Toward a unified theory of the radiation by relativistic particles in crystals

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A quantum theory of the electromagnetic emission by relativistic particles incorporating channeling and the thermal vibrations of the crystal nuclei is derived. A general expression for the emission probability is found after an average over the initial polarizations of the particles and a summation over the final polarizations of the particles and over the polarizations of the photons. An average is carried out over the crystal states of the nuclei in the cases with and without excitation of phonons. The total emission is made up of channeling emission and bremsstrahlung, which are related to each other. During scattering by thermal vibrations, incoherent bremsstrahlung is produced. Some particular cases which determine the properties of the emission in the case of channeling are derived from the general expression and analyzed.

1. INTRODUCTION

The passage of relativistic particles with velocities $v<c/n$ (c is the phase velocity of light in vacuum, and n is the refractive index of the medium) through crystals results in the emission of electromagnetic radiation which is distinct from ordinary bremsstrahlung in an amorphous target. Theories for channeling radiation[1] and for coherent and incoherent bremsstrahlung[2] have been derived independently. Each type of radiation has been treated as a process unrelated to the radiation of other types in these theories. In the case of the bremsstrahlung the problem has been solved in the Born approximation.[3,4] The interaction of a particle with the crystal over the "coherence length" results in the appearance of coherence and interference effects in bremsstrahlung.[2,4]

A more general radiation problem has recently been solved[5] by methods of classical electrodynamics. The interaction of a beam of relativistic particles with a crystal was analyzed there. The particles were assumed to be incident at small angles with respect to a crystallographic axis or plane. This assumption corresponds to a transitional type of channeling, in which channeling radiation and coherent bremsstrahlung are manifested simultaneously. An important property was established in Ref. 5: The coherent bremsstrahlung and the channeling radiation are interrelated and determine the resultant emission by the particle. In deriving a general theory it is necessary to consider the interplay among all radiation mechanisms (channeling radiation and coherent and incoherent bremsstrahlung) as the particle interacts with atomic planes and rows, the discrete nature of the crystal lattice, the thermal vibrations of the nuclei, and electronic excitations.

The radiation emitted during spontaneous transitions of the channeled particles—the channeling radiation—is dominant in only part of the spectrum.[6,7] The coherent bremsstrahlung is dominant at disorientation angles greater than the critical angle for channeling. The theory derived for this radition by Ter-Mikaélyan[2] is applicable only under those conditions. Since it is important for interpreting experimental data to have a theory for the entire bremsstrahlung spectrum during the motion of particles in a channeling regime, it is necessary to derive a theory which would describe the entire bremsstrahlung spectrum during channeling.

We derive such a theory in the present paper. This theory predicts the probability for electromagnetic emission, taking into account the recoil and the spin polarization during inelastic scattering, under channeling conditions. This radiation dominates the hard part of the bremsstrahlung spectrum, at electron or position energies from ~ 1 MeV to 10 GeV; it is important far from the resonances of the channeling radiation in the soft part of the spectrum, at low energies ~ 1–10 MeV.

2. FORMULATION OF THE PROBLEM

A beam of relativistic charged particles in incident on a crystal at a small angle with respect to a crystallographic axis or plane. The thermal vibrations of the nuclei are of small amplitude, so the probability density P for a certain configuration of lattice nuclei can be written

$$P(u_1, u_2, \ldots, u_{3N}) = [\det(A_{kl})]^{-1/2}(2\pi)^{-3N/2}\exp\left(-\frac{1}{2}\sum_{k,l}A_{kl}u_ku_l\right), \quad (1)$$

where u_k, u_l ($k, l = 1, 2,\ldots, 3N$) are the coordinates of the thermal displacements of the nuclei, and N is the number of lattice sites in the crystal. The multidimensional distribution (1) is valid for both classical and quantum statistics because of its Gaussian form; the Gaussian form for the quantum-mechanical probability for the coordinate of an oscillator follows from the Bloch theorem.[8] It is not difficult to generalize this theorem to our case of a multidimensional distribution which is determined by a quadratic form with coefficient A_{kl}.

The interaction of a particle of energy E and spin 1/2 with a scalar field V is described in the Dirac picture by the system of equations

$$\begin{aligned} &\boldsymbol{\sigma}\mathbf{p}\Psi_1+(V-1)\Psi_2=E\Psi_2,\\ &\boldsymbol{\sigma}\mathbf{p}\Psi_2-(V+1)\Psi_1=E\Psi_1, \end{aligned} \quad (2)$$

where we are using a system of units with $c=\hbar=m_0=1$, the momentum operator is $\mathbf{p}=-i\nabla$, the interaction operator is

$$V=V_0+\Delta V, \quad \text{where} \quad \Delta V=V_d+V_{th}+V_e,$$

V_0 is the average potential of an atomic row, and V_d, $V_{\rm th}$,

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and V_e, respectively, are perturbations resulting from the discrete nature of the crystal lattice, the thermal vibrations of the nuclei, and the excitation of atomic electrons.

A solution of Eqs. (2) is a bispinor of the type

$$\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}.$$

The meaning of the interaction operator can be understood from the expressions

$$V_0=\sum_n\int\langle V_a(|\mathbf{r}-\mathbf{R}_n|)\rangle_{th}\,d\tau, \qquad (3)$$

$$V_d=\sum_n[\langle V_a(|\mathbf{r}-\mathbf{R}_n|)\rangle_{th}-\int\langle V_a(|\mathbf{r}-\mathbf{R}_n|)\rangle_{th}\,d\tau], \qquad (4)$$

$$V_{th}=\sum_n[V_a(|\mathbf{r}-\mathbf{R}_n|)-\langle V_a(|\mathbf{r}-\mathbf{R}_a|)\rangle_{th}], \qquad (5)$$

$$V_e=\sum_n\Big[V_{en}(|\mathbf{r}-\mathbf{R}_n|)+\sum_s V_{ee}(|\mathbf{r}-\mathbf{R}_n-\mathbf{R}_{ns}|)\Big]$$
$$-\sum_n V_a(|\mathbf{r}-\mathbf{R}_n|), \qquad (6)$$

