Potential distribution in a quantum point contact

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A quantum ballistic point contact in the form of a bridge connecting expanding horn-shaped banks is analyzed. In this model the conducting channel and the reservoirs to which the external potentials are applied can be treated as a single unit. This model allows a meaningful interpretation of effects associated with the transition of electrons from a reservoir to the conducting channel. An equation which determines the potential distribution in the contact and an expression for the conductivity which incorporates electron diffraction at the junction between a reservoir and the conducting channel are derived. A previous analysis of resistance measurements by the four-probe method is reexamined.

1. INTRODUCTION

The discovery of quantization of the conductance of a channel in a two-dimensional electron gas (2DEG)[1,2] was followed by a sharp increase in interest in quantum ballistic point contacts, i.e., entities in which the width of the conducting channel, d, is comparable to the characteristic wavelength λ and in which the channel length L is shorter than the momentum mean free path l (we are assuming $L \gtrsim d$, as is usually the case). Entities of this type ("short channels") actually figured in experiments even before the discovery of the quantization of conductance.[3–5]

The flow of a current through a quantum ballistic point contact has several distinguishing features (waveguide and nonlocal properties). All of these features stem from the circumstance that the resistance of an entity of this sort is determined not by the scattering of electrons by random inhomogeneities but by diffraction of the electron wave at a constriction, i.e., at the entrance to and exit from the channel.

In the present paper we analyze a model of a ballistic point contact in the form of an electron gas in a sample which has a bridge—the constriction—and expanding horn-shaped banks (Fig. 1). The current density falls off with distance into the banks, so it can be assumed that the potential $\varphi(\mathbf{r})$ approaches a constant value deep in the banks. The bank is thus a "reservoir" to which the given external potential is applied. This model makes it possible to describe the conducting channel and the reservoir as a single unit and thus to find a valid description of effects which occur as electrons go from the reservoir into the channel.

We will derive an equation for the potential distribution $\varphi(\mathbf{r})$ which arises in the point contact when a given potential difference V is applied to its banks, and a current proportional to V flows through the contact. We will also calculate the conductance of the point contact, which explicitly incorporates the diffraction of electrons as they pass from the bank (reservoir) into the channel and vice versa.

We assume that both dimensions d and L, as well as the electron wavelength λ, are large in comparison with the lattice constant (a semiconductor or a semimetal), so we can use the effective-mass method in describing the electrons.

The conductance of a quantum ballistic point contact has also been calculated previously, for more-restricted models: for an aperture in a screen,[6] for an adiabatically expanding channel,[7] and for a long channel between two screens.[8] The potential distribution is a topic which has been discussed qualitatively by Landauer[9] repeatedly. There are qualitative discussions in papers by Büttiker[10] but they refer to one–dimensional channels without banks.

The ideas regarding the potential distribution which developed in the course of the present study led us to reexamine Engquist and Anderson's analysis[11] of the results of the four–probe method for measuring resistance.

2. CLASSICAL POINT CONTACT

Under the conditions $\lambda \ll d \lesssim L$ a point contact is "classical." A theory for a point contact of this sort was derived by Kulik, Shekhter, *et al.*[12,13] We will present this theory below in a slightly altered form, more convenient for comparison with the theory of a quantum point contact, and we will focus on the potential distribution.

In equilibrium at $V = 0$, the state of the electron gas is described by a spatially uniform (within the sample) distribution function $f_0(\mathbf{k}) = f_T(\varepsilon_{\mathbf{k}})$, where f_T is the Fermi function. The electron charge density is neutralized by the ion charge density. Under the condition $V \neq 0$, a nonequilibrium and nonuniform distribution $f(\mathbf{r},\mathbf{k})$ is established, the electron charge is redistributed in space, and a net electron charge density arises:

$$\delta\rho(\mathbf{r}) = 2e\int \frac{d^3k}{(2\pi)^3}\,[f(\mathbf{r},\mathbf{k}) - f_0(\mathbf{k})] \qquad (1)$$

(the factor of 2 results from the summation over spin orientations). This density screens out the external field which is the source of the potential difference V, with the result that

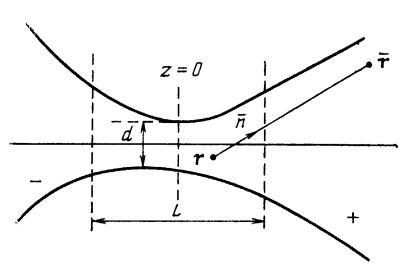

FIG. 1.

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some potential distribution $\varphi(\mathbf{r})$ is established in the point contact. This distribution satisfies the Poisson equation

$$\nabla^2\varphi = -\frac{4\pi}{\epsilon}\delta\rho, \tag{2}$$

where ε is the dielectric constant if free electrons are ignored, and it satisfies the boundary condition

$$\varphi(z\to+\infty)-\varphi(z\to-\infty)=V. \tag{3}$$

To close the system of equations (1), (2), we use a kinetic equation for the distribution function in the electric field $\mathbf{E} = -\nabla\varphi$

$$\frac{\partial f}{\partial t} = -v_{\mathbf{k}}\nabla f + e\nabla\varphi\frac{\partial f}{\partial \mathbf{k}} - \frac{f-f_0}{\tau} = 0, \tag{4}$$

where τ is the relaxation time of the nonequilibrium electron distribution.

A solution of Eq. (4) by the method of characteristics yields

$$f(\mathbf{r},\mathbf{k}) = \int_{-\infty}^{0}\frac{dt}{\tau}e^{t/\tau}f_T[\varepsilon_{\mathbf{k}}+e\varphi(\mathbf{r})-e\varphi(\mathbf{r}(t)_{r,k})]. \tag{5}$$

This integral has the following meaning: For given $(\mathbf{r},\mathbf{k})$ we find the path traced out in the field $\mathbf{E}$ (elastic reflections from the boundaries of the sample are taken into account) by an electron which starts from infinity and arrives at the point $\mathbf{r}$, where it has a momentum $\mathbf{k}$. The law of motion along this path is $\mathbf{r}(t)_{r,k}$; the time $t = -\infty$ corresponds to the infinitely remote point, and $t = 0$ to the point $\mathbf{r}$.

