Anomalous Doppler resonance of electrons with a one-dimensional monochromatic electromagnetic wave

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A kinetic and hydrodynamic theory of emission from electron beams in a finite magnetic field under conditions of anomalous Doppler resonance with a one-dimensional monochromatic electromagnetic wave is developed. The transition from the hydrodynamic to the kinetic regime is investigated. The instability suppression mechanisms and the evolution of the distribution function of the emitting electrons are described. Analytic solutions of the problem of emission from the beam under conditions of the anomalous Doppler effect are obtained in the purely hydrodynamic and kinetic limits. It is shown that the physics of the kinetic instabilities under the conditions of the anomalous Doppler effect differ substantially from the physics of kinetic instabilities due to inverse collisional Landau damping.

1. We know that a wave and an electron exchange energy effectively under conditions of Cherenkov resonance. The electron is retarded or accelerated, depending on the field phase in which it moves, and correspondingly delivers or draws energy to or from the wave. The net result depends on which of the electrons are in the majority, i.e., it is determined by the derivative of the distribution function in velocity, $\partial f/\partial v_{\parallel}$, at the resonance point $v_{\parallel} = \omega/k$. This is reflected in Landau's known equation for the decay rate of collisionless wave damping in a plasma.

The following general statement can be made: the collisionless damping decay rate (or growth rate in the case of instability) is proportional to the derivative of the distribution function $\partial f/\partial v_{\parallel}$ only for processes accompanied by phasing of the electrons in the wave field, i.e., for wave–particle interaction processes in which the electrons can, depending on their phase in the field, give up as well as draw energy. There are, however, interactions that are not accompanied by phasing of the particles. The present paper is devoted to a kinetic and hydrodynamic treatment of such processes.

Let us define clearly the processes we have in mind. Consider electrons moving in the field of certain gyroscopic or conservative external forces, i.e., acting as oscillators which we assume to be not excited in the initial state. Among the mechanisms whereby such oscillators interact with waves are the anomalous and normal Doppler effects.[2,3] In the normal Doppler effect the unperturbed oscillator can only be accelerated (going simultaneously into the excited state), i.e., draw energy from the wave. Collisionless damping of the wave should then be observed, with a decay rate proportional to the distribution function itself at the resonance point (but not to its derivative).[4] In the anomalous Doppler effect, on the other hand, the unexcited oscillator can only slow down (and become excited thereby) i.e., give up energy to the wave. The wave amplitude should increase then at a rate proportional likewise to the resonant value of the distribution function itself.

One of the possible processes of this type is resonant interaction of transverse electromagnetic waves with a linear electron beam moving along a constant external magnetic field. We specify the spectrum of the electromagnetic waves in the form $\omega^2 = k^2 c_0^2$, where the phase velocity c_0 is assumed to be lower than the longitudinal electron velocity $u (c_0 < u)$. It is the latter which ensures the existence of the instability, considered below, in the anomalous Doppler effect.

It is known that in the case of a beam with a small velocity spread

$$\Delta v_{\parallel}/u \ll \delta\omega/\omega, \tag{1}$$

the instability in the anomalous Doppler effect has a hydrodynamic character and a growth rate[5]

$$\delta\omega = (\omega_b^2 \omega_B / 2\omega)^{1/2}. \tag{2}$$

For a hot beam, however, when inequality (1) is violated, the instability is kinetic with a growth rate[6]

$$\delta\omega = \frac{\pi}{2} \frac{\omega_b^2 \omega_B}{|k| \omega} f_0(v_{\parallel}) \big|_{v_{\parallel} = (\omega + \omega_B)/k}, \tag{3}$$

i.e., proportional, as noted above, to the distribution function itself. Here $\Delta v_{\parallel}$ is the longitudinal-velocity spread of the beam electrons, ω_b and ω_B are respectively the electron Langmuir and cyclotron frequencies, and $f_0(v_{\parallel})$ is the unperturbed electron distribution function in the longitudinal velocities. We shall make it clear also that the instability in question evolves at frequencies and wave numbers satisfying the system

$$\omega = k c_0, \qquad \omega = k v_{\parallel} - \omega_B. \tag{4}$$

We proceed now to the rigorous treatment.

2. Let a non-monoenergetic neutralized electron beam propagate along an external magnetic field B_0 directed along the Z axis, in a space filled by a homogeneous isotropic dielectric with a constant $\varepsilon_0 > 1$, and let it interact with a circularly polarized electromagnetic wave propagating in the same direction. In this case the equations of motion of the electrons for the velocity components $v_{\parallel}$ and $v_{\perp} = v_x + i v_y$, and also the equation for the transverse components $A_{\perp} = A_x + i A_y$ of the wave vector potential, are

$$\begin{gathered}
\frac{dv_{\perp}}{dt} + i\omega_B v_{\perp} = -\frac{e}{mc}\left(\frac{\partial A_{\perp}}{\partial t} + v_{\parallel}\frac{\partial A_{\perp}}{\partial z}\right), \\
\frac{dv_{\parallel}}{dt} = \frac{1}{2}\frac{e}{mc}\left(v_{\perp}\frac{\partial A_{\perp}^*}{\partial z} + \text{c.c.}\right), \\
\frac{\partial^2 A_{\perp}}{\partial t^2} - c_0^2 \frac{\partial^2 A_{\perp}}{\partial z^2} = \frac{4\pi c}{\varepsilon_0} j_{\perp},
\end{gathered} \tag{5}$$

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where $c_0^2 = c^2/\varepsilon_0^2$. The transverse component of the beam current is given by[7]

$$j_\perp = en_b \iint f_0(v_0) v_\perp(t, z_0, v_0) \delta[z - z(t, z_0, v_0)] dz_0 dv_0, \quad (6)$$

where n_b is the unperturbed density of the electron beam; $f_0(v_0)$, just as in Eq. (3), is the unperturbed electron distribution function in the longitudinal velocities, while $z(t, z_0, v_0)$ is the trajectory of the electron leaving the point z_0 at $t = 0$. The first two equations of the system (5) are supplemented by the initial conditions

$$v_\parallel|_{t=0} = v_0, \qquad v_\perp|_{t=0} = 0, \quad (7)$$

the second of which means that there are no transverse velocities in the unperturbed beam (beam of unexcited oscillators).