where

$$V_a=V_{en}(|\mathbf{r}-\mathbf{R}_n|)+\langle V_{ee}(|\mathbf{r}-\mathbf{R}_n-\mathbf{R}_{ns}|)\rangle,$$
$$V_{en}=e_1Ze/|\mathbf{r}-\mathbf{R}_n|,\ V_{ee}=e_1e/|\mathbf{r}-\mathbf{R}_n-\mathbf{R}_{ns}|,$$
$$\langle V_{en}\rangle_{th}=\int V_{en}(|\mathbf{r}-\mathbf{R}_n|)P(\mathbf{u})\,d\mathbf{u}_n\langle V_{ee}\rangle_{th}$$
$$=\int V_{ee}(|\mathbf{r}-\mathbf{R}_n-\mathbf{R}_{ns}|)P(\mathbf{u})\,d\mathbf{u}_n,$$
$$\langle V_{ee}\rangle=\int V_{ee}(|\mathbf{r}-\mathbf{R}_n-\mathbf{R}_{ns}|)\rho(\mathbf{R}_{ns})\,d\mathbf{R}_{ns},$$

$\mathbf{r}$, $\mathbf{R}_n$, and $\mathbf{R}_{ns}$, respectively, are the radius vectors of the particle, the nucleus at site n, and the sth electron in atom n; $s=1,2,...,z$; e_1 and Ze are the charges of the particle and the nucleus, respectively; and $\langle...\rangle_{\rm th}$ is an average over the thermal vibrations of the lattice nuclei. The radius vector of the nucleus in site n can be written in the form $\mathbf{R}_n=\mathbf{R}_n^0+\mathbf{u}_n$, where $\mathbf{R}_n^0$ is the equilibrium position of the nucleus, and $\mathbf{u}_n=(u_{3n-2}, u_{3n-1}, u_{3n})$ is the thermal displacement of the nucleus from its equilibrium position. The configuration of the crystal is determined by the multidimensional thermal-displacement vector $\mathbf{u}=(u_1, u_2, ..., u_{3n})$. We will assume below that the distribution function of the electrons in atom n, i.e., $\rho(\mathbf{R}_{ns})$ is identical for each lattice site. If the particle beam is incident at a small angle from the z axis, the integration variable satisfies $d\tau=dz$, and the coordinate dependence of expressions (3)–(6) can be described by

$$V_0=V_0(x,y),\quad V_d=V_d(x,y,z),$$
$$V_{th}=V_{th}(x,y,z;u),\quad V_e=V_e(x,y,z).$$

If the particle beam is incident nearly in a plane, e.g., the xz plane, then we have $d\rho=d\rho$, $\pi=(x,z)$, and an interaction $V_0=V_0(y)$.

We will solve the problem in a coordinate system whose polar axis is parallel to the longitudinal projection of the initial momentum of the particle onto the channeling plane or axis. A distinctive structural feature of the interaction operator is an integration over the variable $d\tau$, which is always the same as the variables which run along the crystallographic axis or plane.

We transform the Dirac equation (2) into an equation with a single unknown spinor:

$$[-{}^1/_2\Delta+EV+1]\,\Psi_1=E^2\Psi_1. \qquad (7)$$

Since the energy of the particle satisfies $E\gg V$, the problem reduces in this case to one of solving the Schrödinger equation

$$\Delta\Phi(\mathbf{r})+2m[E-V(\mathbf{r})]\,\Phi(\mathbf{r})=0, \qquad (8)$$

where $\Phi(\mathbf{r})$ is the coordinate part of the spinor Ψ.

Using the solution of Eqs. (8) and (2), we can calculate the radiation in first-order perturbation theory from the known[9] expression for the probability for a spontaneous transition from state to state i:

$$W_{if}=\frac{e^2}{2\pi}\int d^3\varkappa\,\omega^{-1}\delta(\omega-\omega_{if})\,\Phi_\lambda, \qquad (9)$$

$$\Phi_\lambda=|\alpha_{if}\beta_\lambda|^2, \qquad (10)$$

$$\alpha_{if}=\int e^{-i\varkappa r}\Psi_i^+\alpha\Psi_f\,d^3r, \qquad (11)$$

where the operator α is given by

$$\alpha=\begin{pmatrix}0&\sigma\\ \sigma&0\end{pmatrix},$$

σ are the Pauli matrices, $\varkappa$ is the wave number of the photon, λ is a polarization index, and β_λ is the polarization unit vector.

3. SOLUTION OF THE PROBLEM IN FIRST-ORDER PERTURBATION THEORY

The state of the system consisting of the particle and the crystal can be described by

$$\Psi(\mathbf{r},\mathbf{u})=\frac{A_0}{L^{l/2}}\begin{pmatrix}\mathbf{v}\\ (\sigma\mathbf{p})\mathbf{v}/(E+1)\end{pmatrix}\Phi(\mathbf{r})\chi(\mathbf{u}),$$

where $\mathbf{v}$ is the two-dimensional spin polarization vector of the particle, which satisfies the normalization condition $\mathbf{v}^+\mathbf{v}=1$; $\chi(\mathbf{u})$ is a wave function which describes the crystal; $A_0=[(E+1)/2E]^{1/2}$; and we have $l=1$ and 2 for axial and planar channeling, respectively.

Using perturbation theory, we find a general solution in the form

$$\Psi(\mathbf{r},\mathbf{u})=\Psi_0(\mathbf{r},\mathbf{u})+[\Delta V(\mathbf{r},\mathbf{u})/(E-H_0)]\,\Psi_0(\mathbf{r},\mathbf{u}), \qquad (12)$$

where the bispinor Ψ_0 is found from the equation

$$H_0\Psi_0=E_0\Psi_0,\quad H_0=\alpha\mathbf{p}+\beta+V_0+H_1,$$

H_1 is the Hamiltonian of the crystal,

$$\Psi_0=\frac{A_0}{L^{l/2}}\begin{pmatrix}\mathbf{v}\\ (\sigma\mathbf{p})\mathbf{v}/(E+1)\end{pmatrix}\varphi_0(\rho)\chi_0(\mathbf{u})\exp(i\mathbf{p}_\parallel\mathbf{r}_\parallel),$$

and the four-dimensional matrix β is

$$\beta=\begin{pmatrix}I&0\\0&I\end{pmatrix},\quad I=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