Linearizing (5) in terms of φ, we find $\delta f = f - f_0$ in the form

$$\delta f(\mathbf{r},\mathbf{k}) = \left(-\frac{\partial f_T}{\partial\varepsilon_{\mathbf{k}}}\right)[\langle e\varphi\rangle_{r,k} - e\varphi(\mathbf{r})], \tag{6}$$

$$\langle\varphi\rangle_{r,k} = \int_{-\infty}^{0}\frac{dt}{\tau}e^{t/\tau}\varphi(\mathbf{r}(t)_{r,k}). \tag{7}$$

The path can be taken at $\mathbf{E} = 0$ here. If $l = v\tau$ is sufficiently large, the average in (7) is dominated by remote points on the path, at which $\varphi(\mathbf{r})$ differs only slightly from one of the asymptotic values $\varphi(z\to\pm\infty)$. We can thus write

$$\langle\varphi\rangle_{r,k} = \varphi(+\infty)\theta_{r,k}+\varphi(-\infty)\theta^{-}_{r,k}. \tag{8}$$

Here $\theta^{+}_{r,k} = 1$ if the path arrives at point $\mathbf{r}$ with momentum $\mathbf{k}$ from the right bank ($z = +\infty$), while if the path arrives from the left bank ($z = -\infty$) we have $\theta^{+}_{r,k} = 0$. There is a corresponding definition for $\theta^{-}_{r,k}$. We obviously have

$$\theta^{+}_{r,k}+\theta^{-}_{r,k} = 1. \tag{9}$$

It can be seen from (6) that at the banks ($z\to\pm\infty$) the momentum distribution is an equilibrium distribution, as it should be in the region in which a potential whose value is constant over space is given. From (1) and (6) we find

$$\delta\rho(\mathbf{r}) = \delta\rho_\infty(\mathbf{r})+\delta\rho_e(\mathbf{r}), \tag{10}$$

where the second term depends locally on the potential,

$$\delta\rho_e(\mathbf{r}) = -e^2 g_T\varphi(\mathbf{r}), \tag{11}$$

$$g_T = 2\int\frac{d^3k}{(2\pi)^3}\left(-\frac{\partial f_T}{\partial\varepsilon_{\mathbf{k}}}\right), \tag{12}$$

while the first term depends on only the asymptotic values of the potential,

$$\delta\rho_\infty(\mathbf{r}) = e^2 g_T[\varphi(+\infty)w^{+}(\mathbf{r})+\varphi(-\infty)w^{-}(\mathbf{r})], \tag{13}$$

$$w^{\pm}(\mathbf{r}) = \frac{2}{g_T}\int\frac{d^3k}{(2\pi)^3}\left(-\frac{\partial f_T}{\partial\varepsilon_{\mathbf{k}}}\right)\theta^{\pm}_{r,k}. \tag{14}$$

Here g_T is the effective density of states which are active in transport processes, and $w^{\pm}(\mathbf{r})$ are the probabilities that the path arrives at point $\mathbf{r}$ from the right and left banks. According to (9) we have

$$w^{+}(\mathbf{r})+w^{-}(\mathbf{r}) = 1. \tag{15}$$

It can be seen from the latter equations that if we have $\varphi(\mathbf{r})\equiv$const then we have $\delta\rho(\mathbf{r})\equiv 0$, as we should. With increasing distance into the banks, the functions $w^{\pm}(\mathbf{r})$ become equal to zero and unity. Far inside the banks we thus have $\delta\rho = 0$.

Breaking $\delta\rho$ up into two terms as in (10), we can rewrite the Poisson equation (2) in the following form:

$$\nabla^2\varphi - \frac{\varphi}{a^2} = -\frac{4\pi}{\epsilon}\delta\rho_\infty, \quad a^{-2} = 4\pi g_T e^2/\epsilon. \tag{16}$$

Here a is the screening length (Debye or Thomas–Fermi). In the region occupied by the electron gas, Eq. (16) should be supplemented with the Poisson equation in the surrounding medium and with the appropriate conditions on the potential at the boundary of the electron gas.

Characteristic distances for the variation of the function $\delta\rho_\infty(\mathbf{r})$ are d and L. For $|z|\gg d, L$, the density $\delta\rho_\infty$ approaches the asymptotic value $e^2 g_T\varphi(\pm\infty)$. In the case of strong screening, with $a\ll d, L$, we can thus ignore the term $\nabla^2\varphi$ in (16); we find

$$\varphi(\mathbf{r}) = \varphi(+\infty)w^{+}(\mathbf{r})+\varphi(-\infty)w^{-}(\mathbf{r}). \tag{17}$$

This result corresponds to the "quasineutrality approximation," in which we can adopt $\delta\rho = 0$ as a condition for determining $\varphi(\mathbf{r})$. In this approximation, the potential difference V decreases in a region $\Delta z\sim L, d$, as can be seen from (17).

In the opposite limit, of weak screening, $a\gg L, d$, we cannot discard the term φ/a^2 (as can be verified), so the resulting equation has no solution which satisfies boundary condition (3). In the case of weak screening the potential difference V decreases over a region $\Delta z\sim a$ (it can be verified that this assertion is valid not only for the model of a point contact in the form of an aperture[13] but also for a model in the form of a constriction).

The validity of (8) requires that the quantity $l = v\tau$ be greater than the region over which the potential decreases. In the case of strong screening, the condition under which the point contact is of a ballistic nature is therefore $l\gg d, L$, while in the case of weak screening the corresponding condition is $l\gg a$.

3. BASIC EQUATIONS FOR A QUANTUM POINT CONTACT

We describe the behavior of the electrons in the quantum point contact in the self-consistent-field approximation. In equilibrium for $V = 0$, each electron is in a potential $U(\mathbf{r})$ which is the sum of the potential created by the ions and the self-consistent potential of the electrons. The poten-

tial U determines the electron states $\psi_n(\mathbf{r})$ and their energies ε_n. States n are filled in accordance with the Fermi distribution $f_T(\varepsilon_n)$. The electron charge density is

$$\rho(\mathbf{r})=2e\sum_n f_T(\varepsilon_n)\,|\psi_n(\mathbf{r})|^2. \tag{18}$$

Near the constriction, with $d\lesssim\lambda$, the states ψ_n are quite different from plane waves $e^{i\mathbf{kr}}$, and the density ρ is spatially nonuniform. In the banks, this nonuniformity prevails only over distances on the order of λ from the boundaries of the sample.

In the case $V\neq 0$, a change occurs in the electron density, $\delta\rho(\mathbf{r})$, and a change occurs in the self-consistent potential, $\delta U(\mathbf{r})$. If we write $\delta U(\mathbf{r})=e\varphi(\mathbf{r})$, then $\varphi(\mathbf{r})$ is that electrostatic potential which arises from the applied potential difference V. Obviously, φ and $\delta\rho$ are related by the Poisson equation (2).

In the quantum-mechanical case, the kinetic equation (4) should be replaced by an equation for a one-electron (spin-zero) density matrix f:

$$\frac{\partial f}{\partial t}=-i[H,f]-\frac{f-f_0}{\tau}=0, \tag{19}$$

where

$$H=H_0+e\varphi,\quad H_0=-\frac{1}{2m}\nabla^2+U,\quad f_0=f_T(H_0). \tag{20}$$

Calculating $\delta f=f-f_0$ for a given φ from (19), we find the change in the electron charge density,

$$\delta\rho(\mathbf{r})=2e(\mathbf{r}|\delta f|\mathbf{r}), \tag{21}$$

as a functional of the potential: $\delta\rho\{\varphi\}$. Substituting this functional into the right side of the Poisson equation (2), we find an equation for seeking φ.

