Riemann surfaces with boundaries and string theory

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The operations of cutting and sewing Riemann surfaces allow one to express the path integral on a Riemann surface in terms of integrals over its pieces, which are Riemann surfaces with boundaries. This yields an expression for the determinant of the Laplace operator on a Riemann surface in terms of the Krichever maps for its pieces. Possible applications of the proposed methods to the investigation of the string perturbation theory series in terms of the universal modulus space are indicated.

I. INTRODUCTION

Open Riemann surfaces (i.e., Riemann surfaces with boundaries) are of interest in string theory for a variety of reasons. The most obvious subject is perturbation theory for open strings in the formalism of "first quantization." Other applications are string propagators and correlators (correlation functions), which are defined as integrals over open surfaces with prescribed boundary conditions. A third subject is related to mappings of the Krichever type, which establish a correspondence between open Riemann surfaces and the points of certain infinite-dimensional Grassmann manifolds.

The Krichever mappings define certain special coordinates on the modulus spaces given by the A matrices (defined below in Sec. 2). The use of these matrices is not free of problems, since the matrices A depend on the choice of coordinates on the Riemann surface. However, in our opinion, at the present stage of development it makes sense to do some calculations in terms of these matrices. We have shown recently in Refs. 1, 2 that by means of A one can relate the determinants on open surfaces and their doubles, and thus express the measures for open strings in terms of the Mumford measures for closed strings. (Similar relations were obtained by means of other methods in the papers Refs. 3.)

In Refs. 1 and 2 the matrix A was used in the intermediate steps of the calculations but did not enter into the final result. Now we intend to use the technique of Refs. 1, 2 derive expressions for the string propagators, in which A occurs explicitly. We thus obtain expressions for the determinants in terms of the matrices A. Unfortunately, these formulas depend on the genus of the surface. Therefore, before applying them to the construction of the string measure on the universal modulus space in terms of the two-dimensional field (as proposed in Ref. 4), one has to handle some additional problems which are posed in Subsection 3.4. In spite of this, the resulting formula looks quite attractive and we think that the proposed approach deserves further study.

2. EXPRESSIONS FOR THE STRING PROPAGATORS

We recall the main ideas of Refs. 1 and 2 in a form suitable for the discussion that follows. We consider a Riemann surface S with p_S handles and an $(M+1)$-component boundary $\partial S = \Gamma_0 + \cdots + \Gamma_M$. Two closed Riemann surfaces are associated with S.

The first surface D is the double of S, i.e., a closed Riemann surface of genus $p_D = 2p_S + M$ with an antiholomorphic Z_2-isometry $\xi \mapsto \xi^*$, such that $S = D/Z_2$. The Z_2-invariant points of D form the boundary ∂S so that $\Gamma_0 + \cdots + \Gamma_M$ are geodesics of the Z_2-invariant metric on D.

Another closed Riemann surface $\widetilde{S}$ of genus p_S is obtained from S by gluing on $(M+1)$ hemispheres (disks) $d_0 \cdots, d_M$ to the components $\Gamma_0, \cdots, \Gamma_M$ of the boundary. Let $\xi_{(\mu)}$ be a complex coordinate on S, such that Γ_μ coincides with the circle $|\xi_{(\mu)}|^2 = 1$ and d_μ is inside the region $|\xi_{(\mu)}| \leqslant 1$.

We now consider the space of holomorphic functions on S. They can be expanded in terms of a basis consisting of functions which have meromorphic continuations to $\widetilde{S}$. The basis can be fixed by prescribing the principal parts of these functions on d_μ. More precisely, we represent the holomorphic function F on S in the following form

$$F = \text{const} + \sum_{\mu=0}^{M} \sum_{n}^{+\infty} a_n^{(\mu)} f_n^{(\mu)}, \tag{1}$$

where the basis functions $f_n^{(\mu)}$ have the following behavior near Γ_ν:

$$f_n^{(\mu)} \approx \frac{\delta^{\mu\nu}}{\xi_{(\nu)}^n} + \sum_{m>0}^{\infty} A_{nm}^{(\mu\nu)} \xi_{(\nu)}^m. \tag{2}$$

The matrix $A = \{A_{mn}^{(\mu\nu)}\}$ (considered as a matrix consisting of $(M+1)^2$ infinite-dimensional blocks) will be the central object of our discussion.[1)] One can similarly describe the space of sections of any line bundle over S, e.g., the bundle of j-differentials. We also note that the space of functions $(j=0)$ is closed with respect to multiplication, i.e., the functions form a ring, whereas the sections of other line bundles are modules over that ring. This property of holomorphic functions appears to be quite essential. In particular, this is what distinguishes the matrices A corresponding to Riemann surfaces in the space of all possible matrices of that type. However, the details are outside the scope of the present article.