We seek the solution of the system (5) in the class of monochromatic waves

$$A_\perp = A(t) \exp(i\omega t - ikz), \quad (8)$$

where $A(t)$ is a slowly varying amplitude. In addition, recognizing that a circularly polarized wave does not modulate an electron beam in density[3,8] [and the system (5) is valid only under this condition], we use the relations

$$v_\parallel(t, z_0, v_0) = v_\parallel(t, v_0),$$
$$z(t, z_0, v_0) = z_0 + \int_0^t v_\parallel(t', v_0) dt'. \quad (9)$$
$$v_\perp = \frac{e}{mc} V(t, v_0) \exp(i\omega t - ikz(t, z_0, v_0)).$$

Carrying out next an elementary integration with respect to z_0 in (6), we obtain for $j_\perp$

$$j_\perp = \frac{e^2 n_b}{mc} \int f_0(v_0) V(t, v_0) dv_0 \exp(i\omega t - ikz), \quad (10)$$

where z, just as in (8), is an independent variable. Substituting furthermore (8), (9), and (10) in (5), we have the following system of equations and of initial conditions

$$\frac{dA}{dt} = -\frac{1}{2} i \frac{\omega_b^2}{\omega} \int f_0(v_0) V(t, v_0) dv_0,$$
$$\frac{dV}{dt} + i(\omega - kv_\parallel + \omega_B) V = -i(\omega - kv_\parallel) A,$$
$$\frac{dv_\parallel}{dt} = -\frac{1}{2} ik \left(\frac{e}{mc}\right)^2 (V^* A - VA^*), \quad (11)$$
$$A|_{t=0} = A_0, \quad V|_{t=0} = 0, \quad v_\parallel|_{t=0} = v_0,$$

which generalizes the equations obtained in Ref. 8 to include a non-single-velocity beam. We note in addition that since the electrons strongly interacting with the wave have $v_\parallel = (\omega + \omega_B)/k$ [see Eq. (4)], the second equation of (11) can be rewritten in the form

$$dV/dt + i(\omega - kv_\parallel + \omega_B) = i\omega_B A, \quad (12)$$

which we shall use henceforth. It is also easy to show that the system (11) has integrals

$$v_\parallel = v_0 - \frac{1}{2}\left(\frac{e}{mc}\right)^2 \frac{k}{\omega_B} |V|^2,$$
$$|A|^2 - \frac{\omega_b^2}{2\omega\omega_B} \int f_0(v_0) |V|^2 dv_0 = |A_0|^2, \quad (13)$$
$$\frac{k\omega}{\omega_b^2}\left(\frac{e}{mc}\right)^2 |A|^2 + \int v_\parallel f_0(v_0) dv_0$$
$$= \frac{k\omega}{\omega_b^2}\left(\frac{e}{mc}\right)^2 |A_0|^2 + \int v_0 f_0(v_0) dv_0,$$

the second and third of which are the energy and momentum conservation laws, while one of the integrals in (13) is a consequence of the other two.

We obtain now a relation for the beam-electron distribution function in the longitudinal velocities. In the initial state this function is $f_0(v_0)$. In the succeeding instants of time the distribution function is given by

$$f(v_\parallel) = \int f_0(v_0) \delta[v_\parallel - v_\parallel(t, v_0)] dv_0, \quad (14)$$

where $v_\parallel(t, v_0)$ is either the solution of the last equation of the system (11) or, equivalently, the first expression of (13). Since

$$\delta[v_\parallel - v_\parallel(t, v_0)] = \sum_j \delta[v_0 - v_{0j}] \left| \frac{\partial v_\parallel}{\partial v_0}(t, v_{0j}) \right|^{-1}, \quad (15)$$

where v_{0j} are the roots of the equation $v_\parallel = v_\parallel(t, v_0)$, we can rewrite (14) in the form

$$f(v_\parallel) = \sum_j f_0(v_{0j}) \left| \frac{\partial v_\parallel}{\partial v_0}(t, v_{0j}) \right|^{-1}. \quad (16)$$

Equations (11), the integrals (13), and relation (16) will be used below for a numerical analysis; it is therefore expedient to change in them to nondimensional variables and to stipulate a specific function $f_0(v_0)$. We introduce the following quantities:

$$\tau = \omega_B t, \quad \eta = \frac{|k|}{\omega_B} v_\parallel, \quad \eta_0 = \frac{|k|}{\omega_B} v_0, \quad \langle\eta_0\rangle = \frac{k}{\omega_B} \frac{\omega + \omega_B}{k},$$
$$\varepsilon = \frac{|k|}{\omega_B}\left(\frac{e}{mc}\right) A, \quad a = \frac{|k|}{\omega_B}\left(\frac{e}{mc}\right) V, \quad (17)$$
$$\varphi(\eta_0) = \frac{\omega_B}{|k|} f_0(v_0), \quad \varkappa^2 = \frac{\omega_b^2}{2\omega_B|\omega|}.$$