A solution of (19) is

$$f=\int_{-\infty}^{0}\frac{dt}{\tau}e^{t/\tau}\hat{f}_0(t), \tag{22}$$

where for any operator we have, by definition,

$$\hat{v}(t)=e^{iHt}ve^{-iHt}, \tag{23}$$

and in particular we have

$$\hat{f}_0(t)=f_T(\hat{H}_0(t))=f_T(H-e\hat{\varphi}(t)). \tag{24}$$

Substituting (24) into (22), we find

$$f=\int_{-\infty}^{0}\frac{dt}{\tau}e^{t/\tau}f_T(H_0+e\varphi-e\hat{\varphi}(t)), \tag{25}$$

in complete analogy with the classical expression (5).

We now linearize f in (25) with respect to φ. First linearizing $\hat{f}_0(t)$, we find

$$\delta\hat{f}_0(t)=-i\int_0^t dt'\,[f_0,e\tilde{\varphi}(t-t')]=-i\int_0^t ds\,[f_0,e\tilde{\varphi}(s)],$$
$$\tilde{v}(t)=e^{iH_0t}ve^{-iH_0t}. \tag{26}$$

We introduce a convergence factor $e^{\eta s}$, where $\eta\to+0$, within the integral (we recall that we are interested in $t<0$). We can then use the representation

$$\int_0^t ds\ldots=\int_0^{-\infty}ds\ldots+\int_{-\infty}^t ds\ldots. \tag{27}$$

We define the superoperator $\mathscr{P}$ by

$$\mathscr{P}v=-i\int_0^\infty ds\,e^{-\eta s}[f_0,\tilde{v}(-s)],\quad \eta\to+0. \tag{28}$$

Now using the replacement $s\to -s$ in the first of integrals (27), and $s\to s+t$ in the second, we find

$$\delta\hat{f}_0(t)=\mathscr{P}[e\tilde{\varphi}(t)-e\varphi]. \tag{29}$$

Using (22) and (29), we finally find

$$\delta f=\mathscr{P}[\langle e\varphi\rangle-e\varphi], \tag{30}$$

where we are using the operator

$$\langle\varphi\rangle=\int_{-\infty}^{0}\frac{dt}{\tau}e^{t/\tau}\tilde{\varphi}(t). \tag{31}$$

The last two expressions constitute the quantum-mechanical generalization of (6) and (7). The replacement of $\hat{\varphi}$ by $\tilde{\varphi}$ corresponds to a calculation of the path without consideration of the field $\mathbf{E}=-\nabla\varphi$. The superoperator $\mathscr{P}$ is the quantum-mechanical analog of multiplication by $(-\partial f_T/\partial\varepsilon)$.

The meaning of the representation of the correction δf to the density matrix in the form (30) can be explained as follows: If, after the potential $e\varphi$ is turned on, the electron system remained at equilibrium (i.e., if no current flowed), the correction to f would have been $-\mathscr{P}e\varphi$. That this is true can be seen quite clearly from the classical expression (6). It can also be seen from this expression that the corresponding term in δf does not give rise to a current. This result can also be verified in the quantum-mechanical case. A current arises only by virtue of the term $\mathscr{P}\langle e\varphi\rangle$.

The correction to the charge density is broken up similarly. From (21) we find

$$\delta\rho(\mathbf{r})=\delta\rho_\infty(\mathbf{r})+\delta\rho_e(\mathbf{r}),$$
$$\delta\rho_\infty(\mathbf{r})=2e^2(\mathbf{r}|\mathscr{P}\langle\varphi\rangle|\mathbf{r}),\quad \delta\rho_e(\mathbf{r})=-2e^2(\mathbf{r}|\mathscr{P}\varphi|\mathbf{r}). \tag{32}$$

4. ELECTRON CHARGE DENSITY INDUCED BY THE POTENTIAL FIELD

In this section we express the density components (32) in terms of quantities associated with the retarded Green's function of Hamiltonian H_0. To keep the calculations free of the complexities which stem from the continuous spectrum, we restrict the size of the banks to some large value R such that $R\gg v\tau$ (it can be assumed that over this distance the current flowing through the constriction has been completely "absorbed" by the reservoir, i.e., the banks). Since H_0 does not contain the magnetic field, all of its eigenfunctions ψ_n can be chosen to be real.

The retarded Green's function is

$$G(\mathbf{r},t|\mathbf{r}')=\theta(t)\sum_n e^{-i\varepsilon_n t}\psi_n(\mathbf{r})\psi_n(\mathbf{r}')=\theta(t)\,(\mathbf{r}|e^{-iH_0t}|\mathbf{r}'). \tag{33}$$

Its Fourier transform

$$G_\varepsilon(\mathbf{r},\mathbf{r}')=-i\int_0^\infty dt\, e^{i\varepsilon t}G(\mathbf{r},t|\mathbf{r}')=\sum_n\frac{\psi_n(\mathbf{r})\psi_n(\mathbf{r}')}{\varepsilon-\varepsilon_n+i0} \quad (34)$$

satisfies the equation

$$[H_0(\mathbf{r})-\varepsilon]G_\varepsilon(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'). \quad (35)$$

The function G_ε can be calculated by removing the boundaries of the banks (i.e., by setting $R\to\infty$), if we require that as $z\to\pm\infty$ the transform G_ε contain only outgoing waves.

Below we will use the quantity

$$g_\varepsilon(\mathbf{r},\mathbf{r}')=\sum_n\delta(\varepsilon-\varepsilon_n)\psi_n(\mathbf{r})\psi_n(\mathbf{r}')=-\frac{1}{\pi}\operatorname{Im}G_\varepsilon(\mathbf{r},\mathbf{r}'). \quad (36)$$

The following orthogonality relation holds here:

$$\int d\bar{\mathbf{r}}\,g_\varepsilon(\mathbf{r},\bar{\mathbf{r}})g_{\varepsilon'}(\mathbf{r}',\bar{\mathbf{r}})=\delta(\varepsilon-\varepsilon')g_\varepsilon(\mathbf{r},\mathbf{r}'). \quad (37)$$

Using the system of functions ψ_n, we can find a representation for the superoperator $\mathscr{P}$. From its definition, (28), we have

$$(\mathbf{r}_1|\mathscr{P}v|\mathbf{r}_2)$$

$$=-\int\int d\mathbf{r}\,d\mathbf{r}'\sum_{nn'}\frac{f_T(\varepsilon_n)-f_T(\varepsilon_{n'})}{\varepsilon_n-\varepsilon_{n'}-i\eta}$$

$$\times(\mathbf{r}|v|\mathbf{r}')\psi_n(\mathbf{r}_1)\psi_n(\mathbf{r})\psi_{n'}(\mathbf{r}_2)\psi_{n'}(\mathbf{r}'). \quad (38)$$

Switching to Green's functions, we find

$$(\mathbf{r}_1|\mathscr{P}v|\mathbf{r}_2)=-\int\int d\mathbf{r}\,d\mathbf{r}'\int\int d\varepsilon\,d\varepsilon'\frac{f_T(\varepsilon)-f_T(\varepsilon')}{\varepsilon-\varepsilon'-i\eta}$$
$$\times(\mathbf{r}|v|\mathbf{r}')g_\varepsilon(\mathbf{r},\mathbf{r}_1)g_{\varepsilon'}(\mathbf{r}',\mathbf{r}_2). \quad (39)$$