For simplicity we will discuss the excitation of phonons here; the excitation of electrons can be treated in a completely analogous way. Using (12) and the notation introduced in Ref. 7, we can write the matrix elements of transition current (11) as the series

$$\alpha_{if}=\overset{(0)}{\alpha_{if}}+\overset{(1)}{\alpha_{if}}+\overset{(2)}{\alpha_{if}}+\ldots,$$

whose terms are

$$\overset{(0)}{\alpha_{if}}=\langle v_i|\mathbf{A}^{(0)}+i[\boldsymbol{\sigma}\mathbf{B}^{(0)}]|v_f\rangle, \tag{13}$$

$$\alpha_{if}^{(1)}=\sum_{n\neq f}\frac{\langle v_i|\mathbf{A}^{(1)}+i[\boldsymbol{\sigma}\mathbf{B}^{(1)}]|v_n\rangle}{E_n^{(0)}-E_f^{(0)}}, \tag{14}$$

$$\alpha_{if}^{(2)}=\sum_{m\neq i}\frac{\langle v_m|\mathbf{A}^{(2)}+i[\boldsymbol{\sigma}\mathbf{B}^{(2)}]|v_f\rangle}{E_m^{(0)}-E_i^{(0)}}, \tag{15}$$

where

$$\mathbf{B}^{(s)}=E_3^{(s)}a_3^{-1}I_3^s-E_2^{(s)}a_sI_2^{(s)},\qquad s=0,1,2,$$

$$\mathbf{A}^{(s)}=E_2^{(s)}a_sI_2^{(s)}+E_3^{(s)}a_3^{-1}I_3^{(s)},\qquad s=0,1,2,$$

$$E_2^{(0,1)}=E_f,\quad E_2^{(2)}=E_n,\quad E_3^{(0,1)}=E_i,\quad E_3^{(2)}=E_m,$$

$$a_0=[(1+E_i)/(1+E_f)]^{1/2},\quad a_1=[(1+E_i)/(1+E_m)]^{1/2},$$

$$a_2=[(1+E_m)/(1+E_f)]^{1/2}.$$

Here are the explicit expressions for the first three terms of the series (13)–(15), which we will be using below:

$$\mathbf{I}_2^{(0)}=\frac{1}{E_fL^l}\int e^{-i\varkappa r}\Phi_i^*(\mathbf{r})\mathbf{p}_\perp\Phi_f(\mathbf{r})\,d^3r, \tag{16}$$

$$\mathbf{I}_3^{(0)}=-\frac{1}{E_iL^l}\int e^{-i\varkappa r}\Phi_f(\mathbf{r})\mathbf{p}_\perp\Phi_i^*(\mathbf{r})\,d^3r, \tag{17}$$

$$E_f\mathbf{I}_2^{(0)}-E_i\mathbf{I}_3^{(0)}=\varkappa I_1^{(0)},\qquad I_1^{(0)}=\frac{1}{L^l}\int e^{-i\varkappa r}\Phi_i^*(\mathbf{r})\,\Phi_f(\mathbf{r})\,d^3r, \tag{18}$$

$$\mathbf{I}_2^{(1)}=\frac{1}{E_nL^l}\int e^{-i\varkappa r}\Phi_i^*(\mathbf{r})\,\Delta V_{nf}\mathbf{p}_\perp\Phi_n(\mathbf{r})\,d^3r, \tag{19}$$

$$\mathbf{I}_3^{(1)}=-\frac{1}{E_iL^l}\int e^{-i\varkappa r}\Phi_n(\mathbf{r})\mathbf{p}_\perp\Delta V_{nf}\Phi_i(\mathbf{r})\,d^3r, \tag{20}$$

$$E_n\overset{(1)}{\mathbf{I}_2}-E_i\overset{(1)}{\mathbf{I}_3}=\varkappa\overset{(1)}{I_1}, \tag{21}$$

$$I_1^{(1)}=\frac{1}{L^l}\int e^{-i\varkappa r}\Phi_i^*(\mathbf{r})\,\Delta V_{nf}\Phi_n(\mathbf{r})\,d^3r, \tag{22}$$

$$\mathbf{I}_2^{(2)}=\frac{1}{E_fL^l}\int e^{-i\varkappa r}\Delta V_{mi}^*\Phi_m^*(\mathbf{r})\mathbf{p}_\perp\Phi_f(\mathbf{r})\,d^3r, \tag{23}$$

$$\mathbf{I}_3^{(2)}=-\frac{1}{E_mL^l}\int e^{-i\varkappa r}\Phi_f(\mathbf{r})\,\Delta V_{mi}^*\mathbf{p}_\perp\Phi_m^*(\mathbf{r})\,d^3r, \tag{24}$$

$$E_f\overset{(2)}{\mathbf{I}_2}-E_m\overset{(2)}{\mathbf{I}_3}=\varkappa\overset{(2)}{I_1}, \tag{25}$$

$$I_1^{(2)}=\frac{1}{L^l}\int e^{-i\varkappa r}\Phi_m^*(\mathbf{r})\,\Delta V_{mi}^*\Phi_f(\mathbf{r})\,d^3r. \tag{26}$$

The matrix elements of the perturbation operator are

$$\Delta V_{nf}=\int\Phi_n^*(\mathbf{r})\chi_{i'}(\mathbf{u})\,[V_d(\mathbf{r})+V_{th}(\mathbf{r},\mathbf{u})]\,\Phi_f(\mathbf{r})\chi_{f'}(\mathbf{u})\,d^3r\,d^3u,$$

and the indices on the wave functions Φ_1 and $\chi_{i'}$ have the following meaning: i is the quantum-mechanical state of the particle, and i' is the state of the crystal.