In the new notation, Eqs. (11) become

$$\frac{d\varepsilon}{d\tau} = -i\varkappa^2 \int \varphi(\eta_0) a(\tau, \eta_0) d\eta_0, \quad \varepsilon|_{\tau=0} = \varepsilon_0,$$
$$\frac{da}{d\tau} + i(\langle\eta_0\rangle - \eta) a = -i\varepsilon, \quad a|_{\tau=0} = 0, \quad (18)$$
$$\frac{d\eta}{d\tau} = -\frac{1}{2} i(a^*\varepsilon - a\varepsilon^*), \quad \eta|_{\tau=0} = \eta_0,$$

from which it follows that $\langle\eta_0\rangle$ corresponds to the dimensionless velocity at which the electron is in exact cyclotron resonance with the wave. We shall assume that the distribution function $\varphi(\eta_0)$ has a maximum at $\eta_0 = \langle\eta_0\rangle$ and is an even function of the difference $\eta_0 - \langle\eta_0\rangle$. Very useful, as will be shown below, for analytic estimates and numerical computations is the following distribution function:

$$\varphi(\eta_0)=\frac{1}{\pi}\frac{\Theta}{(\eta_0-\langle\eta_0\rangle)^2+\Theta^2}, \qquad (19)$$

where the parameter Θ can be regarded as the "temperature" of the beam. Introducing now the quantities

$$x_0=\eta_0-\langle\eta_0\rangle, \quad x=\eta-\langle\eta_0\rangle, \qquad (20)$$

we write down the system (18) in a form most convenient for numerical integration:

$$\begin{aligned}
&\frac{d\varepsilon}{d\tau}=-i\varkappa^2\int\varphi(x_0)a(\tau,x_0)\,dx_0, \quad \varepsilon|_{\tau=0}=\varepsilon_0;\\
&\frac{da}{d\tau}-ixa=i\varepsilon, \quad a|_{\tau=0}=0,\\
&\frac{dx}{d\tau}=-\frac{1}{2}i(a^*\varepsilon-a\varepsilon^*), \quad x|_{\tau=0}=x_0,\\
&\varphi(x_0)=\frac{1}{\pi}\frac{\Theta}{x_0^2+\Theta^2}.
\end{aligned} \qquad (21)$$

We present also a nondimensional form of the two independent integrals in (13):

$$\begin{aligned}
&x=x_0-\frac{1}{2}|a|^2,\\
&|\varepsilon|^2-\varkappa^2\int\varphi(x_0)|a|^2\,dx_0=|\varepsilon_0|^2,
\end{aligned} \qquad (22)$$

which are needed to monitor the computation accuracy. In the same variables, Eq. (16) for the distribution function becomes

$$\varphi(x)=\sum_j\varphi(x_{0j})\left|\frac{\partial x}{\partial x_0}(\tau,x_{0j})\right|^{-1}, \qquad (23)$$

where x_{0j} is the root of the equation $x=x(\tau,x_0)$.

3. The suitability of the chosen distribution $\varphi(x_0)$ becomes apparent in the linear analysis which we shall perform prior to discussing and analyzing the numerical computations. In the linear approximation, the system (21) takes the form

$$\begin{aligned}
&\frac{d\varepsilon}{d\tau}=-i\varkappa^2\int\varphi(x_0)a(\tau,x_0)\,dx_0, \quad \varepsilon|_{\tau=0}=\varepsilon_0,\\
&\frac{da}{d\tau}-ix_0a=i\varepsilon, \quad a|_{\tau=0}=0,\\
&\varphi(x_0)=\frac{1}{\pi}\frac{\Theta}{x_0^2+\Theta^2}.
\end{aligned} \qquad (24)$$

Assuming $\varepsilon, a\sim e^{\delta\tau}$, we obtain the dispersion equation

$$1-\varkappa^2\frac{\Theta}{\pi}\int\frac{dx_0}{(x_0^2+\Theta^2)(x_0^2+\delta^2)}=0, \qquad (25)$$

the integral in which is easily calculated. As a result we have

$$\delta(\delta+\Theta)=\varkappa^2, \qquad (26)$$

with one of the roots of (26) always positive, indicating the onset of instability

$$\delta_{1,2}=\pm(\varkappa^2+\Theta^2/4)^{1/2}-\Theta/2. \qquad (27)$$

Let us consider the limiting expressions for the growth rates:

$$\delta=\begin{cases}\varkappa & \varkappa\gg\Theta\\ \varkappa^2/\Theta, & \varkappa\ll\Theta\end{cases}. \qquad (28)$$

The first growth rate here is hydrodynamic and coincides with (2). The second is kinetic, is given in dimensional form by

$$\delta\omega=\omega_b^2/2|\omega|\Theta \qquad (29)$$

and coincides with (3). Indeed, in the case of our chosen distribution function we get

$$f_0\left(v_0=\frac{\omega+\omega_B}{k}\right)=\frac{|k|}{\omega_B}\frac{1}{\pi\Theta},$$

as can be seen from (17).

Obviously, the boundary of the hydrodynamic and kinetic regimes is determined by the relation $\varkappa\sim\Theta$. It must be emphasized that the dispersion equation (26) is algebraic not only in hydrodynamics but also in kinetics; this is an exceptional property of (19).