It is then an easy matter to find the second term in the density (32):

$$\delta\rho_e(\mathbf{r})=-e^2\int d\mathbf{r}'\,\Pi(\mathbf{r},\mathbf{r}')\varphi(\mathbf{r}'), \quad (40)$$

where

$$\Pi(\mathbf{r},\mathbf{r}')=-2\int\int d\varepsilon\,d\varepsilon'\frac{f_T(\varepsilon)-f_T(\varepsilon')}{\varepsilon-\varepsilon'}g_\varepsilon(\mathbf{r},\mathbf{r}')g_{\varepsilon'}(\mathbf{r},\mathbf{r}'). \quad (41)$$

We turn now to the first term in (32). We first calculate $\langle\varphi\rangle$. Using the definition (31), we find

$$(\mathbf{r}|\langle\varphi\rangle|\mathbf{r}')=\int d\bar{\mathbf{r}}\,\varphi(\bar{\mathbf{r}})(\mathbf{r}|K(\bar{\mathbf{r}})|\mathbf{r}'), \quad (42)$$

where

$$(\mathbf{r}|K(\bar{\mathbf{r}})|\mathbf{r}')=\int_{-\infty}^0\frac{dt}{\tau}e^{t/\tau}(\mathbf{r}|e^{iH_0t}|\bar{\mathbf{r}})(\bar{\mathbf{r}}|e^{-iH_0t}|\mathbf{r}'). \quad (43)$$

Using (33) to express the matrix elements in terms of Green's functions, we find

$$(\mathbf{r}|K(\bar{\mathbf{r}})|\mathbf{r}')=\int_0^\infty\frac{dt}{\tau}e^{-t/\tau}G(\bar{\mathbf{r}},t|\mathbf{r})G^*(\bar{\mathbf{r}},t|\mathbf{r}'). \quad (44)$$

In the integration over t, values $t\sim\tau$ are important. If τ is large, then the functions G and also the function K are essentially zero everywhere except at points $\bar{\mathbf{r}}$ far from the source, where the potential $\varphi(\bar{\mathbf{r}})$ takes on one of its asymptotic values $\varphi(\pm\infty)$. As $\tau\to\infty$ we thus find from (42), by analogy with (8),

$$(\mathbf{r}|\langle\varphi\rangle|\mathbf{r}')=\varphi(+\infty)(\mathbf{r}|\theta^+|\mathbf{r}')+\varphi(-\infty)(\mathbf{r}|\theta^-|\mathbf{r}'), \quad (45)$$

where

$$(\mathbf{r}|\theta^\pm|\mathbf{r}')=\int_\pm d\bar{\mathbf{r}}(\mathbf{r}|K(\bar{\mathbf{r}})|\mathbf{r}'). \quad (46)$$

The $\pm$ on the integral means that the integration is over the half-space $z>0$ or $z<0$. Substituting the Green's functions in the form of expansions (33) into (44), we find

$$(\mathbf{r}|K(\bar{\mathbf{r}})|\mathbf{r}')=\int\int d\varepsilon\,d\varepsilon'\frac{1}{1+i(\varepsilon-\varepsilon')\tau}g_\varepsilon(\mathbf{r},\bar{\mathbf{r}})g_{\varepsilon'}(\mathbf{r}',\bar{\mathbf{r}}). \quad (47)$$

To find $\delta\rho_\infty(\mathbf{r})$, we must calculate $\mathscr{P}\theta^\pm$. Using (39) and (47) and the orthogonality condition (37), we find

$$(\mathbf{r}|\mathscr{P}\theta^\pm|\mathbf{r})=-\int_\pm d\bar{\mathbf{r}}\int\int d\varepsilon\,d\varepsilon'\frac{f_T(\varepsilon)-f_T(\varepsilon')}{\varepsilon-\varepsilon'}$$
$$\times\frac{1}{1+i(\varepsilon-\varepsilon')\tau}g_\varepsilon(\mathbf{r},\bar{\mathbf{r}})g_{\varepsilon'}(\mathbf{r},\bar{\mathbf{r}}). \quad (48)$$

It follows from the discussion above that remote values of $\bar{\mathbf{r}}$ are important in this integral. We thus write the Green's function in terms of the radiation amplitude:

$$G_\varepsilon(\mathbf{r},\bar{\mathbf{r}})=A_\varepsilon(\mathbf{r},\bar{\mathbf{n}})\frac{\exp(ik_\varepsilon\bar{r})}{\bar{r}},\quad k_\varepsilon=(2m\varepsilon)^{1/2}. \quad (49)$$

Here $\bar{\mathbf{n}}$ is a unit vector in the $\bar{\mathbf{r}}$ direction (Fig. 1). Using this representation, we write

$$g_\varepsilon g_{\varepsilon'}=-\frac{1}{4\pi^2}\frac{1}{\bar{r}^2}(Ae^{ik\bar{r}}-A^*e^{-ik\bar{r}})(A'e^{ik'\bar{r}}-A'^*e^{-ik'\bar{r}}). \quad (50)$$

Here the prime means that the quantity refers to the energy ε'. As we will see below, small values of $\varepsilon-\varepsilon'\approx 1/\tau$ are important in the integral over the energies. We thus need retain only the slowly oscillating terms in product (50):

$$g_\varepsilon g_{\varepsilon'}\to\frac{1}{4\pi^2}\frac{1}{\bar{r}^2}|A|^2[e^{i(k-k')\bar{r}}+\text{ c.c. }]. \quad (51)$$

Furthermore, we can use the replacement

$$\frac{f_T(\varepsilon)-f_T(\varepsilon')}{\varepsilon-\varepsilon'}\to\frac{\partial f_T(\varepsilon)}{\partial\varepsilon}. \quad (52)$$

Writing $k-k'=(\varepsilon-\varepsilon')/v_\varepsilon$, and setting $\varepsilon-\varepsilon'=\omega$, we can put the integral (48) in the form

$$(\mathbf{r}|\mathscr{P}\theta^\pm|\mathbf{r})=\frac{1}{2\pi^2}\int_0^\infty d\bar{r}\int_\pm d\bar{o}\int d\varepsilon\int_{-\infty}^{+\infty}d\omega\left(-\frac{\partial f_T}{\partial\varepsilon}\right)$$
$$\times\frac{1}{1+i\omega\tau}|A_\varepsilon|^2\cos\frac{\omega\bar{r}}{v_\varepsilon}. \quad (53)$$

Here the $\pm$ on the integral over the solid angle $d\bar{o}$ near the direction $\bar{\mathbf{n}}$ shows that the integration is carried out along those directions $\bar{\mathbf{n}}$ which go off to the right or left bank. The integration over $\bar{r}$ and ω is rendered dimensionless by means of the quantities $v_\varepsilon\tau$ and $1/\tau$; our assumptions that $\bar{r}$ is large and ω small are thus justified. As a result we find

$$(\mathbf{r}|\mathscr{P}\theta^\pm|\mathbf{r})=\frac{1}{2\pi}\int d\varepsilon\left(-\frac{\partial f_T}{\partial\varepsilon}\right)v_\varepsilon\int_\pm d\bar{o}\,|A_\varepsilon(\mathbf{r},\bar{\mathbf{n}})|^2. \quad (54)$$