The matrix A satisfies the important relation

$$nA_{mn}^{(\mu\nu)} = mA_{nm}^{(\nu\mu)}, \tag{3}$$

which follows from the relation

$$\oint_{\partial S} f_m^{(\mu)} df_n^{(\nu)} = \sum_{\lambda=0}^{M} \oint_{\Gamma_\lambda} f_m^{(\mu)} df_n^{(\nu)} = 0.$$

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Indeed, $f_m^{(\mu)} d f_n^{(\nu)}$ is a holomorphic 1-differential on S, and the integral of any closed form over a null-homologous cycle vanishes.

We consider the contribution of scalars on the world-sheet to the string action. For this purpose we calculate the integral.

$$I_S\{\phi\} = \left\{ \int D\Phi\, D\bar{\Phi}\, \exp\|\Phi\|^2 \right\}^{-1} \times \int D\Phi\, D\bar{\Phi}\, \exp \int_S (\partial\Phi\, \overline{\partial\bar{\Phi}} + \bar{\partial}\Phi\, \partial\bar{\Phi}) \quad (4)$$

with respect to the anticommuting fields $\Phi, \bar{\Phi}$ with fixed boundary conditions on ∂S: $\Phi|_{\Gamma_\mu} = \phi_\mu$. We parametrize the boundary conditions in the following manner:

$$\phi_{(\mu)} = \text{const} + \sum_{n=1}^{\infty} [c_n^{(\mu)} \xi_{(\mu)}^{-n} + d_n^{(\mu)} \bar{\xi}_{(\mu)}^{-n}]\,|_{\Gamma_\mu} \quad (5)$$

(note that on $\Gamma_\mu \xi_{(\mu)} = \bar{\xi}_{(\mu)}^{-1}$).

There exist two important orthonormal bases which agree with the norm $\|\Phi\|$: the first consists of the eigenfunctions of the Laplacian, and the second consists of δ functions. In the second basis the integral I_D for the surface D split into its parts S_i can be represented as a product of the I_{S_i} integrated in a certain manner over the boundary conditions. Here is an important example of such a relation:

$$I_D = \left\{ \int D\phi\, D\bar{\phi}\, \exp\|\phi\|^2 \right\}^{-1} \int D\phi\, D\bar{\phi}\, I_S\{\phi\} I_{S^*}\{\phi\} \quad (6)$$

for the double $D = S + S^*$ of an open Riemann surface S (S^* is a second copy of S equipped with the opposite complex structure). The norm $\|\phi\|^2$ of the fields ϕ on the boundary ∂S is determined by a contour integral along ∂S, whereas $\|\Phi\|^2$ is expressed in terms of the integral over the surface S. (We also note that $I_{S^*} = \overline{I_S}$, and $I_D = \det'_D \Delta_0 / \det N_0$). The equation (6) expresses a naive unitarity property in the first quantized formalism. (It is clear that some effort should be made to relate this condition with the usual unitarity concept, in the spirit of second quantization.)

Since it is easier to compute the integral (6) in the orthonormal basis $\{\xi_{(\mu)}^{-n}, \bar{\xi}_{(\mu)}^{-n}\}$ for the boundary conditions on

$$\partial S = \sum_\mu \Gamma_\mu,$$

it is reasonable to express $I_S\{\phi\}$ in terms of the coefficients $\{c_n^{(\mu)}, d_n^{(\mu)}\}$. For this purpose one must find the (unique) solution of the classical equations of motion, i.e., the harmonic function Φ_{cl} on S, satisfying the boundary conditions

$$\Phi_{cl}|_{\Gamma_\mu} = \phi_{(\mu)}.$$

Then by means of a change of variables the integral (4) reduces to the integral $I_S\{0\}$ with respect to the fields

$$\Phi_{qu} = \Phi - \Phi_{cl},$$

which vanish on the boundary. We introduce a natural notation for this integral:

$$I_S\{0\} = \det{}_S \Delta_-.$$

It is clear that the integral equals the determinant of the Laplacian with zero boundary conditions on S and at the same time it coincides with product of the eigenvalues of the Laplacian on the double D, corresponding to the Z_2-antisymmetric eigenfunctions. We therefore have

$$I_S\{\phi\} = I_S\{0\} \exp S_{cl}\{\phi\} = \det{}_S \Delta_- \exp S_{cl}\{\phi\}. \quad (7)$$