We obtain now more general linear-approximation equations. To this end we reduce the problem (24) to an integral equation with delay, without specifying for the time being the form of the distribution function. Solving the equation for a with the zero initial conditions

$$a(\tau)=i\int_0^\tau\varepsilon(\tau')\exp\{ix_0(\tau-\tau')\}\,d\tau' \qquad (30)$$

and substituting the solution into the equation for ε, we get

$$\begin{aligned}
&\frac{d\varepsilon}{d\tau}=\varkappa^2\int_0^\tau\varepsilon(\tau')k(\tau-\tau')\,d\tau',\\
&k(\tau-\tau')=\int\varphi(x_0)\exp\{ix_0(\tau-\tau')\}\,dx_0.
\end{aligned} \qquad (31)$$

We consider three distribution functions: previous, Maxwellian, and "step":

$$\begin{aligned}
&\varphi(x_0)=\frac{1}{\pi}\Theta(x_0^2+\Theta^2)^{-1} && \text{I},\\
&\varphi(x_0)=(\pi\Theta)^{-1/2}\exp\left(-\frac{1}{\Theta}x_0^2\right) && \text{II},\\
&\varphi(x_0)=(2\Theta)^{-1}, \quad x_0\in[-\Theta,\Theta]. && \text{III}.
\end{aligned} \qquad (32)$$

For these distributions we have from (31)

$$\begin{aligned}
&k(\tau)=e^{-\Theta\tau} && \text{I},\\
&k(\tau)=e^{-\Theta\tau^2/4} && \text{II},\\
&k(\tau)=\frac{\sin\Theta\tau}{\Theta\tau} && \text{III},
\end{aligned} \qquad (33)$$

The distribution I is the simplest. Indeed, from the first equation of (31) and from (33) it follows that

$$\frac{d\varepsilon}{d\tau}=\varkappa^2\int_0^\tau\varepsilon(\tau)\exp\{-\Theta(\tau-\tau')\}\,d\tau', \qquad (34)$$

the last equation not being an intgral one. We differentiate it once with respect to τ and substitute again (34) in the results of the differentiation. We obtain then the ordinary differential equation

$$d^2\varepsilon/d\tau^2+\Theta\,d\varepsilon/d\tau-\varkappa^2\varepsilon=0, \qquad (35)$$

which leads to (26).

The situation with the remaining distribution functions is somewhat more complicated. To solve Eq. (31) in the general case we use a Laplace transformation. We have then for the Laplace transform $\varepsilon(p)$ of the function $\varepsilon(\tau)$

$$\varepsilon(p)=\frac{\varepsilon_0}{p-\varkappa^2K(p)},\quad K(p)=\int_0^\infty e^{-p\tau}k(\tau)\,d\tau, \tag{36}$$

where $\varepsilon|_{\tau=0}=\varepsilon_0$. For the chosen distribution functions (32) we have

$$\begin{aligned} &K(p)=(p+\Theta)^{-1} && \text{I},\\ &K(p)=\left(\frac{\pi}{\Theta}\right)^{1/2}\exp\left(\frac{p^2}{\Theta}\right)\left[1-\Phi\left(\left(\frac{p^2}{\Theta}\right)^{1/2}\right)\right] && \text{II},\\ &K(p)=\frac{1}{\Theta}\operatorname{arctg}\frac{\Theta}{p} && \text{III}, \end{aligned} \tag{37}$$

where $\Phi(x)$ is the probability integral. Taking the inverse Laplace transform we get

$$\varepsilon(\tau)=\frac{\varepsilon_0}{2\pi i}\int_C\frac{e^{p\tau}\,dp}{p-\varkappa^2K(p)}, \tag{38}$$

where the contour C passes on the P plane to the right of all the singularities of the integrand. As expected, the integral (38) cannot be calculated exactly for the distributions II and III. For I, however, we have

$$\varepsilon=\varepsilon_0\frac{p_2e^{p_1\tau}-p_1e^{p_2\tau}}{p_2-p_1}, \tag{39}$$

where

$$p_{1,2}=\pm(\varkappa^2+\Theta^2/4)^{1/2}-\Theta/2,$$

which coincides with (27). The solution (39) can be obtained also from Eq. (35) by supplementing it with the initial conditions

$$\varepsilon(0)=\varepsilon_0,\quad \varepsilon'(0)=0.$$

Let us explain why distribution I of (32) is exceptional. In the solution of the initial problem there is specified, besides $\varepsilon|_{\tau=0}$ also some initial perturbation of the electron-distribution function, e.g., it is assumed to be exactly zero. This leads to vanishing of all the moments of the distribution-function perturbation. Since they form, in general, an infinite set, they can be made to vanish only if an infinite set of linearly independent solutions of the problem (31) is on hand. It follows hence that in the general case the linear dispersion equation should have an infinite set of solutions, i.e., it must be transcendental. In the case of distribution I of (32), however, all the higher moments of the distribution-function perturbations are expressed in terms of the first moment. We know of no other distributions with this property.

4. We obtain now an analytic solution of the system (21) in the hydrodynamic approximation and use it to estimate the maximum value of the wave amplitude $|\varepsilon_{\max}|$ and to explain the mechanism that eliminates the instabilities in this regime. Putting $\varphi(x_0)=\delta(x_0)$ in (21) we obtain[8]

$$\begin{aligned} &d\varepsilon/d\tau=-i\varkappa^2a,\quad da/d\tau-ixa=i\varepsilon,\\ &dx/d\tau=-{}^1/{}_2i(a^*\varepsilon-a\varepsilon^*). \end{aligned} \tag{40}$$

Taking now the integrals (22) into account, we rewrite Eq. (40) for an adiabatic turning-on of the field in the past ($\varepsilon|_{\tau\to-\infty}=0$) in the form

$$d\varepsilon/d\tau=-i\varkappa^2a,\quad da/d\tau+{}^1/{}_2i|a|^2a=i\varepsilon. \tag{41}$$

The term cubic in a in (41) describes the cyclotron-frequency nonlinear frequency shift due to the beam stopping. It is this stopping which leads to saturation of the instability in the hydrodynamic approximation.