The integral over angles multiplied by v_ε in (54) is the flux of particles of energy ε which is emitted into the right or left bank by a point source at the point $\mathbf{r}$. If the source is in an unbounded space, the total flux in all directions is $m^2 v_\varepsilon/\pi$. The ratio of these fluxes is by definition the emissivity of point $\mathbf{r}$ for emission into the right or left bank:

$$\eta_\varepsilon^{\pm}(\mathbf{r}) = \frac{\pi}{m^2}\int_{\pm} d\bar{o}\,|A_\varepsilon(\mathbf{r},\bar{\mathbf{n}})|^2. \tag{55}$$

Using this concept, we find the final expression

$$\delta\rho_\infty(\mathbf{r}) = e^2[q^+(\mathbf{r})\varphi(+\infty)+q^-(\mathbf{r})\varphi(-\infty)], \tag{56}$$

$$q^{\pm}(\mathbf{r}) = \int d\varepsilon\, g_\varepsilon\left(-\frac{\partial f_T}{\partial\varepsilon}\right)\eta_\varepsilon^{\pm}(\mathbf{r}). \tag{57}$$

Here we have introduced the state density in the banks:

$$g_\varepsilon = 2\frac{m^{3/2}}{\sqrt{2}\pi^2}\sqrt{\varepsilon}. \tag{58}$$

Using (48), and using orthogonality condition (37), we can easily verify the following equation:

$$q^+(\mathbf{r})+q^-(\mathbf{r}) = \int d\varepsilon\left(-\frac{\partial f_T}{\partial\varepsilon}\right) g_\varepsilon(\mathbf{r},\mathbf{r}) \equiv g_T(\mathbf{r}). \tag{59}$$

This quantity is the effective density of kinetically active states at point $\mathbf{r}$. On the other hand, from (41) and (37) we easily find

$$\int d\mathbf{r}'\,\Pi(\mathbf{r},\mathbf{r}') = g_T(\mathbf{r}). \tag{60}$$

We now see that if we have $\varphi(\mathbf{r})\equiv$const then we have $\delta\rho(\mathbf{r})\equiv 0$, as we should.

5. POISSON'S EQUATION

Before we consider the Poisson equation, let us take a more detailed look at the components of the induced charge density. We begin with the component $\delta\rho_\infty$, which is associated with the flow of a current through the point contact. The quantities $q^{\pm}(\mathbf{r})$ in (56) show the extent to which the point $\mathbf{r}$ is coupled with the banks. With a reciprocity principle in mind, we can say that $q^{\pm}(\mathbf{r})$ is the "probability" that the electron waves emitted by the right or left bank (reservoir) reach point $\mathbf{r}$. In this sense, $q^{\pm}(\mathbf{r})$ is a quantum-mechanical analog of the quantity $g_T w^{\pm}(\mathbf{r})$ in (13). There is, however, an important distinction between the classical and quantum quantities. It follows from (15) that in a classical point contact the total coupling with the two banks is identical for all points $\mathbf{r}$. In a quantum point contact, this is not the case, as we see from (59). Those points at which $|\psi_n(\mathbf{r})|^2$ is small for actual states are coupled only poorly with the banks. One could thus say that the density $\delta\rho_\infty(\mathbf{r})$, which is associated with the current flow, is large in places into which the electron waves arriving from the banks can easily penetrate.

We turn now to the density $\delta\rho_e$ in (40), which corresponds to a redistribution of the charges in the field $e\varphi$ under equilibrium conditions, as we have already stated. To strengthen this interpretation, we consider the result of the conversion of the kernel Π in the banks far from the constriction and far from the boundaries of the sample, where the system can be assumed spatially homogeneous, and where the functions ψ_n can be assumed to be plane waves. The kernel $\Pi(\mathbf{r},\mathbf{r}')$ then depends only on the difference $\mathbf{r}-\mathbf{r}'$, and we can write the Fourier transform as follows:

$$\Pi_{\mathbf{q}} = \int d^3r\, e^{-i\mathbf{q}\mathbf{r}}\Pi(\mathbf{r},0) = -2\int\frac{d^3k}{(2\pi)^3}\frac{f_T(\varepsilon_{\mathbf{k}})-f_T(\varepsilon_{\mathbf{k}+\mathbf{q}})}{\varepsilon_{\mathbf{k}}-\varepsilon_{\mathbf{k}+\mathbf{q}}}. \tag{61}$$

We thus have

$$(\delta\rho_e)_{\mathbf{q}} = -e^2\Pi_{\mathbf{q}}\varphi_{\mathbf{q}}, \tag{62}$$

where Π_q is a polarization loop of the screening of the Coulomb interaction. We thus see in particular that our self-consistent-field approximation corresponds somewhat to the random phase approximation, since the total potential, rather than the potential of external sources, appears in (62) (Ref. 14).

Far from the constriction, $\Pi_{q=0}$ (which is $g_T(\mathbf{r})$ according to (60)) is the same as g_T from (12).

In the coordinate representation, $\Pi(\mathbf{r})=\Pi(\mathbf{r},0)$ is a kernel with a range λ, as can be seen from, for example, the known expression for a degenerate gas at $T=0$:

$$\Pi(r) = \frac{mk_F^2}{4\pi^2}\frac{1}{r^2}j_1(2k_Fr), \quad j_1(x) = x^{-2}\sin x - x^{-1}\cos x. \tag{63}$$

The kernel $\Pi(r)$ has a singularity r^{-1} in the limit $r\to 0$. This singularity is integrable. In the limit $r\to\infty$, we find $\Pi(r)\sim r^{-3}\cos 2k_Fr$. We thus see that the range of kernel $\Pi(r)$ is λ_F and that the characteristic value of Π is $\sim g_F k_F^3$.

It is clear from these properties of kernel Π that deep in the banks, where the potential $\varphi(\mathbf{r})$ takes on its asymptotic values, we have

$$\delta\rho_e(\mathbf{r})|_{z\to\pm\infty} = -e^2 g_T\varphi(\pm\infty). \tag{64}$$

On the other hand, as we go off into (for example) the right bank we find $\eta^+\to 1$ and $\eta^-\to 0$ and thus $g^+\to g_T$ and $g^-\to 0$. We thus have

$$\delta\rho_\infty(\mathbf{r})|_{z\to\pm\infty} = e^2 g_T\varphi(\pm\infty); \tag{65}$$

i.e., the total density change is $\delta\rho=0$ in the banks far from the constriction.

There is another important distinction between $\delta\rho_\infty$ and $\delta\rho_e$, which is manifested in the case of a degenerate gas. Only the kinetically active states near the Fermi surface participate in the formation of $\delta\rho_\infty$, as can be seen from (57). For the density $\delta\rho_e$, on the other hand, the kernel Π which determines this density is formed by all states, as can be seen from (41) and (61).