We will be using (16)–(26) to find the probability

$$W_{if}=\frac{e^2}{2\pi}\sum_{s,\lambda}\int d^3\varkappa\,\omega^{-1}\delta(\omega-\omega_{if})\,|\boldsymbol{\alpha}^{(s)}\beta_\lambda|^2. \tag{27}$$

Summing this expression over the final polarizations of the particle, averaging it over the initial polarizations,[10] and summing over the photon polarizations β_λ, we find

$$W_{if}=\sum_{s=0}^{2}W^{(s)}$$

$$=\frac{e^2}{2\pi}\int\{(\alpha_0^2+\omega_0^2\beta_0^2)\,|\mathbf{I}_2^{(0)}|^2+(\omega_0^2\beta_0^2-\alpha_0^2)\,|\mathbf{I}_2^{(0)}\mathbf{n}_\perp|^2$$

$$+G_0|\overset{(0)}{I_1}|^2-2T_0[\mathrm{Re}(\overset{(0)}{\mathbf{I}_2}\mathbf{n}_\perp)\overset{(0)*}{I_1}]\}\,\omega^{-1}\delta(\varkappa-\varkappa_{if})\,d^3\varkappa$$

$$+\frac{e^2}{2\pi}\sum_{s=1}^{2}\sum_{\substack{n\neq f\\(m\neq i)}}\Big\{(E^0_{n(m)}-E^0_{f(i)})^{-2}\int[(\alpha_s^2+\beta_s^2\omega_s^2)\,|\mathbf{I}_2^{(s)}|^2$$

$$+(\beta_s^2\omega_s^2-\alpha_s^2)\,|\mathbf{I}_2^{(s)}\mathbf{n}_\perp|^2+G_s|I_1^{(s)}|^2-2T_s(\mathrm{Re}(\mathbf{I}_2^{(s)}\mathbf{n}_\perp)I_1^{(s)*})]$$

$$\cdot\omega^{-1}\delta(\varkappa-\varkappa_{if})\,d^3\varkappa+(E_n^0-E_f^0)^{-1}(E_m^0-E_i^0)^{-1}\int[F_1(\mathbf{I}_2^{(1)}\mathbf{I}_2^{(2)*}$$

$$+F_2(I_1^{(1)}I_1^{(2)*})-2F_3(\mathrm{Re}(\mathbf{I}_2^{(1)}\mathbf{n}_\perp)I_1^{(2)*})$$

$$-2F_4(\mathrm{Re}(\mathbf{I}_2^{(2)}\mathbf{n}_\perp)I_1^{(1)*})]\,\omega^{-1}\delta(\varkappa-\varkappa_{if})\,d^3\varkappa\Big\}, \tag{28}$$

where

$$G_s=2\delta_s^2\omega_s^2+(\alpha_s^2+\omega_s^2\beta_s^2)\,(v_\parallel^{(s)})^2-4\beta_s\delta_s\omega_s^2v_\parallel^{(s)}n_\parallel$$

$$+(\omega_s^2\beta_s^2-\alpha_s^2)\,(v_\parallel^{(s)}n_\parallel)^2,\qquad s=0,1,2, \tag{29}$$

$$T_s=2\beta_s\delta_s\omega_s^2+(\alpha_s^2-\beta_s^2\omega_s^2),\qquad s=0,1,2, \tag{30}$$

$$F_1=\alpha_1\alpha_2(1-n_\perp^2)+\omega_1\omega_2\beta_1\beta_2(1+n_\perp^2), \tag{31}$$

$$F_2=\{2\omega_1\omega_2\delta_1\delta_2+v_\parallel^{(1)}v_\parallel^{(2)}[\alpha_1\alpha_2(1-n_\parallel^2)+\omega_1\omega_2\beta_1\beta_2(1+n_\parallel^2)]$$

$$-2\omega_1\omega_2(v_\parallel^{(1)}n_\parallel\beta_1\delta_2+v_\parallel^{(2)}n_\parallel\delta_1\beta_2)\}, \tag{32}$$

$$F_{3(4)}=2\omega_1\omega_2\beta_{1(2)}\delta_{2(1)}v_\parallel^{(2)}n_\parallel(\alpha_1\alpha_2-\omega_1\omega_2\beta_1\beta_2), \tag{33}$$

$\mathbf{n}$ is a unit vector in the emission direction, and

$$\alpha_0={}^1/_2(E_f/E_i)^{1/2}(a_0^2+1)/a_0,$$

$$\alpha_{1(2)}={}^1/_2(E_{n(f)}/E_{i(m)})^{1/2}(a_s^2+1)/a_s,\qquad s=1,\ 2,$$

$$\beta_0={}^1/_2(E_i/E_f)^{1/2}(1-a_0^2)/a_0,$$

$$\beta_{1(2)}={}^1/_2(E_{n(f)}/E_{i(n)})^{1/2}(1-a_s^2)/a_s,\quad s=1,2,$$

$$\delta_s=(2a_s)^{-1}(E_i/E_f)^{1/2},\quad s=0,1,2;\qquad \omega_0=E_f-E_i,$$

$$\omega_{1(2)}=E_{n(f)}-E_{i(m)},\qquad v_\parallel^{(1)}=p_{\parallel\,n}/E_n,\qquad v_\parallel^{(0,2)}=p_{\parallel,f}/E_f.$$

The first term ($s=0$, i.e., the zeroth approximation) in (28) obviously corresponds to the known emission which accompanies spontaneous transitions of a channeled particle (Ref. 7, for example) and is of no particular interest here.

At large angles of incidence the tranverse energy is large, $E_\perp\gg V_0$, so the channeling can be treated in the Born approximation:

$$\Phi_j(\mathbf{r})=\exp(i\mathbf{p}_j\mathbf{r})+\sum_{n\neq j}\frac{V_{nj}}{E_j-E_n}\exp(i\mathbf{p}_n\mathbf{r}),\quad j=i,f.$$

In addition, we can replace the wave functions by plane waves in the integrals in (19)–(26). The zeroth and first approximations in (28) also give us the known probability for coherent bremsstrahlung with a discrete transfer of transverse and longitudinal momenta, respectively, and for incoherent bremsstrahlung.[2]