The solution of (41) is

$$|\varepsilon|^2=\varkappa^2|a|^2=8\varkappa^3[\operatorname{ch}2\varkappa\tau]^{-1},\quad -\infty<\tau<+\infty. \tag{42}$$

It follows from (42) that the maximum dimensionless amplitude of the electromagnetic is in the hydrodynamic case

$$\left|\varepsilon_{\max}^{\text{hydr}}\right|=8^{1/2}\varkappa^{3/2}. \tag{43}$$

5. We discuss now the results of a numerical solution of Eqs. (21) that were integrated at $\varkappa^2=0.01$ for different values of the "temperature" Θ. Figure 1 shows the values of $|\varepsilon|$, $|a|$, and x (the last two values for resonant electrons for which $x|_{\tau=0}\equiv x_0=0$) as functions of τ calculated at $\Theta=0.01$; this can be regarded as a good hydrodynamic approximation. The maximum wave amplitude calculated from (43) is $|\varepsilon_{\max}|=0.09$, in good agreement with that obtained by numerical methods. We see that the solution is almost periodic and that the stabilizing factor, as noted above, is the nonlinear frequency shift. As should be the case under the conditions of the anomalous Doppler effect, a growth of $|\varepsilon|$ and $|a|$ is accompanied by a decrease of x, i.e., the energy of the longitudinal motion of the beam goes to increase the transverse energy of the electron oscillations and to radiation. The same is observed also for particles with $x_0\neq0$, i.e., which are not at resonance with the wave from the very outset, but the larger $|x_0|$ the smaller $|a|$ and $|x-x_0|$, this being due to the resonant character of the instability. It can be stated that with increase of $|x_0|$, i.e., with increasing deviation of the velocity of the electrons from the

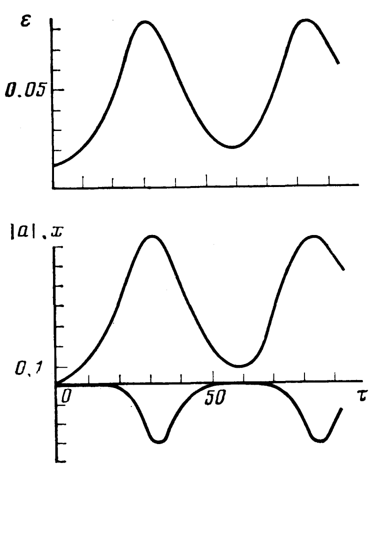

FIG. 1.

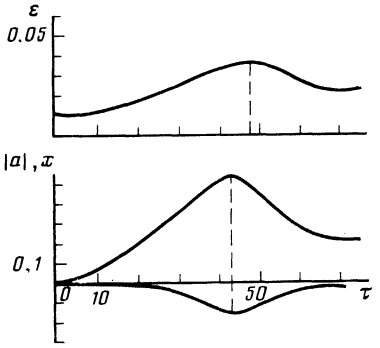

FIG. 2.

resonance region, their contribution to the radiation decreases.

Figure 2 shows the same values for the case $\Theta = 0.25$, which can be regarded as kinetic (the boundary between the kinetic and hydrodynamic regimes is $\Theta = 0.1$, i.e., when $\Theta \sim \varkappa$). It should be noted that a distinctive property of solutions with finite Θ is their non-periodicity: the amplitude $|\varepsilon|$ decreases after saturation not to the initial value ε_0 but to a higher one, and this tendency grows with increase of Θ. It is probable that for very large Θ the amplitude $|\varepsilon|$ assumes with time a quasistationary value.

Figure 3 shows, for two successive instants of time and at the same "temperature," the distribution functions $\varphi(x)$ [$\varphi(x_0)|_{\tau=0}$ is shown dashed], $x(x_0,\tau)$, and $a(x_0,\tau)$. It can be seen that the maximum of the distribution function shifts to the left with time, and its half width decreases, and that significant perturbations of $|x - x_0|$ and $a(x_0)$ are observed only for resonance electrons for which $|x_0| \leqslant \delta$ (where δ for $\Theta = 0.25$ is of the order of 0.035). Nonresonant electrons with $|x_0| > \delta$ do not take part in the interaction. Subsequently (Fig. 3b) the half-width of the distribution function decrease even more strongly, and a dip is formed on the distribution function at velocities equal to the resonance value. It will be shown below that this last circumstance is the cause of the kinetic-instability saturation. In addition, in view of the decrease of the growth rate of the kinetic instability, the time of saturation of this instability becomes longer than the corresponding time in hydrodynamic instability, as is confirmed numerically. The abrupt decrease of the distribution-function half width in the course of the evolution of the kinetic instability allows us to state that a single-velocity beam is separated from all the available electrons and has a temperature much lower than in the initial beam. This phenomenon for the normal Doppler interaction between a beam of neutrals and radiation was noted in Ref. 9. It is also seen from an analysis of Fig. 2 that electrons with $x_0 = 0$ go off-resonance before $|\varepsilon|$ saturates, i.e., the further growth of the latter is due to electrons that were not in exact resonance with the wave at the initial instant, but landed in the resonance region as a result of deceleration.

It must be stated that at $\Theta = 0.25$ the strong-kinetics condition $\Theta \gg \varkappa$ is not met. The figures presented pertain to a transition regime, when the kinetic effects are already very substantial but are not yet fully predominant. Numerical computations with even larger Θ, however, are very difficult.

