as can be seen from a comparison of the graphs of the functions Im V_1 and Im S_1, the region of validity of the correspondence (6) is much wider: It is valid at least for $0 < \lambda^{1/2} \sim (Rw_0)^{-1} \lesssim 3$, and on the entire numerical axis ($I = 1,0$) for the real parts of V_I and S_I.

3. DISPERSION RELATION

The above observed analogy between the two approaches to the description of the mass shift becomes even deeper because of the existence, in our case as well, of a dispersion relation (DR) between the real and imaginary parts of the shift.

The DR for $\Delta W_{aI}(R)$ ($I = 0,1$) has the form

$$\operatorname{Im} \Delta W_{aI}(R) = -\frac{2}{\pi} \int_0^\infty \frac{\operatorname{Re} \Delta W_{aI}(Rx)}{x^2-1} \frac{dx}{x}, \tag{20}$$

$$\operatorname{Re} \Delta W_{aI}(R) = \frac{2}{\pi} \int_0^\infty \frac{\operatorname{Im} \Delta W_{aI}(Rx)}{x^2-1} dx. \tag{21}$$

The validity of Eqs. (20) and (21) is readily verified by direct evaluation of the integrals with Eqs. (12), (13) and (16) taken into account. The DR were obtained with the help of the relation ($z^2 = (x - x')^2 < 0$)

$$\Delta(z,R) \equiv \frac{i}{4\pi^2} \frac{1}{R^2+z^2+i0\cdot R} = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\Delta(z,R')dR'}{R'-R}, \tag{22}$$

which may be used, in view of Eq. (8), to replace $D_B^c(x,x')$ in Eq. (11) by

$$\Delta(z,0) - \Delta(z,R) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{\Delta(z,R'-R) - \Delta(z,R')}{R'-R} dR'. \tag{23}$$

Substitution of expression (23) into Eq. (11) and the evaluation first of the integral over the difference of the proper times, leads to the expression

$$\Delta W_{aI}(R) = \frac{\alpha w_0 \tau}{2\pi^2} \int_{-\infty}^{\infty} \frac{dR'}{R'-R} [\Phi(R'-R) - \Phi(R')].$$

in which ($M = 1 + w_0^2 R^2/2$), for example, for $I = 1$

$$\Phi(R) = -M(M^2-1)^{-1/2} \operatorname{arch} M + i\pi[M \operatorname{sign} R \cdot (M^2-1)^{-1/2} - (w_0 R)^{-1}]$$

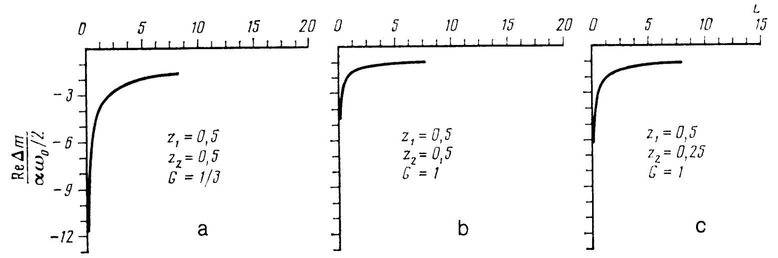
and is connected with $V_1(Rw_0)$ in Eq. (12). After an obvious change of variable of integration and utilization of the asymptotic (as $R \to \infty$) properties of $\Phi(R)$ we obtain the desired result, Eqs. (20) and (21) after separating the real and imaginary parts.

If we make in the DR (20) and (21) the replacement $kRx \to y^{-1}$ and denote $\mu = 1/kR$ ($k > 0$, an arbitrary number; to correspond with Eq. (6) one may take $k = \gamma/2$), then these relations coincide in form with the DR written in the two formulas (32) of Ref. 3 (after the real and imaginary parts have been separated); for example, our formula (21) becomes

$$\operatorname{Re} \Delta W_{aI}(1/k\mu) = \frac{-2\mu}{\pi} \int_0^\infty \frac{\operatorname{Im} \Delta W_{aI}(1/ky)}{y^2-\mu^2} dy. \tag{24}$$