We turn now to the total emission probability (28), for small angles of incidence, at which the channeling causes the substantial change in the wave function of the particle. For soft radiation—the region where we find the peaks of channeling radiation—i.e., under the condition $\omega \ll E$, with $E_i \approx E_f$, $I_2^s \approx I_3^s$, and $I_1^s \approx 0$ $(s = 0, 1, 2)$, the general expression (28) takes the following form when we take the thermal vibrations of the nuclei (and also the polarization) into account:

$$W_{if} = \frac{e^2}{2\pi}\left\{\int\left[\ |\mathbf{I}_2^{(0)}\beta_\lambda|^2 + \left|\sum_{n\neq f}\frac{1}{E_n^0 - E_f^0}\mathbf{I}_2^{(1)}\beta_\lambda\right.\right.\right.$$
$$\left.\left.\left. + \sum_{m\neq i}\frac{1}{E_m^0 - E_i^0}\mathbf{I}_2^{(2)}\beta_\lambda\right|^2\right]\right\}\delta(\omega-\omega_{if})\frac{d^3\varkappa}{\omega}. \qquad (34)$$

Following Ref. 11, we expand expressions (4) and (5) for the perturbation in a Fourier integral

$$V_d = \frac{e_i Ze}{2\pi^2}\sum_n\left\{\int\left\langle\frac{\exp[i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)]}{\mathbf{q}^2}\right\rangle_{th} d^3q - \int\left\langle\frac{\exp[i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)]}{\mathbf{q}^2}\right\rangle_{th} d^3q\, d\tau\right\},$$

$$V_{th} = \frac{e_i Ze}{2\pi^2}\sum_n\left\{\int\frac{\exp[i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)]}{\mathbf{q}^2}d^3q - \int\left\langle\frac{\exp[i\mathbf{q}(\mathbf{r}-\mathbf{R}_n)]}{\mathbf{q}^2}\right\rangle_{th} d^3q\right\},$$

and we write these expressions in a form convenient for the calculations below:

$$V_d = c_0\int\left\{\left[\frac{\exp(i\mathbf{q}\mathbf{r})}{\mathbf{q}^2} - \int\frac{\exp(i\mathbf{q}\mathbf{r})}{\mathbf{q}^2}d\tau\right]d^3q \times\left\langle\sum_n\exp(-i\mathbf{q}\mathbf{R}_n)\right\rangle_{th}\right\}, \qquad (35)$$

$$V_{th} = c_0\int\frac{\exp(i\mathbf{q}\mathbf{r})}{\mathbf{q}^2}d^3q \times\left[\sum_n\exp(-i\mathbf{q}\mathbf{R}_n) - \left\langle\sum_n\exp(-i\mathbf{q}\mathbf{R}_n)\right\rangle_{th}\right], \qquad (36)$$

where $c_0 = e_i Ze/2\pi^2$.

From the conservation laws we can find the momentum $\mathbf{q}$ transferred to the crystal along the plane or axis,

$$\mathbf{p}_{\parallel,f} = \mathbf{q}_\parallel + \boldsymbol{\varkappa}_\parallel + \mathbf{p}_{\parallel,i},$$

and that in the transverse direction,

$$\mathbf{p}_{\perp,f} = \mathbf{q}_\perp + \boldsymbol{\varkappa}_\perp + \mathbf{p}_{\perp,i} + \mathbf{g}_\perp$$

(here $\mathbf{g}$ is a reciprocal-lattice vector). Since the scattering probability decays rapidly for $q > 2\pi/d$ (d is the lattice constant), we can ignore the Umklapp process.

To separate the coherent and incoherent parts of (34), we sum and average over all of the crystal atoms an expression of the type

$$\left|\sum_n\langle|\exp(i\mathbf{q}\mathbf{R}_n)|\rangle\right|^2 = \left|\sum_n\exp(i\mathbf{q}\mathbf{R}_n^0)\right|^2|\langle|Q|\rangle|^2 + N(\overline{|\langle|Q|\rangle|^2} - \overline{|\langle|Q|\rangle|}^2), \qquad (37)$$

where $\langle|Q|\rangle = \langle i|\exp(i\mathbf{q}\mathbf{u}_n)|f\rangle$ is a generalized dipole moment, and $\mathbf{R}_n = \mathbf{R}_n^0 + \mathbf{u}_n$. This procedure is legitimate since the energy of the phonon is significantly smaller than the change in the transverse energy of the particle.

In the dipole approximation in the scattering $(\mathbf{q}\cdot\mathbf{u} \ll 1)$, the right side of expression (37) can be written as

$$\left|\sum_n\exp(i\mathbf{q}\mathbf{R}_n^0)\right|^2\exp(-\mathbf{q}^2\overline{\mathbf{u}^2}) + N[1-\exp(-\mathbf{q}^2\overline{\mathbf{u}^2})].$$

After division by $N\to\infty$, the latter expression takes the form

$$\delta(\mathbf{q}-\mathbf{g})\exp(-\mathbf{q}^2\mathbf{u}^2) + [1-\exp(-\mathbf{q}^2\overline{\mathbf{u}^2})]. \qquad (38)$$

It is worthwhile to take a detailed look at the case of emission accompanying elastic scattering (without a transfer of energy to the crystal) and the case of emission accompanying inelastic scattering (in which the crystal is excited).

4. ELASTIC SCATTERING

In this case the emission results exclusively from a perturbation which has nonvanishing diagonal matrix elements between states of the crystal. Using the results of the average over the thermal displacements of the nuclei found above, (37) and (38), we can write the probability for a nonphonon emission in first-order perturbation theory as follows:

$$W_{if} = \frac{e^2}{2\pi}\int\left[\ |\mathbf{I}_2^{(0)}\beta_\lambda|^2 + \left|\sum_{n\neq f}\frac{1}{E_n^0-E_f^0}\int\beta_\lambda\mathbf{I}_2^{(1)}(\mathbf{q},\boldsymbol{\varkappa}_\perp)d^3q\right.\right.$$
$$\left.\left. + \sum_m\frac{1}{E_m^0-E_i^0}\int\beta_\lambda\mathbf{I}_2^{(2)}(\mathbf{q},\boldsymbol{\varkappa}_\perp)d^3q\right|^2\right]\delta(\omega-\omega_{if})\frac{d^3\varkappa}{\omega}$$
$$\times\left\{\sum_g\exp(-\mathbf{q}^2\overline{\mathbf{u}^2})\delta(\mathbf{q}-\mathbf{g}) + [1-\exp(-\mathbf{q}^2\overline{\mathbf{u}^2})]\right\}. \qquad (39)$$