5. Let us make clear now the main nonlinear mechanism that suppresses the kinetic instability, and obtain the corresponding analytic solution. To this end we solve formally Eq. (12):

$$V = \frac{\omega_B}{\omega - kv_\parallel + \omega_B} A, \tag{44}$$

where $\omega \to \omega - i\partial/\partial t$. Substituting next (44) in the first equation, multiplied by A^*, of (11) and adding the complex conjugate, we get

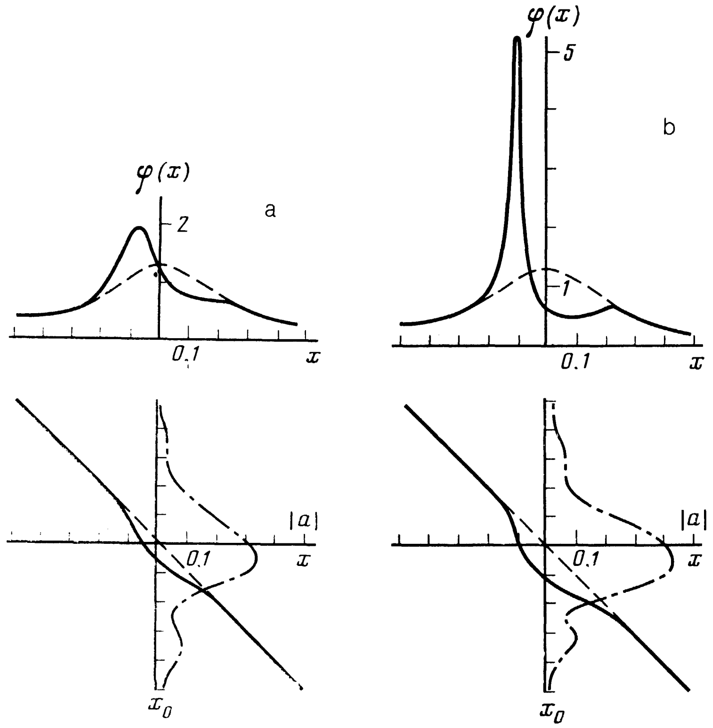

FIG. 3.

$$\frac{d|A|^2}{dt}=\frac{\omega_b^2\omega_B}{\omega|k|}\left\{\int f_0(v_0)\frac{\hat{\delta}}{[v_p-v_\parallel(t,v_0)]^2+\hat{\delta}^2}dv_0\right\}|A|^2, \tag{45}$$

where $\hat{\delta}=|k|^{-1}\partial/\partial t$, while $v_p=(\omega+\omega_B)/k$ is the resonant velocity of the electrons. In the kinetic approach we have $\delta\to 0$, so that we can rewrite (45) as

$$\frac{d|A|^2}{dt}=\pi\frac{\omega_b^2\omega_B}{\omega|k|}\left\{\int f_0(v_0)\delta[v_p-v_\parallel(t,v_0)]dv_0\right\}|A|^2. \tag{46}$$

Integrating in (46) with respect to v_0 we obtain ultimately

$$\frac{d}{dt}|A|^2=2\delta\omega_n|A|^2. \tag{47}$$

Here

$$\delta\omega_n=\frac{\pi}{2}\frac{\omega_b^2\omega_B}{\omega|k|}f_0(\hat{v})\left|\frac{\partial v_\parallel}{\partial v_0}(\hat{v})\right|^{-1} \tag{48}$$

is the nonlinear growth rate, and $\hat{v}$ is the root of the equation

$$v_p=v_\parallel(t,v_0), \tag{49}$$

which must be solved for v_0. In the kinetic approach, Eq. (49) has most likely one root (see, e.g., Fig. 3), a fact reflected in the form of Eq. (48). In the linear approximation we have $v_\parallel(t,v_0)=v_0$, meaning $\hat{v}=v_p$, and from (48) follows the linear growth rate

$$\delta\omega=\frac{\pi}{2}\frac{\omega_b^2\omega_B}{\omega|k|}f_0(v_p), \tag{50}$$

which coincides with (3).

Equation (48), of course, does not eliminate the problem's main mathematical difficulties connected with finding the function $v_\parallel(t,v_0)$ and determining the root $\hat{v}$. It leads, however, explicitly to the nonlinear factors that stabilize the variation of the amplitude $|A|$.

We assume, as before, that f_0 reaches a maximum at the resonant velocity v_p. Since $\hat{v}\neq v_p$, it follows that $f_0(\hat{v})<f_0(v_p)$ which leads to a decrease of the growth rate, corresponding to a shift of the distribution function in velocity space. This shift, however, takes place in a velocity interval $\delta\omega/k$ in which the distribution function in the kinetic regime varies little, and therefore $f_0(\hat{v})\approx f_0(v_p)$. This nonlinear effect can therefore be neglected and we can write (48) in the form

$$\delta\omega_n=\delta\omega\left|\frac{\partial v_\parallel}{\partial v_0}(\hat{v})\right|^{-1}, \tag{51}$$

where $\delta\omega$ is the linear growth rate (50).

The main stabilizing factor is due to the multiplier

$$\left|\frac{\partial v_\parallel}{\partial v_0}(\hat{v})\right|^{-1}.$$

As seen from Fig. 3, the derivative $\partial v_\parallel/\partial v_0$ for $v_0=\hat{v}$ is larger than unity and increases with time (it is equal to unity only if $t=0$). It is this which decreases the growth rate. The physical meaning is that electrons with resonant values of the velocity v_p transfer energy to the wave intensely and shift in velocity space towards lower velocities (are slowed down). As a result, the distribution function acquires a dip at $v_\parallel=v_p$ and a peak at lower velocities—as is clearly seen from Fig. 3. Decreasing the number of resonant electrons suppresses the instability.