4. MASS SHIFT IN A RECTANGULAR WAVEGUIDE

We assume that the axis of a rectangular waveguide of dimension $a_1 \times a_2$ is directed along the x_3-axis, which is the direction of motion of the UA charge. The calculation scheme, Eq. (11), remains unchanged but the expression



(7) for $D_B^c(x, x')$ becomes more complicated due to the appearance of an infinite number of image charges. If only the third and fourth components of the 4-velocity $\dot{x}_\alpha(\tau)$ are different from zero, we may take for $D_B^c(x, x')$ the following expression (see Ref. 5):

$$D_B^c(x,x') = \frac{i}{4\pi^2}\sum_{p_1=-\infty}^{\infty}\sum_{p_2=-\infty}^{\infty}\Big[\frac{1}{R_1^2+(x_3-x_3')^2-(x_0-x_0')^2} - \frac{1}{R_2^2+(x_3-x_3')^2-(x_0-x_0')^2} - \frac{1}{R_3^2+(x_3-x_3')^2-(x_0-x_0')^2} + \frac{1}{R_4^2+(x_3-x_3')^2-(x_0-x_0')^2}\Big]. \quad (25)$$

Here

$$R_1^2=4(p_1a_1)^2+4(p_2a_2)^2, \quad R_2^2=4(p_1a_1)^2+4(p_2a_2-x_2)^2,$$
$$R_3^2=4(p_1a_1-x_1)^2+4(p_2a_2)^2, \quad R_4^2=4(p_1a_1-x_1)^2+4(p_2a_2-x_2)^2, \quad (26)$$

and x_1 and x_2 are the coordinates of the particle in the transverse section of the waveguide (the origin of the coordinates is at one of the edges; the waveguide is assumed to be of infinite length).

Further calculations with Eq. (8) taken into account are a repetition of those performed in the case of the plane, so that we obtain in the same notation as before[3)]

$$\Delta m_a = -\frac{\alpha w_0}{2\pi}\sum_{p_1}\sum_{p_2}(V_1(R_1w_0)-V_1(R_2w_0)-V_1(R_3w_0) + V_1(R_4w_0)]. \quad (27)$$

The expression for Δm_C in the case of the waveguide is readily obtained. As in the derivation of Eqs. (9) and (10) we make use of $D_B^c(x, x') - D^c(x-x')$ and set $w_0 = 0$ to obtain

$$\Delta m_C = \frac{\alpha}{2}\sum_{p_1}\sum_{p_2}\Big[\frac{1-\delta_{0p_1}\delta_{0p_2}}{R_1} - R_2^{-1} - R_3^{-1} + R_4^{-1}\Big]. \quad (28)$$

The presence here of the factor $1-\delta_{0p_1}\delta_{0p_2}$ indicates the omission of the corresponding term for $p_1 = 0 = p_2$. It is seen that Δm_C is real, see Eq. (26). Introducing the notation

$$M_i = 1 + w_0^2R_i^2/2, \quad i = 1, 2, 3, 4, \quad (29)$$

we write the real and imaginary parts of the total shift $\Delta m = \Delta m_a + \Delta m_C$, using the definition of the function V_1:

$$\mathrm{Im}\,\Delta m = \frac{\alpha w_0}{2\pi}\sum_{p_1}\sum_{p_2}\Big[\frac{M_1 \operatorname{arch} M_1}{(M_1^2-1)^{1/2}} - \frac{M_2 \operatorname{arch} M_2}{(M_2^2-1)^{1/2}} - \frac{M_3 \operatorname{arch} M_3}{(M_3^2-1)^{1/2}} + \frac{M_4 \operatorname{arch} M_4}{(M_4^2-1)^{1/2}}\Big], \quad (30)$$

$$\mathrm{Re}\,\Delta m = \frac{\alpha w_0}{2}\sum_{p_1}\sum_{p_2}[M_1(1-\delta_{0p_1}\delta_{0p_2})/(M_1^2-1)^{1/2} - M_2(M_2^2-1)^{-1/2} - M_3(M_3^2-1)^{-1/2} + M_4(M_4^2-1)^{-1/2}]. \quad (31)$$

With increasing p_1 and p_2 the individual terms in the square brackets of expressions (30) and (31) do not tend to zero separately, but owing to the different signs the sums in the square brackets rapidly decrease as $p_1 \to \infty$ or $p_2 \to \infty$, and the double sums in (30) and (31) converge.