In, for example, the case in which the particles are incident along an axis, the integrals $I_2^{(1)}$ and $I_2^{(2)}$ here are given by

$$\mathbf{I}_2^{(1)}(\mathbf{q},\boldsymbol{\varkappa}_\perp) = \frac{cc_0}{L^t E_n}\int\frac{\exp[-i(\boldsymbol{\varkappa}_\perp\boldsymbol{\rho} - \mathbf{q}_\perp\boldsymbol{\rho}')]}{q^2} \times\Phi_i^*(\boldsymbol{\rho})\Phi_n^*(\boldsymbol{\rho}')\Phi_f(\boldsymbol{\rho}')\mathbf{p}_\perp\Phi_n(\boldsymbol{\rho})d\boldsymbol{\rho}\,d\boldsymbol{\rho}', \qquad (40)$$

$$\mathbf{I}_2^{(2)}(\mathbf{q},\boldsymbol{\varkappa}_\perp) = \frac{cc_0}{L^t E_f}\int\frac{\exp[-i(\boldsymbol{\varkappa}_\perp\boldsymbol{\rho} - \mathbf{q}_\perp\boldsymbol{\rho}')]}{\mathbf{q}^2}\Phi_m(\boldsymbol{\rho}') \times\Phi_i^*(\boldsymbol{\rho}')\Phi_m^*(\boldsymbol{\rho})\mathbf{p}_\perp\Phi_f(\boldsymbol{\rho})d\boldsymbol{\rho}\,d\boldsymbol{\rho}'. \qquad (41)$$

It follows from the form of the perturbations (3)–(6) that there is no interference between the different orders of perturbation theory. We see from (39) that there is coherent emission only in the case of scattering by an ideal crystal $(u = 0)$. In principle, incoherent scattering can also arise in the case $u = 0$, if the crystal lattice consists of different isotopes. In this case the longitudinal momentum transferred by the particle is nonzero. In the general case with $q_\parallel \neq 0$, the conservation laws for the nonphonon emission can be written

$$\Delta E_\parallel + \Delta E_\perp = \hbar\omega, \quad \Delta p_\parallel = q_\parallel + \varkappa_z. \qquad (42)$$

In the case of ultrarelativistic particles, for radiation in the forward direction $(\varkappa = \varkappa_z)$, we find from (42)

$$\Delta E_\perp = q_\parallel - \omega[2\gamma^2(1-\omega/E)]^{-1}, \quad E \gg 1. \qquad (43)$$

During channeling, the left side of this expression is significantly smaller than the reciprocal-lattice vector, so for $q_\parallel \sim 2\pi/d$ hard photons with an energy $\omega \sim 2E^2/d$ are emitted (in the case $E < d/\lambda_c$, where λ_c is the Compton wavelength). These photons are formed over a single lattice period, and no amplification occurs in this region. If, on the other hand, there is strong scattering, with $\Delta E_\perp \sim 1/d$, soft photons can be emitted in the case $q_\parallel \neq 0$ with a large formation length. In this case, however, the probability for the process is low, because the momentum transfer is large. If the momentum transfer $\mathbf{q}$ is nearly perpendicular to the momentum $\mathbf{p}$ of the particle the longitudinal momentum $q_\parallel$ can be a small quantity, on the left side of expression (43). The family of reciprocal-lattice vectors $\mathbf{g}$ determines a crystallographic axis which is nearly paralled to $\mathbf{p}$. This case of a so-called transitional channeling was originally analyzed by the methods of classical mechanics and electrodynamics in Ref. 5.

5. INELASTIC SCATTERING

It follows from expression (28) for the probability that when phonons are excited we need to sum over the final states of the crystal in expressions of the type

$$\left|\sum_{n\neq f}\frac{\langle i|\exp(-i\varkappa\mathbf{r})V_q|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle}{\Delta E_{i'f'}+\Delta E_{\perp,fn}-cq_\parallel}\right|^2 \times\left|\sum_j\langle i'|\exp(i\mathbf{qR}_j)|f'\rangle\right|^2 \quad (44)$$

$$\sum_{n\neq f}\frac{\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_\perp|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle}{(\Delta E_{i'f'}+\Delta E_{\perp,fn}-cq_\parallel)^2} (\langle i|\exp(-i\varkappa\mathbf{r})|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle)^* \times\left|\sum_j\langle i'|\exp(i\mathbf{qR}_j)|f'\rangle\right|^2, \quad (45)$$

$$\sum_{m\neq n}\frac{\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_\perp|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle}{(\Delta E_{i'f'}+\Delta E_{\perp,fn}-cq_\parallel)(\Delta E_{i'f'}+\Delta E_{\perp,fm}-cq_\parallel)} \times(\langle i|\exp(i\mathbf{qr})|m\rangle V_q\langle m|\exp(-i\varkappa\mathbf{r})\mathbf{p}_\perp|f\rangle)^* \times\left|\sum_j\langle i'|\exp(i\mathbf{qR}_j)|f'\rangle\right|^2, \quad (46)$$

$$\sum_{m,n}\frac{\langle i|\exp(-i\varkappa\mathbf{r})|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle}{(\Delta E_{i'f'}+\Delta E_{\perp,fn}-cq_\parallel)(\Delta E_{i'f'}+\Delta E_{\perp,fm}-cq_\parallel)} \times(\langle i|\exp(i\mathbf{qr})|m\rangle V_q\langle m|\exp(-i\varkappa\mathbf{r})|f\rangle)^* \times\left|\sum_j\langle i'|\exp(i\mathbf{qR}_j)|f'\rangle\right|^2, \quad (47)$$

where $V_q = c_0/\mathbf{q}^2$, $|f\rangle$ and $|i\rangle$ are the final and initial states of the particle, $|f'\rangle$ and $|i'\rangle$ are the corresponding states of the crystal nuclei, j is the index of a nucleus, and $|m\rangle$ and $|n\rangle$ are intermediate states of the particle.