It must be noted that the calculation of (51) is a very complicated task. We formulate therefore certain simplified equations by using a linear approximation, or more accurately the solution (44) with $\omega=\omega-i\delta\omega$, where $\delta\omega$ is the growth rate (50). Substitution of (44) in the last equation of (11) gives

$$\frac{dv_\parallel}{dt}=-\left(\frac{e}{mc}\right)^2 k\frac{\omega_B\delta\omega}{[\omega+\omega_B-kv_\parallel(t,v_0)]^2+\delta\omega^2}|A|^2 \tag{52}$$

The system (52), (47), and (51) is closed. We express it in terms of the dimensionless variables:

$$\tau=\delta\omega t,\quad x_\parallel=\frac{kv_\parallel}{\delta\omega},\quad x_p=\frac{\omega+\omega_B}{\delta\omega},\quad x_0=\frac{kv_0}{\delta\omega},$$
$$B=\left(\frac{e}{mc}\right)^2 k^2\frac{\omega_B}{\delta\omega^3}|A|^2, \tag{53}$$

and obtain

$$\frac{dB}{d\tau}=2\left|\frac{\partial x_\parallel}{\partial x_0}(\hat{x})\right|^{-1}B,\quad B|_{\tau=0}=B_0,$$
$$\frac{dx_\parallel}{d\tau}=-\frac{1}{[x_p-x_\parallel(\tau,x_0)]^2+1}B,\quad x_\parallel|_{\tau=0}=x_0. \tag{54}$$

Here $\hat{x}$ is the root of the equation $x_p=x_\parallel(\tau,x_0)$. Since the system (54) contains not even a single parameter, the characteristic value of the saturation amplitude B is of the order of unity. This leads to the estimate

$$\left(\frac{e}{mc}\right)^2|A|^2_{max}\approx\frac{\delta\omega^3}{k^2\omega_B}, \tag{55}$$

where $\delta\omega$ is the kinetic growth rate (50).

The system (54) can be further simplified. First, without loss of generality, we can put $x_p=0$, which corresponds to a change of the origin in velocity space. Second, introducing a new function $B=da/d\tau$, we integrate the second equation of (54) and obtain

$$\tfrac{1}{3}x_\parallel^3+x_\parallel-\tfrac{1}{3}x_0^3-x_0=-a. \tag{56}$$

Since $x_p=0$. The root $\hat{x}$ is determined from the equation $x_\parallel(\tau,\hat{x})=0$, and putting next $x_0=\hat{x}$ in (56) we get

$$\hat{x}+\tfrac{1}{3}\hat{x}^3=a. \tag{57}$$

Differentiating furthermore (56) with respect to x_0 and substituting $\hat{x}$ for x_0 in the result, we arrive at

$$\frac{\partial x_\parallel}{\partial x_0}(\hat{x})=1+x^2, \tag{58}$$

with allowance for which the first equation of the system (54) can be rewritten in the form

$$B=\frac{da}{d\tau}=2\int_0^a\frac{da'}{1+\hat{x}(a')^2}+B_0,\quad a|_{\tau=0}=0, \tag{59}$$

where $\hat{x}$ is determined from (57). If a are small enough (strictly speaking, the simplified system (54) can be used only for such a), the last equation becomes much simpler:

$$B=da/d\tau=2\,\mathrm{arctg}\,a+B_0. \tag{60}$$

This shows once more that $B_{max} \sim 1$ (for $B_0 \ll 1$). Further analytic integration of (60) is impossible, but the instability suppression due to depletion of the distribution functions at velocities close to $v_p = (\omega + \omega_B)/k$ can nonetheless be described even in the framework of the crude model (54).

In the linear approximation it follows from (60) that $B = B_0 e^{2\tau}$ as it should. If, however $B_0 \gg 1$, then $B \approx B_0$, i.e., the radiation under the conditions of the anomalous Doppler effect is insignificant.

Figure 4 shows the result of numerical integration of Eq. (59) for various B_0. For small B_0, the growth of the wave remains exponential for a long time. At large B_0, the initial exponential section is rapidly replaced by a more gently sloping one. This is due to departure of resonant electrons into the region of lower velocity and depletion of the resonant part of the distribution function, a depletion that is stronger the larger B_0.

It should be noted that Eq. (59) with $\hat{x}$ determined from (57) does not describe the total saturation of the amplitude B. At some instant of time the exponential amplitude variation simply gives way to a substantially slower one. This is in fact the instant of saturation. The absence of total saturation of B is the consequence of the approximations used in the derivation of these equations.

The estimate (55) can be obtained by another method. It follows from Ref. 10 that in the anomalous Doppler effect each of the resonant electrons transfers to the wave an energy equal to

$$\delta W = -(\omega/ku)\,\delta W_e, \tag{61}$$

where δW_e is the change of the energy of the electron's translational motion. Obviously, $\delta W_e \approx mu\delta\omega/k$. The total wave energy, on the other hand consists of the changes δW_e of all the resonant electrons. The number of such electrons per unit volume is

$$N = n_b \int\limits_{v_p-\delta\omega/k}^{v_p+\delta\omega/k} f_0(v_0)\,dv_0 \approx 2n_b \frac{\delta\omega}{k} f_0(v_p). \tag{62}$$

From the last two relations we obtain then

$$W = n_b m (k^{-1}\delta\omega)^2 \omega k^{-1} f_0(v_p). \tag{63}$$

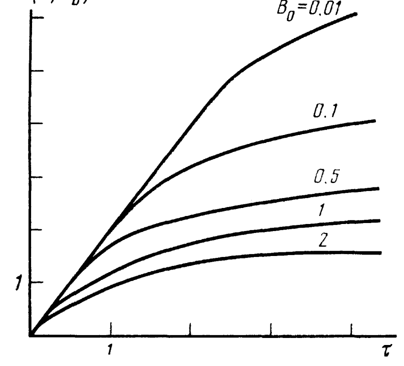

FIG. 4.