Here

$$S_{cl}\{\phi\} = \int_S (|\partial \Phi_{cl}|^2 + |\bar{\partial} \Phi_{cl}|^2)$$

is the value of the action on the classical solution Φ_{cl}. Expressed in terms of the expansion (5) the classical action S_{cl} becomes a quadratic function of the Fourier coefficients

$$S_{cl}\{c, d\} = \overline{(c_m^{(\mu)}, d_m^{(\mu)})}\, \hat{M} \begin{pmatrix} c_n^{(\nu)} \\ d_n^{(\nu)} \end{pmatrix},$$

where the matrix $\hat{M}$ depends only on the Riemann surface S but not on the boundary conditions.

We now plan to express the matrix $\hat{M}$ in terms of the matrix A introduced in Eq. (2). Such a representation is quite natural, since any harmonic function $\Phi_{cl}(\xi, \bar{\xi})$ can be written locally as a sum of a holomorphic and an antiholomorphic function. However, these functions need not admit continuations to global single-valued functions. Therefore we have globally on S

$$\Phi_{cl}(\xi, \bar{\xi}) = F(\xi) + \overline{G(\xi)} + H(\xi, \bar{\xi}), \quad (8)$$

where F, G are single-valued holomorphic functions on S and H is a linear combination of integrals

$$\int^{\xi} \omega_N, \quad \overline{\int^{\xi} \omega_N}$$

(here ω_N is a basis of holomorphic 1-differentials on the double D), which is a single-valued function on S. The contribution of the functions H, which will be neglected in most of the cases below in order to simplify the formulas, is responsible, in particular for factors such as the volume of the Jacobian in the determinant formulas.[1,2] We reintroduce these contributions in the final answers, without entering into the technical details (cf., e.g., Ref. 2).

The globally defined holomorphic functions F, G in Eq. (8) can be written in the form of the expansions (1), (2) considered above:

$$F(\xi) = \sum_{\mu, m} a_m^{(\mu)} f_m^{(\mu)}(\xi), \quad G(\xi) = \sum_{\mu, m} \overline{b_m^{(\mu)} f_m^{(\mu)}}(\xi). \quad (9)$$

Making use of Eq. (2) we obtain the following formula for the action on the classical solution $\Phi_{cl} = F + \bar{G}$:

$$\begin{aligned} \int_S (|\partial \Phi_{cl}|^2 + |\bar{\partial} \Phi_{cl}|^2) &= i \sum_{\lambda=0}^{M} \left(\oint_{\Gamma_\lambda} \bar{F} \partial F + \oint_{\Gamma_\lambda} \overline{\partial G} G \right) \\ &= \sum_{\lambda} \sum_{\mu, \nu, m, n} \left\{ \overline{a_m^{(\mu)}} a_n^{(\nu)} \left[\delta^{\mu\nu} \delta_{mn} n - \sum_{\lambda, l} \overline{A_{ml}^{(\mu\lambda)}} l A_{nl}^{(\nu\lambda)} \right] \right. \\ &\quad \left. + \overline{b_m^{(\mu)}} b_n^{(\nu)} \left[\delta^{\mu\nu} \delta_{mn} n - \sum_{\lambda, l} A_{ml}^{(\mu\lambda)} l \bar{A}_{nl}^{(\nu\lambda)} \right] \right\} \\ &= \bar{a}^{tr} (i\partial_t - \bar{A} i\partial_t A^{tr})\, a + \bar{b}^{tr} (i\partial_t - A i\partial_t \bar{A}^{tr})\, b \\ &= \bar{a}^{tr} (1 - \bar{A} A)\, i\partial_t a + \bar{b}^{tr} (1 - A\bar{A})\, i\partial_t b\,. \end{aligned} \quad (10)$$

In the last equations we have used matrix notation). We have made use here of the relation (3) which in matrix form looks like

$$A\partial_t=\partial_t A^{tr}.$$

The superscript *tr* denotes matrix transposition and the matrix

$$\partial_t=-i\{m\delta^{\mu\nu}\delta_{mn}\}$$

can be interpreted as the operator of differentiation along the boundary.