The potential distribution in a quantum point contact is determined by the integrodifferential equation

$$\nabla^2\varphi(\mathbf{r}) - \frac{4\pi e^2}{\epsilon}\int d\mathbf{r}'\,\Pi(\mathbf{r},\mathbf{r}')\varphi(\mathbf{r}') = \frac{4\pi}{\epsilon}\rho_\infty(\mathbf{r}) \tag{66}$$

with a given right side. This equation should be supplemented with Poisson's equation in the region around the point contact, along with the corresponding boundary conditions at the boundary of the point contact.

The estimate for the kernel Π far from the constriction which we presented above obviously continues to hold near the constriction if $d\gtrsim\lambda$. Accordingly, the order-of-magnitude estimate of the integral term on the left side of (66) is the same as in the classical equation (16), i.e., φ/a^2. This result means that the considerations expressed at the end of Sec. 2 regarding the region over which the applied potential

difference decreases are also valid for a quantum point contact if the condition $d \gtrsim \lambda$ holds. The condition under which the point contact can be regarded as ballistic, also stated at the end of Sec. 2, also continue to hold.

There are some obvious generalizations here. The potential U can include external fields which bound the region in which the electrons move (e.g., the field of a gate in the experiments of Refs. 1 and 2). It may also include the potential of elastic scatterers in the vicinity of the constriction, $|z| \lesssim L$ (e.g., individual impurities).

If the point contact is multipolar rather than a two–pole, i.e., if there are many massive banks, the right side of Poisson's equation (66) takes the form

$$\delta\rho_\infty(\mathbf{r}) = e^2 \sum_s q_s(\mathbf{r})\varphi_s, \tag{67}$$

where φ_s is the value of the potential specified in bank s, and $q_s(\mathbf{r})$ is constructed by analogy with (57) with the help of the emissivity $\eta_\varepsilon^s(\mathbf{r})$ corresponding to the emission into bank s.

Equation (66) can be compared with Eq. (45) of Ref. 10 for the potential in a one–dimensional channel. In Eq. (45) of Ref. 10, the screening was local, while in (66) it is nonlocal over distances on the order of λ_F. More important is the circumstance that the term which is nonuniform in φ in Eq. (45) of Ref. 10 contains instead of the emissivities $q^\pm$ in (56) the wave functions $|\psi_\pm|^2$ for electrons arriving from the right and left reservoirs. For this reason, as was pointed out in Ref. 10, the potential in the channel does not go over smoothly to the reservoir potential. A smooth transition does occur in the case of Eq. (66).

6. POTENTIAL VARIATION IN A QUANTUM CHANNEL

As an example we consider a point contact in the form of a long channel ($L \gg d$) with expanding banks (Fig. 2). For simplicity we assume that the point contact is symmetric with respect to the $z = 0$ plane, and we set $\varphi(\pm\infty) = \pm V/2$. We begin with a discussion of the potential distribution in the classical picture, with $d \gg \lambda_F$. It is easy to see from geometric considerations that in the case of specular reflection from the boundaries of the sample the following relation holds for points inside the channel (at distances greater than d from the ends of the channel):

$$\theta_{\mathbf{r},\mathbf{k}}^{\pm} = \theta(\mp k_z), \tag{68}$$

where $\theta(x)$ is the unit step function. It then follows from (14) that inside the channel we have $w^\pm(\mathbf{r}) = 1/2$, and from (13) we find $\delta\rho_\infty(\mathbf{r}) = 0$. In the quasineutrality approximation, (17), this result means that we have $\varphi(\mathbf{r}) = 0$ inside the channel. If we abandon the quasineutrality approximation but assume $L \gg a$, we find from (16) that we have $\varphi(\mathbf{r}) = 0$ at points in the channel at distances greater than a from the ends of the channel. We thus find $\varphi(\mathbf{r}) = \text{const}$ inside a sufficiently long channel in the classical description.

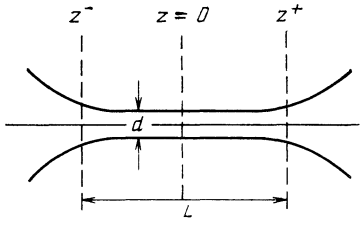

FIG. 2.

In the quantum–mechanical case, with $d \approx \lambda_F$, this situation is different. To demonstrate the point, we first examine the right side of Eq. (66):

$$\delta\rho_\infty(\mathbf{r}) = {}^1/_2 e^2 V g[\eta^+(\mathbf{r}) - \eta^-(\mathbf{r})], \tag{69}$$

where g and $\eta^\pm$ are calculated for $\varepsilon = \varepsilon_F$. We assume a single–mode channel; i.e., we assume that below ε_F there is only a single transverse–quantization level ε_1. For points far from the ends of the channel, the emissivities $\eta^\pm(\mathbf{r})$ are determined by this mode exclusively and can be calculated quite easily. As a result we find

$$\eta^+(\mathbf{r}) - \eta^-(\mathbf{r}) = -\frac{16\pi}{k_F k_1}[\psi_1(x,y)]^2 \frac{(1-r^2)r}{|2-r^2e^{2i\alpha}|^2} \times \sin\alpha \sin(2k_1 z). \tag{70}$$

Here $\psi_1(x,y)$ is the wave function of the transverse motion for level ε_1,

$$k_1^2/2m = \varepsilon_F - \varepsilon_1, \quad \alpha = k_1 L + \varphi, \quad R = re^{i\varphi}, \tag{71}$$

where R is the coefficient of the reflection of the waveguide mode from the boundary planes z^+ and z^- (the phase a is independent of the way in which these planes are chosen).

With regard to the kernel $\Pi(\mathbf{r},\mathbf{r}')$, we note that even for a single–mode channel it is a complicated matter to evaluate this kernel, since it receives contributions from all modes, including nonpropagating modes, i.e., modes which correspond to transverse–quantization levels ε_2, ε_3,..., which lie below ε_F. It can be verified, however, that for points $\mathbf{r}$ and $\mathbf{r}'$ which are farther than λ_F from the ends of the channel the kernel Π does not sense these ends and depends on the difference between z and z'. This is due to interference between states with different energies, which arises during the integration in (41), even at $T = 0$. The dependence of Π on x, y and x', y' is extremely complicated.

We thus see that the kernel Π is a function of the difference between z and z' in Eq. (66) for the potential inside a channel and far from the ends of the channel, while the right side has a z dependence $\sin 2k_1 z$. Obviously, a solution with $\varphi(\mathbf{r}) = \text{const}$ inside the channel would be impossible in this case. Also impossible would be a solution in which $\varphi(\mathbf{r})$ depends only on x, y. We thus see that in a quantum channel at $T = 0$ the reflections from the ends of the channel unavoidably lead to a variation of the potential along the channel.

We turn now to the case $T \neq 0$. It can be seen from (57) that in this case the difference in (70), which appears in (69), should be "averaged over ε_F" in an interval $\Delta\varepsilon_F \approx T$. In this case we have $\Delta k_1 \approx T/v_F$, and if $L \gg v_F/T$ then (70) vanishes after the averaging. Consequently, if $T \neq 0$ the variations in the potential penetrate a distance v_F/T from the ends of the channel into the channel.