Since we have $\Delta E_{\perp fn}, \Delta E_{\perp fm} \gg \Delta E_{i'f'}$, we can ignore the phonon energy $\Delta E_{i'f'}$ and sum over the final states of the crystal in expressions of the type

$$\left|\sum_{j=1}^{N}\langle i'|\exp(i\mathbf{qR}_j)|f'\rangle\right|^2 = \langle i'|\left|\sum_j\exp(i\mathbf{qR}_j)\right|^2|i'\rangle - \left|\langle i'|\sum_j\exp(i\mathbf{qR}_j)|i'\rangle\right|^2. \quad (48)$$

After summation, expression (48) can be put in the form

$$N(1-\overline{|\langle i'|\exp(i\mathbf{qu})|i'\rangle|^2}). \quad (49)$$

The superior bar in (49) means an average over the initial states. In the dipole approximation ($qu \ll 1$), expression (49) take the following form after averaging:

$$N[1-\exp(-\mathbf{q}^4\overline{\mathbf{u}^4}/4)]. \quad (50)$$

It follows from the analysis above [see (44)–(50)] that when phonons are excited there will be only incoherent electromagnetic emission.

In the soft-photon approximation, the emission accompanied by the excitation of phonons can be put in the form [see (28), (34), and (50)]

$$W_{if}=\frac{e^2}{2\pi}\left\{\iint\left[\left|\mathbf{I}_2^{(0)}\beta_\lambda\right|^2+\left|\sum_{n\neq f}\frac{1}{E_n^0-E_f^0}\int\beta_\lambda\mathbf{J}_2^{(1)}(\mathbf{q},\varkappa_\perp)d^3q + \sum_{m\neq i}\frac{1}{E_m^0-E_i^0}\int\beta_\lambda\mathbf{J}_2^{(2)}(\mathbf{q},\varkappa_\perp)d^3q\right|^2\right]\delta(\omega-\omega_{if})\frac{d^3\varkappa}{\omega}\right\} \times N[1-\exp(-\mathbf{q}^4\overline{\mathbf{u}^4}/4)], \quad (51)$$

$$\mathbf{J}_2^{(1)}=\frac{c_0}{L^4E_n}\int\frac{\exp[-i(\varkappa_\perp\rho'-\mathbf{q}_\perp\rho)]}{\mathbf{q}^2} \times\Phi_i^*(\rho')\Phi_n^*(\rho)\Phi_f(\rho)\mathbf{p}_\perp\Phi_n(\rho')d\rho\,d\rho', \quad (52)$$

$$\mathbf{J}_2^{(2)}=\frac{c_0}{L^4E_f}\int\frac{\exp[-i(\varkappa_\perp\rho'-\mathbf{q}_\perp\rho)]}{\mathbf{q}^2} \times\Phi_m(\rho')\Phi_i^*(\rho')\Phi_m^*(\rho)\mathbf{p}_\perp\Phi_f(\rho)d\rho\,d\rho'. \quad (53)$$

In certain particular cases, it is straightforward to sum over the intermediate states of the particle in expressions (44)–(47). To determine this possibility, we write the energy conservation law:

$$q_\parallel v_\parallel+\varkappa_\parallel v_\parallel=\Delta E_{\perp,if}+\Delta E_{i'f'}+\omega. \quad (54)$$

We consider the case in which a particle is moving along a crystallographic axis or plane under the condition $v\approx 1$. Using (54), we can write equations for the resonant frequency ω_0,

$$\omega_0=(\Delta E_{\perp,if}+\Delta E_{i'f'})/(1-v_\parallel\cos\theta), \quad (55)$$

and for the frequency deviation,

$$q_\parallel=(\omega-\omega_0)(1-\cos\theta).$$

We thus see that for a large frequency deviation, $q_\parallel \gg \Delta E_{\perp,if}$, we can ignore $\Delta E_{\perp,if}$ and $\Delta E_{i'f'}$ in the denominators of expressions of the type in (44)–(47). We can thus sum over intermediate states; for example

$$\left|\sum_{n\neq f}\frac{\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_\perp|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle}{\Delta E_{i'f'}+\Delta E_{\perp,fn}-q_\parallel}\right|^2 \approx\frac{V_q^2}{\mathbf{q}_\parallel^2} \times|\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_\perp\exp(i\mathbf{qr})|f\rangle|^2, \quad (56)$$

$$\sum_{m,n}\frac{\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_\perp|n\rangle V_q\langle n|\exp(i\mathbf{qr})|f\rangle}{(\Delta E_{i'f'}+\Delta E_{\perp,fn}-q_\parallel)(\Delta E_{i'f'}+\Delta E_{\perp,fm}-q_\parallel)}(\langle i|\exp(i\mathbf{qr})|m\rangle \times V_q\langle m|\exp(-i\varkappa\mathbf{r})|f\rangle)^* =\frac{V_q^2}{\mathbf{q}_\parallel^2}|\langle i|\mathbf{q}_\perp\exp[-i(\varkappa_\perp-\mathbf{q}_\perp)\mathbf{r}_\perp]|f\rangle|^2. \quad (57)$$

The coherent term contains a factor $\delta(\mathbf{q}\text{-}\mathbf{g})$, so we find fixed frequencies for a given direction of $\varkappa$. This term corresponds to coherent bremsstrahlung with channeling, and for it we can write an energy conservation law, with the help of (54):

$$\Delta E_{\perp,\,if}+\omega=q_{\parallel}+\varkappa_{\parallel}. \tag{58}$$

A combinational term $\Delta E_{\perp,if}$, which was derived in Ref. 5 by a classical approach, has appeared in dispersion relation (58). The expressions for the coherent and incoherent bremsstrahlung contain matrix elements of the type

$$|\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_{\perp}\exp(i\mathbf{q}\mathbf{r})|f\rangle|^2,$$
$$[\langle i|\exp(-i\varkappa\mathbf{r})\mathbf{p}_{\perp}\exp(i\mathbf{q}\mathbf{r})|f\rangle](\langle i|\exp[-i(\varkappa-\mathbf{q})\mathbf{r}]|f\rangle)^*,$$

which describe the effect of channeling on these types of radiation.