In order to determine $S_{cl}\{c,d\}$ one must express the right-hand side of Eq. (10) in terms of $c_m^{(\mu)}, d_m^{(\mu)}$. The equations (2), (5), and (9) applied to $\phi_{(\mu)} = (F+\overline{G})|_{\Gamma_\mu}$ yield

$$c_m^{(\mu)} = a_m^{(\mu)} + \sum_{l,\lambda} b_l^{(\lambda)}\overline{A_{lm}^{(\lambda\mu)}}, \quad d_m^{(\mu)} = b_m^{(\mu)} + \sum_{l,\lambda} a_l^{(\lambda)} A_{lm}^{(\lambda,\mu)},$$

or, in matrix notation,

$$(c^{tr}, d^{tr}) = (a^{tr}, b^{tr})\begin{pmatrix} 1 & A \\ \bar{A} & 1 \end{pmatrix},$$

$$(a^{tr}, b^{tr}) = (c^{tr}, d^{tr})\begin{pmatrix} 1 & -A \\ -\bar{A} & 1 \end{pmatrix}\begin{pmatrix} (1-A\bar{A})^{-1} & 0 \\ 0 & (1-\bar{A}A)^{-1} \end{pmatrix}.$$

Substituting these expressions into Eq. (1) we obtain for the action

$$S_{cl}\{c,d\} = \overline{(c^{tr}, d^{tr})}\mathscr{S}i\partial_t\begin{pmatrix} c \\ d \end{pmatrix}, \tag{11a}$$

where

$$\mathscr{S}i\partial_t=$$

$$\begin{pmatrix} 1 & -\bar{A} \\ -A & 1 \end{pmatrix}\begin{pmatrix} (1-\bar{A}A)^{-1} & 0 \\ 0 & (1-A\bar{A})^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\bar{A} \\ -A & 1 \end{pmatrix} i\partial_t$$

$$=\begin{pmatrix} (1-\bar{A}A)^{-1}(1+\bar{A}A) & -2(1-\bar{A}A)^{-1}\bar{A} \\ -2(1-A\bar{A})^{-1}A & (1-A\bar{A})^{-1}(1+A\bar{A}) \end{pmatrix} i\partial_t. \tag{11b}$$

In the derivation of Eq. (11) we have made use of the relations

$$\partial_t A^{tr}=A\partial_t$$

and

$$(1-A\bar{A})^{-1}A=A(1-\bar{A}A)^{-1}.$$

The equations (7) and (11) form the basis of the discussion that follows.

We note that the functional

$$I_S\{\phi,\bar{\phi}\}=I_S\{c,d\}$$

may be considered as a generalized string propagator (generalized, since it corresponds to the correlator not of two, but of $M+1$ structure states on the boundaries $\Gamma_0,\ldots,\Gamma_M$ of the world sheet). It is important that we have defined it only for a limited class of metrics on S, since the curves Γ_μ are assumed to be geodesics.

3. SOME APPLICATIONS OF EQUATION (11)

The expression for the string propagator given by Eqs. (7) and (11) can be used for solving various problems in string theory. We discuss below some of these applications.

3.1. Open strings

The most obvious application, already considered in Refs. 1 and 2 is to the theory of open strings. In this case scalar fields on an open Riemann surface S with vanishing normal derivatives on the boundary are of interest. The corresponding determinant of the Laplacian, $\det'_S\Delta_+$ can be expressed in terms of the determinant $\det'_D\Delta$ on the double D and $I_S\{0,0\} = \det_S\Delta_-$. Specifically,

$$\det_D{}'\,\Delta/\det N_0=\det_S{}'\,\Delta_+/\det N_0 \det_S \Delta_-. \tag{12}$$

It is known that the left-hand side of this expression equals

$$\frac{\det_D{}'\,\Delta}{\det N_0}=(\det \operatorname{Im} T_D)\,|\det_D\bar{\partial}|^2\exp(S_{\mathscr{L}}). \tag{13}$$

Here T_D is the period matrix of the double and $S_{\mathscr{L}}$ is the Liouville action. The volume of the Jacobian

$$\det \operatorname{Im} T_D=V_+V_-$$

equals the product of its Z_2-symmetric and its Z_2 antisymmetric subtori. As was already mentioned, in this paper we shall not worry about factors like $V_\pm$, but will simply introduce them into the equations without dwelling on the details of the derivations (these can be found in Ref. 2). From Eqs. (6) and (7) we conclude

$$\det{}'\,\Delta_D/\det N_0=(\det_S\Delta_-)^2\int D\phi\, D\bar{\phi}\exp(2S_{cl}\{\phi,\bar{\phi}\}).$$