7. CONDUCTANCE OF A POINT CONTACT

The current density in a point contact is

$$\mathbf{j}(\mathbf{r}) = \int\int d\mathbf{r}'\, d\mathbf{r}''(\mathbf{r}'|\mathbf{j}(\mathbf{r})|\mathbf{r}'')(\mathbf{r}''|\delta f|\mathbf{r}'), \tag{72}$$

where the matrix element of the current density operator is

$$(\mathbf{r}'|\mathbf{j}(\mathbf{r})|\mathbf{r}'')=\frac{ie}{2m}[\delta(\mathbf{r}-\mathbf{r}'')\nabla\delta(\mathbf{r}-\mathbf{r}')-\delta(\mathbf{r}-\mathbf{r}')\nabla\delta(\mathbf{r}-\mathbf{r}'')]. \tag{73}$$

Substituting the density-matrix correction δ_f from (30) into (72) and writing the superoperator $\mathscr{P}$ as in (38), we can show that the real nature of the functions ψ_n has the consequence that the term $-\mathscr{P}e\varphi$ in (30) does not contribute to $\mathbf{j}(\mathbf{r})$. In other words, the term in δf which is responsible for the "equilibrium" screening charge $\delta\rho_e$ does not give rise to a current. This point was brought out by Landauer.[9] The only contribution to the current density comes from the term $\mathscr{P}\langle e\varphi\rangle$, which is completely determined by the asymptotic value of the potential in the banks, according to (45). The current density in the point contact can thus be calculated without knowledge of the potential distribution in it.

To calculate $\mathbf{j}(\mathbf{r})$, we should substitute the following into (72):

$$(\mathbf{r}''|\delta f|\mathbf{r}')=\varphi(+\infty)(\mathbf{r}''|\mathscr{P}\theta^{+}|\mathbf{r}')+\varphi(-\infty)(\mathbf{r}''|\mathscr{P}\theta^{-}|\mathbf{r}'). \tag{74}$$

The off-diagonal matrix elements $(\mathbf{r}|\mathscr{P}\theta^{\pm}|\mathbf{r}')$ are given by the expression found from (48) by replacing $\mathbf{r}$ by $\mathbf{r}'$ in g_ε'. Since we are interested in only the total current through the point contact, we find it convenient to evaluate $\mathbf{j}(\mathbf{r})$ far inside the banks. For implicity we set $\varphi(+\infty)=0$, and we evaluate $\mathbf{j}(\mathbf{r})$ in the right bank as $z\to+\infty$. The points $\mathbf{r}'$ and $\mathbf{r}''$ in the matrix element of $\mathscr{P}\theta^{-}$ are then far away within the right bank, and the point of integration in (48) is far away within the left bank. Clearly, a Green's function $G_\varepsilon(\mathbf{r},\mathbf{r}')$, in which the points $\mathbf{r}$ and $\mathbf{r}'$ go off to infinity, $\mathbf{r}$ in the left bank and $\mathbf{r}'$ in the right, arises in the course of the calculation. A Green's function of this sort has the asymptotic representation

$$G_\varepsilon(\mathbf{r},\mathbf{r}')\Big|_{\substack{z\to-\infty\\ z'\to+\infty}}=\frac{e^{ik_\varepsilon r}}{r}\frac{e^{ik_\varepsilon r'}}{r'}\frac{1}{v_\varepsilon}T_\varepsilon(\mathbf{n},\mathbf{n}'), \tag{75}$$

where $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors which are in the directions along which $\mathbf{r}$ and $\mathbf{r}'$ go off to infinity. A calculation similar to that which resulted in (54) leads to the following result for the conductance of the point contact:

$$G=2\frac{e^2}{h}\int d\varepsilon\left(-\frac{\partial f_T}{\partial\varepsilon}\right)\int_{-}do\int_{+}do'|T_\varepsilon(\mathbf{n},\mathbf{n}')|^2. \tag{76}$$

The integration over do and do' here is carried out along directions which go off into the left and right banks, respectively; the factor of 2 stems from the summation over the spin orientations; and $|T_\varepsilon(\mathbf{n},\mathbf{n}')|^2$ is the probability that a plane wave arriving from the left bank along the $-\mathbf{n}$ direction goes off into the right bank along the $\mathbf{n}'$ direction.

If the point contact is multipolar, by which we mean that it has more than two banks, then the currents J_s entering banks $s=1, 2,\ldots$ are related linearly to the potentials (φ_s) in these banks[10]:

$$J_s=\sum_{s'}G_{ss'}\varphi_{s'}. \tag{77}$$

It can be shown that the off-diagonal conductance matrix elements G_{ss}' $(s\neq s')$ are given by (76) again, if the integration over do and do' is carried out along directions going off into banks s and s'.

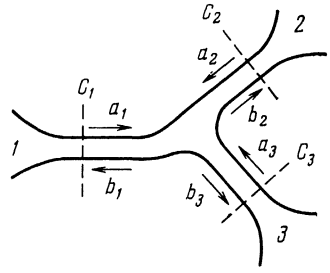

FIG. 3.

We now consider a multipole point contact of a configuration such that each horn goes over to a long waveguide channel (Fig. 3). For a configuration of this sort, the off-diagonal conductances G_{ss}' can be expressed in terms of a scattering matrix for a waveguide joint with truncated horns (between cross sections C_s in Fig. 3) and the reflection coefficients for waves going from the waveguide into the horn. To avoid an overload of indices, we assume that the waveguide channels are single-mode channels. We can then derive the following expression:

$$G_{ss'}=2\frac{e^2}{h}\mathrm{Av}\{(1-|R_s|^2)(1-|R_{s'}|^2)|(Q^{-1}S)_{ss'}|^2\}. \tag{78}$$

Here S is the scattering matrix of the waveguide joint; i.e.,

$$b_s=\sum_{s'}S_{ss'}a_{s'}, \tag{79}$$

where a_s and b_s are the amplitudes of respectively the incoming and outgoing waves in channel s, and the origin for the phase scale is in the cross section C_s. If the amplitudes are normalized so that the particle fluxes in channel s are equal to $|a_s|^2$ and $|b_s|^2$, the matrix S becomes unitary and symmetric. The quantity R_s is the reflection coefficient for a wave which is coming out of waveguide s and going into the horn, with the origin for the phase scale again in the cross section C_s. In addition, Q is a matrix with the elements

$$Q_{ss'}=\delta_{ss'}-S_{ss'}R_{s'}. \tag{80}$$

The matrix elements S_{ss}' and the reflection coefficients R_s depend on the energy ε over which the averaging is carried out:

$$\mathrm{Av}(\ldots)=\int d\varepsilon\left(-\frac{\partial f_T}{\partial\varepsilon}\right)(\ldots). \tag{81}$$

For a two-pole we find from (78) the comparatively simple expression

$$G_{12}=2\frac{e^2}{h}\mathrm{Av}\left\{|S_{12}|^2\frac{(1-|R_1|^2)(1-|R_2|^2)}{|1-S_{11}R_1-S_{22}R_2+R_1R_2\,\mathrm{Det}\,S|^2}\right\}. \tag{82}$$

The cross section C_s, which is the origin for the phase scale, can be chosen in such a way that we have $\mathrm{Det}\,S=1$. In the particular case in which the waveguide expands in an adiabatically smooth fashion into the horns ($R_s=0$) we find from (78)

$$G_{ss'}=2\frac{e^2}{h}\mathrm{Av}|S_{ss'}|^2. \tag{83}$$

An expression of this sort (for a temperature $T=0$) was

derived by Pines[14] under the assumption that the potential approaches a constant value as we go off to infinity in the waveguide channel. It follows from the results of Secs. 5 and 6 that the potential does in fact approach a constant value only far within a massive bank. An explicit analysis of the banks reveals why for a channel without scatterers ($|S_{12}|^2 = 1$), even in the case of an adiabatically smooth expansion into banks ($R_1 = R_2 = 0$), the result $G_{12} = 2$ $(e^2/h) \neq \infty$ follows from (82). The reason is that an adiabatically smooth transition from the waveguide to the horn is a nonreflecting transition only for a wave incident from the waveguide; a wave incident from the horn is reflected if it is an "adiabatic continuation" of a waveguide mode.