The probability W_{if} can be then written in the following form, where we are taking account of the intermediate summation, under the condition $cq_{\parallel}\gg\Delta\varepsilon_{if}$:

$$W_{if}=\frac{e^2}{2\pi}\int\Big\{|\langle i|\exp(-i\varkappa_{\perp}\rho)\mathbf{p}_{\perp}\beta_{\lambda}|f\rangle|^2$$
$$+\Big|\sum_{\mathbf{q}_{\perp}}\frac{V_q}{q_{\parallel}}[|\langle i|\exp(-i\varkappa_{\perp}\rho)\mathbf{p}_{\perp}\beta_{\lambda}\exp(i\mathbf{q}_{\perp}\rho)|f\rangle$$
$$-\langle i|\exp[-i(\varkappa_{\perp}-\mathbf{q}_{\perp})\rho]\mathbf{p}_{\perp}\beta_{\lambda}|f\rangle]|^2\Big\}$$
$$\times N[1-\exp(-\mathbf{q}^2\mathbf{u}^2)]\delta(\omega-\omega_{if})\frac{d^3\varkappa}{\omega}. \tag{59}$$

The summation over the final states in the second term can be carried out approximately under the condition $q_{\parallel}\gg 2\pi/d$, in which case we can ignore the dependence of V_q on the quasimomentum of the particle. As a result we find

$$W_i=\frac{e^2}{c^2q_{\parallel}^2}\sum_{\mathbf{q}_{\perp}\mathbf{q}_{\perp}'}|V_q(\beta_{\lambda}\mathbf{q}_{\perp})(\beta_{\lambda}\mathbf{q}_{\perp}')V_{q'}|$$
$$\times|\langle i|\exp[i(\mathbf{q}_{\perp}-\mathbf{q}_{\perp}')\mathbf{r}_{\perp}]|i\rangle|^2$$
$$\times[1-\exp(-\mathbf{q}^2\mathbf{u}^2)]\frac{d^3\varkappa}{\varkappa}.$$

In accordance with the assumption above, expression (59) is applicable in the part of the bremsstrahlung spectrum which lies in the frequency interval $2E^{1/2}V_m/d_p\ll\omega\ll E$, where V_m is the depth of the potential well, and d_p is its width. Both of these conditions hold—i.e., the interval is nonzero—if $E^{1/2}\ll d_p/2V_m^{1/2}$. The value of $d_p V_m^{-1/2}$ is $\sim 10^4$, so we find the condition $E\ll 0.5\cdot 10^8$ MeV. The region of applicability of (60) is vastly narrower: $2\pi E^2/d\ll\omega\ll E$. These conditions lead to the energy restriction $E\ll d_p/2\pi\sim 10^2$ (i.e., about 50 MeV).

6. DISCUSSION OF RESULTS

Several general conclusions can be drawn from this theory for the motion of particles at a small angle with respect to an axis or plane. The dispersion relations which follow from the energy and momentum conservation laws found above show that transitions of different types are important in different ranges of the radiation frequency and in different ranges of the disorientation angle. At a momentum transfer $\mathbf{q}=\mathbf{g}$ there is a coherent bremsstrahlung, because of the discrete nature of the lattice. During channeling, i.e., at disorientation angles below the critical channeling angle, this coherent bremsstrahlung is generally suppressed dynamically, except in the case of transitional channeling, in which the orientation of the crystal is something between the axial and planar cases.

The transitions associated with incoherent scattering with an arbitrary momentum transfer lead to an incoherent bremsstrahlung with a continuous spectrum. The magnitude of the longitudinal momentum transfer, $q_{\parallel}$, determines the deviation of the frequency of the radiation from resonance. If this deviation is large (in the hard part of the spectrum), we can ignore the change in the transverse energy in comparison with that in the longitudinal energy, and we can sum over intermediate states. In this case the incoherent bremsstrahlung is not the same as that in an amorphous medium, because it is described by a factor $1-\exp(-\mathbf{q}^2\mathbf{u}^2)$ and also because, as we see from (59), an average is taken over the distribution of the particle beam in the transverse plane. The latter averaging increases the incoherent bremsstrahlung for electrons and reduces it for positrons. If the frequency deviation is small, i.e., if we are in the region of soft radiation, where we find the peak of spontaneous emission of channeled particles, we need to carry out an exact summation of the nonresonant terms in (51). These terms generate a continuum. The effect of channeling remains qualitatively the same as in the hard part of the spectrum. Analysis of the results shows that the contributions of the various mechanisms to the bremsstrahlung spectrum depend on the range of radiation frequencies under consideration, the energies of the particles, and the distribution of particles among transverse-energy states. The contributions are different for electrons and positrons.

The regions in which coherent bremsstrahlung and channeling radiation are dominant are well known.[6,7] The radiation which results from incoherent scattering is similar in nature and properties to incoherent bremsstrahlung in an amorphous medium: It has a spectrum which falls off monotonically with increasing frequency ($\propto 1/\omega$) at frequencies well above the channeling-radiation maximum. However, this radiation depends on the orientation of the crystal and the sign of the charge of the particle. Because of the redistribution of the particle beam during channeling, this radiation is amplified severalfold for electrons, while it is attenuated for positrons. Correspondingly, there is an increase or decrease in the diffuse background in comparison with that for an amorphous medium. These conclusions are supported by the well-known experimental observations (see, for example, the review in Ref. 7). The level of the diffuse background depends comparatively weakly on the energy. At low energies (1–100 MeV), at which one observes distinct and comparatively monochromatic peaks in the channeled radiation, the background level is several times lower than these peaks. As the energy increases, the height of the peaks of channeled radiation increases ($\propto E^{1/2}$), and these peaks broaden because of the contributions of higher-order multipoles. Consequently, the diffuse background can be observed in this case only at high frequencies, far from the main channeled-radiation peak, specifically, at energies from 100 MeV to 10 GeV. At higher energies, at which the channeled radiation becomes quite different from dipole radiation, it dominates the entire bremsstrahlung spectrum. It follows from (60) that the orientational dependence of the incoherent bremsstrahlung should be similar to that of the probability for processes which require close collisions with nuclei (e.g., Rutherford scattering).

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