Results of numerical calculations of the quantities Re $\Delta m(L)$ and $L = (w_0 a_1)^2$, for fixed $z_1 \equiv x_1/a_1 = \frac{1}{2}$, $z_2 \equiv x_2/a_2 = \frac{1}{2}$ and $\frac{1}{4}$, $G \equiv a_2/a_1 = \frac{1}{3}$ and 1, are given in Fig. 3, a–c. For $L \to 0$ the mass shift tends to the Coulomb limit, Eq. (28), when

$$2\,\mathrm{Re}\,\Delta m/\alpha w_0 = \mathrm{const}/2L^{1/2}$$

(see below), and for $L \to \infty$ to the value found in Ref. 1:

$$2\,\mathrm{Re}\,\Delta m(\infty)/\alpha w_0 = -1.$$

The graphs in Fig. 3, a–c can be approximately (for the same range of L) by the following formula:

$$\mathrm{Re}\,\Delta m(L) = \frac{\alpha w_0}{4}CL^{-1/2} + \frac{3\alpha w_0}{8}C_1L^{+1/2}, \quad (32)$$

in which C and C_1 depend on z_1, z_2, and G (see Appendix). The first term in Eq. (32) is negative [see (33)], is independent of the acceleration w_0, and coincides with Δm_C see Eqs. (26) and (28). The second, also negative, term equals Re Δm_C for small values of $L \lesssim 2C/3C_1$ [$L = 2C/3C_1$ corresponds to the extremum of the right side of Eq. (32)].

A comparison of formulas (31) and (32) is carried out in Table I where we used numerical values of the constants C and C_1 (see the Appendix):

$$C=-8.319, \quad C_1=-0.084 \quad (z_1=z_2=1/2,\ G=1/3),$$
$$C=-3.231, \quad C_1=-0.21 \quad (z_1=z_2=1/2,\ G=1), \quad (33)$$
$$C=-4.403, \quad C_1=-0.167 \quad (z_1=1/2,\ z_2=1/4,\ G=1).$$

As it should, the agreement between formulas (31) and (32) worsens with increasing L.

TABLE I.

$G={}^1/_3$		$G=1,\ z_1={}^1/_2$				L
$z_1=z_2={}^1/_2$		$z_2={}^1/_2$		$z_2={}^1/_4$		
Re Δm, $\alpha w_0/2$		Re Δm, $\alpha w_0/2$				
(31)	(32)	(31)	(32)	(31)	(32)	
−11.787	−11.787	−4.625	−4.625	−6.271	−6.271	0.125
−2.946	−2.945	−1.359	−1.338	−1.706	−1.692	2.125
−2.179	−2.175	−1.156	−1.116	−1.365	−1.338	4.125
−1.843	−1.836	−1.088	−1.043	−1.233	−1.199	6.125
−1.689	−1.679	−1.063	−1.021	−1.179	−1.142	7.625

5. CONCLUSION

In the paper we studied the effect of boundaries on the mass shift of a UA classical charge moving parallel to the boundaries. The mass shift turns out to be a function of the dimensionless parameter Rw_0, where R determines the distance from the charge to the boundary, and w_0^{-1} is the characteristic transverse dimension of the self-field of the UA charge, which appears in the Schott solutions [see, for example, Ref. 2, formula (42)].

In the case of a conducting plane the mass shifts Δm_a of an electric and scalar charge, as functions of the parameter $(Rw_0)^{-1}$, display a close analogy with the mass shifts of the same charges in the absence of a boundary, but with massive self-fields, provided the latter shifts are viewed as functions of μ/w_0 [see Eq. (6), Fig. 2, and the graphs for the real and imaginary parts of the functions S_1, S_0 in Ref. 3].