8. RESISTANCE MEASUREMENTS BY THE FOUR-PROBE METHOD

The fact that the potential undergoes oscillations along a conducting quantum channel and that the potential varies over the cross section of the channel demonstrates that the procedure of measuring the resistance as the ratio of a potential difference and a current must be approached with some caution. We accordingly feel it worthwhile to reexamine the analysis carried out some time ago by Engquist and Anderson[11] of the four–probe method for measuring resistances.

We consider the four–pole in Fig. 4. All of the conducting channels are single–mode channels, and they go over adiabatically into massive banks. We assume that the system is symmetric about the plane A. At the middle of the horizontal channel, along which a current J is flowing, there is a scatterer, whose resistance is to be measured. The current J is induced by the potential difference $\varphi_3 - \varphi_1$ between banks 3 and 1. The potentials of banks 2 and 4 are chosen in such a way that the current in the vertical channels is zero. "By definition," the conductance of the scatterer of interest, G, is $J/(\varphi_4 - \varphi_2)$. Choosing $\varphi_3 = -\varphi_1 = U/2$ and $\varphi_4 = -\varphi_2 = V/2$, for convenience, we easily find from (77)

$$G = \frac{J}{V} = \frac{1}{2}\left\{(G_{14}-G_{12}) + (G_{14}+G_{12}+2G_{13})\frac{G_{14}+G_{12}+2G_{24}}{G_{14}-G_{12}}\right\}. \tag{84}$$

We evaluate the elements G_{ss}' from (83). We assume that the scatterer of interest is characterized by a reflection coefficient R for reflection from plane A and that the joint of the vertical measurement channel with the horizontal channel is characterized by the scattering matrix

$$\begin{vmatrix} R' & T' & t \\ T' & R' & t \\ t & t & R'' \end{vmatrix}. \tag{85}$$

The joint three–pole is assumed to be symmetric about the plane B. Here R' is the reflection coefficient for a wave which is incident from bank 1 from plane B, T' is the transmission coefficient of this wave beyond plane B, t is the transmission coefficient of the wave from bank 2 into the horizontal channel (to the right or left), and R'' is the reflection coefficient of this wave.

Under ideal conditions, the measurement leads do not perturb channels 1–3. We thus assume t, $R' \to 0$ and T', $R'' \to 1$. We then have $G_{13} \to 1 - |R|^2$, while G_{12}, G_{14}, and G_{24} are small. Here G_{24} is a small quantity of higher order. Making use of these relations, we find

$$G = 2\frac{e^2}{h}(1-|R|^2)\frac{G_{14}+G_{12}}{G_{14}-G_{12}}. \tag{86}$$

The fraction here is the ratio of two small quantities. In the limit $|t| \to 0$ we find the following expression for a temperature $T = 0$, making use of the unitary nature of matrix (85):

$$G = 2\frac{e^2}{h}(1-|R|^2)\frac{1-{}^1/_2(Re^{i(\beta+2\sigma)} + \text{c.c.})}{|R|^2+{}^1/_2(Re^{i\beta} + \text{c.c.})}. \tag{87}$$

Here $\beta = 2k_1L$ is the phase shift as the wave propagates from plane B to A and back, and σ is the phase of the transmission coefficient t. The oscillatory terms in (87) result from interference of the wave incident from bank 1 (or 3) with the wave reflected from the scatterer. Engquist and Anderson[11] assert that these terms vanish because the reservoir is "wide" and has many modes. Our own calculations, however, which explicitly incorporate the presence of a horn with a large number of modes, do not confirm that conclusion.

At a temperature $T \neq 0$, according to (83), all of the G_{ss}' should be averaged over ε near ε_F in an interval of width T. Only the phase β depends strongly on ε in this interval, because of the relation $L \gg \lambda_F$. The change in the phase in the course of the averaging is $\Delta\beta \sim \Delta k_1 L \sim (T/v_F)L$. Under the condition $\Delta\beta \gg 1$, the averaging over the energy leads to the disappearance of the interference terms, and we find the Landauer formula

$$G = 2\frac{e^2}{h}\frac{1-|R|^2}{|R|^2}. \tag{88}$$

The assumption that the joint is symmetric about the plane B was also used by Engquist and Anderson.[11] If we forgo that assumption, we find that the "conductance" G depends on not only the phase σ of the coefficient of transmission from the vertical channel into the horizontal channel but also the degree of right–left asymmetry of this coefficient.

We conclude from all this that the four–probe method measures the resistance as described by the Landauer formula (88) only at a sufficiently high temperature, only if the probes are sufficiently far from the scattering object, and only if the coupling of the potential probes is sufficiently weak. It is clear that the first two of these conditions often do

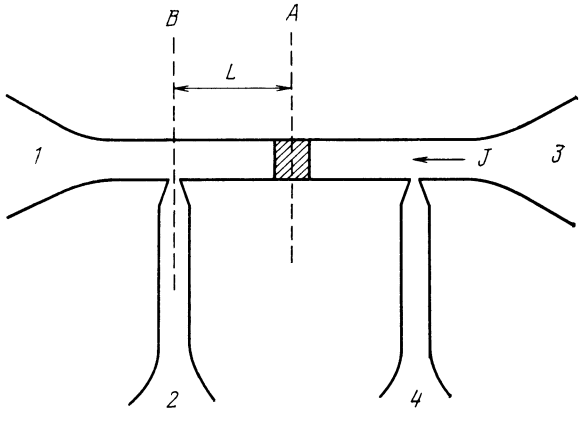

FIG. 4.

not hold. In a 2DEG with $n = 5\times 10^{11}$ cm^{-2}, for example, i.e., with $v_F = 3\times 10^7$ cm/s, at $T = 1$ K we would have a length $\hbar v_F/kt \approx 2\,\mu$m. This length is greater than, for example, the distance between the probes in the experiments of Refs. 3 and 4. The nonadiabatic nature of the coupling of the channels with the banks may also influence the results of measurements by the four-probe method. For example, if a wave propagating along the horizontal channel is reflected from banks 1 and 3, then G_{13} in (84) will convert not into $1 - |R|^2 = |S_{13}|^2$ but into an expression like (82).

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