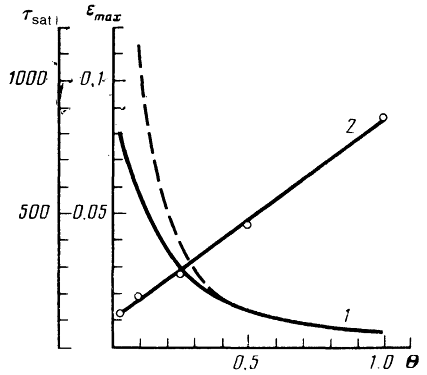

FIG. 5.

Taking now into account the definition of W in terms of the vector potential A and the expression (50) for the growth rate, we again arrive at (55).

Let us compare the maximum wave amplitudes in the hydrodynamic and kinetic regimes under conditions of the anomalous Doppler effect. Taking (43), (17), and (2) into account we can write

$$(e/mc)^2 |A|^{\rm hydr}_{\rm max} \sim \delta\omega^3_{\rm hydr}/k^2\omega_B. \tag{64}$$

We obtain then from (64) and (55)

$$\frac{|A|^{2\rm kin}_{\rm max}}{|A|^{2\rm hydr}_{\rm max}} \sim \frac{\delta\omega^3_{\rm kin}}{\delta\omega^3_{\rm hydr}} \sim \left(\left(\frac{\omega}{\omega_B}\right)^{1/2} \frac{kV}{\omega_B}\right)^3, \tag{65}$$

where V is the half width of the distribution function in velocity space. In dimensional variables this relation is of the form

$$|\varepsilon|^{\rm kin}_{\rm max}/|\varepsilon|^{\rm hydr}_{\rm max} \approx (\varkappa/\Theta)^{3/2} \ll 1, \tag{66}$$

inasmuch as $\Theta \gg \varkappa$ in the kinetic equation. The decrease of the wave amplitude on going from the hydrodynamic to the kinetic regime follows also from Figs. 1 and 2.

Figure 5 shows the dependence of the saturation amplitude $|\varepsilon|_{max}$ on the "temperature" Θ, obtained by numerical integration of Eqs. (21) at $\varkappa = 0.01$ and $\varepsilon_0 = 10^{-6}$ for various Θ (curve 1). The curve agrees well with (66) (dashed line). The same figure shows the dependence of the dimensionless saturation time on Θ (curve 2), obtained by numerical integration of the same equations. As seen from (28), at high "temperatures" we have $\tau_{sat} \sim 1/\delta \sim \Theta$, corresponding to the calculated curve 2.

The main deduction from the analysis of the nonlinear dynamics of the kinetic instability under conditions of the anomalous Doppler effect is the following: the saturation is due to depletion of the distribution function (to formation of a dip on it) in the resonant-velocity region. This mechanism differs from that of the nonlinear stabilization of processes connected with the direct and inverse Landau damping, where the saturation is due to capture of the resonant electrons and to formation of a "plateau" on the distribution function.[11]

[1]L. D. Landau, Zh. Eksp. Teor. Fiz. **16**, 574 (1946).
[2]V. L. Ginzburg, *Theoretical Physics and Astrophysics*, Pergamon, 1979.

[3]M. V. Nezlin, Usp. Fiz. Nauk **120**, 481 (1976) [Sov. Phys. Usp. **19**, 946 (1976)].
[4]B. B. Kadomtsev, A. B. Mikhailovskiĭ, and A. V. Timofeev, Zh. Eksp. Teor. Fiz. **47**, 2266 (1964) [Sov. Phys. JETP **20**, 1517 (1965)].
[5]V. V. Zheleznyakov, Izv. Vyssh. Ucheb. Zaved. Radiofizika **2**, 14 (1959); **3**, 57 (1960).
[6]M. V. Kuzelev, A. A. Rukhadze, and D. S. Fillipychev, Relativistic High-Frequency Electronics [in Russian], Inst. Phys. Prob., Gorky, 1981, p. 170.
[7]M. V. Kuzelev, A. A. Rukhadze, Yu. V. Bobylev, and V. A. Panin, Zh. Eksp. Teor. Fiz. **91**, 1620 (1986) [Sov. Phys. JETP. **64**, 956 (1986)].
[8]A. T. Bogdanov, M. V. Kuzelev, and A. A. Rukhadze, Izv. Vyssh. Ucheb. Zaved. Radiofizika, 1986, **29**, 1431 (1986).
[9]V. S. Letokhov and V. G. Minogin, *Laser-Radiation Pressure on Atoms* [in Russian], Nauka, 1986, p. 224.
[10]M. V. Kuzelev and A. A. Rukhadze, Usp. Fix. Nauk **152** 285 (1987) [Sov. Phys. Usp. **40**, 507 (1987)].
[11]A. V. Galeev and R. Z. Sagdeev, *Reviews of Plasma Physics*, Vol. 7, Consultants Bureau, 1979, p. 1.

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