The presence of the waveguide has a significant effect on the shift Re Δm for $R\sim a_1\ll w_0^{-1}$, since in that case the slef-field near the boundary is not small, and the shift is determined by the Coulomb interaction between the charge and the image charges:

$$\text{Re}\,\Delta m\approx\Delta m_C=\alpha C/4a_1,\quad C<0. \tag{34}$$

In the opposite case, when $R\sim a_1\gg w_0^{-1}$, the self-field of the UA charge, being mainly concentrated near the trajectory (which we have located at the center), is little affected by the presence of the boundary so that the mass shift tends to the value it has in the absence of the waveguide:[1]

$$\text{Re}\,\Delta m\approx-\alpha w_0/2. \tag{35}$$

It is important that the simple sum of the right sides of Eqs. (34) and (35), although giving the correct asymptotics for $a_1w_0\gg 1$ and $a_1w_0\ll 1$, nevertheless deviates substantially in the intermediate region (being too small according to the numerical calculations) from the true value of Re Δm calculated from formula (31). This can also be established with the help of Eqs. (32) and (33):

$$(\text{Re}\,\Delta m-\Delta m_C+\alpha w_0/2)/(\alpha w_0/2)=1+{}^3/_4C_1w_0a_1>0,\quad w_0a_1\lessdot 1.$$

Lastly we note that for $\frac{1}{8}\leqslant L\leqslant 8$, the value of Im Δm in Eq. (30), calculated according to the same recipe as Re Δm [see Eq. A1)], decreases montonically in the following ranges:

$$-5\cdot 10^{-4}>(2\,\text{Im}\,\Delta m/\alpha w_0)>-0.4\quad (G=1,\ z_1=z_2={}^1/_2),$$

$$-3\cdot 10^{-5}>(2\,\text{Im}\,\Delta m/\alpha w_0)>-2\cdot 10^{-3}\quad (G={}^1/_3,\ z_1=z_2={}^1/_2).$$

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APPENDIX

The basis for the derivation of the approximate formula (32) lies in the rapid convergence of the series (31). If it is represented in the truncated off form

$$\sum_{p_1=-N}^{N}\sum_{p_2=-N}^{N}[\ldots], \tag{A1}$$

then, as is shown by numerical calculations, increasing N, starting with $N=6$, changes the sum (A1) only in the 6th to 4th decimal place (in the entire range $\frac{1}{8}\leqslant L\leqslant 8$ under investigation). For infinite $|p_1|$ and $|p_2|\leqslant N$ and sufficiently small L, a general term of the series (31) may be approximated by two terms of a Taylor series, so that the coefficient C is given by the expression

$$C=\sum_{p_1=-N}^{N}\sum_{p_2=-N}^{N}\Big[\frac{1-\delta_{0p_1}\delta_{0p_2}}{(p_1^2+p_2^2G^2)^{1/2}}-(p_1^2+(p_2-z_2)^2G^2)^{-1/2}$$
$$-((p_1-z_1)^2+p_2^2G^2)^{-1/2}+((p_1-z_1)^2+(p_2-z_2)^2G^2)^{-1/2}\Big], \tag{A2}$$

and C_1 by an analogous expression, but with the square roots appearing in the numerator. Both sums are convergent for $N\to\infty$. The same method cannot be used to obtain the coefficients C_2, C_3,... that follow in the expansion (32): The series (31) is not uniformily convergent near the point $L=0$ (see, e.g., Ref. 7, §3.3), and the coeffiecients C_2, C_3,... are represented by divergent series in the limit $N\to\infty$.

[1)] We use a system of units in which $\hbar=c=1$, $\alpha=e^2/4\pi\hbar c$; components of 4-vectors are denoted by $a_\alpha=(\mathbf{a}, ia_0)$.

[2)] We emphasize that the self-field is massless for us in both cases.

[3)] For $p_1=p_2=0$ the parameter $R_1=0$, and the expression for $V_1(0)$ contains the indeterminacy $\infty-\infty$, which is removed by redefining the function $V_1(R_1w_0)$ at zero by continuity: $\lim V_1(x)=0$ as $x\to 0$.

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