The integral in the right-hand side of this equation is easily computed by means of equation (11), and we obtain

$$\det{}'\,\Delta_D/\det N_0 \det \operatorname{Im} T_D=(\det_S\Delta_-/V_-)^2R^2, \tag{14}$$

where

$$R^2=\prod_\mu[\det_\mu{}'\,\partial_t/\det_\mu n_0]\sim 1. \tag{15}$$

As can be seen from Eq. (11b), $\det S=\infty$, R is a divergent constant which is unimportant for most applications, except, perhaps, the problem of relative normalization of surfaces of different topology. (For a given p_D R depends only on the number of components of the boundary.) We conclude from Eqs. (12) and (13) that

$$\det_S\Delta_-=R^{-1}V_-|\det_D\bar{\partial}|\exp({}^1\!/_2S_{\mathscr{L}}), \tag{16}$$

$$\frac{\det_S{}'\,\Delta_+}{\det N_0}=RV_+|\det_D\bar{\partial}|\exp({}^1\!/_2S_{\mathscr{L}}). \tag{17}$$

The measure on the modulus space, which is determined by the ghost determinant, is equal in the theory of open strings to

$$\begin{aligned}(\det_S{}'\Delta_{gh+}/\det N_{\overline{gh+}})^{1/2}&=(\det_D{}'\Delta_{gh}/\det N_{\overline{gh(D)}})^{1/4}\\&=|\det_D\bar{\partial}_2|\exp({}^{13}\!/_2S_{\mathscr{L}}),\end{aligned} \tag{18}$$

where $\bar{\partial}_2$ is the $\bar{\partial}$ operator acting on quadratic differentials. The equations (17) and (18) allow one to express the open string measure in dimension 26 in terms of the Mumford form

$$\left(\frac{\det_S{}'\Delta_+}{\det N_0}\right)^{-13}\left(\frac{\det_S{}'\Delta_{gh+}}{\det N_{\overline{gh+}}}\right)^{1/2}=\frac{|(\det_D\bar\partial)^{-13}\det_D\bar\partial_2|}{V_+^{13}},\tag{19}$$

where the measure $(\det_D\bar\partial)^{-13}\det_D\bar\partial_2$ is in fact real and the modulus sign may be omitted. The perturbation theory series for open strings can thus be written in terms of well-studied quantities from the theory of closed strings. Since in the final form the result (19) does not contain the matrix A, the equation (19) can also be derived by means of other methods[1,3] which make no use of Eq. (11). We now look at more interesting applications of the latter equation.

3.2. Determinants on Riemann surfaces in terms of the matrix A

In the preceding subsection we have discussed the relations between the determinants on open Riemann surfaces and those on their doubles. We now consider a similar relation between the determinants on an open surface S and on the closed surface $\tilde S$ obtained from S by gluing caps d_μ on all the components Γ_μ of the boundary $\partial S=\Gamma_0+\cdots+\Gamma_M$. In this case the relation analogous to Eq. (6) has the form

$$I_{\tilde S}=\frac{\det_{\tilde S}{}'\Delta}{\det N_0}=\left\{\int\prod_{\mu=0}^{M}|D\phi_{(\mu)}|^2\exp\|\phi_{(\mu)}\|^2\right\}^{-1}\times\int\left[\prod_{\mu=0}^{M}|D\phi_{(\mu)}|^2 I_{d_\mu}\{\phi_{(\mu)}\}\right]I_S\{\phi_{(0)}\dots\phi_{(\mu)}\}.\tag{20}$$

Here

$$I_S\{\phi\}=\det_S\Delta_-\exp S_{\rm cl}\{c,d\}$$

where $S_{\rm cl}$ is defined in (11), and $I_{d_\mu}\{\phi_{(\mu)}\}$ are similar functional integrals over the disks d_μ,

$$I_{d_\mu}\{\phi_{(\mu)}\}=I_{d_\mu}\{c^{(\mu)},d^{(\mu)}\}=\det_{d_\mu}\Delta_-\exp\sum_{n=1}^{\infty}n(|c_n^{(\mu)}|^2+|d_n^{(\mu)}|^2).$$

The determinant $\det_{d_\mu}\Delta_-$ on a disk with vanishing boundary conditions reduces to the exponential of the Liouville action. These determinants can be omitted if we agree to introduce some fixed metric on the disks, for example the metric of a hemisphere. The equations (20) and (11) yield

$$\frac{\det_{\tilde S}{}'\Delta}{\det_{\tilde S}N_0}=\frac{\det_S\Delta_-}{V_-}\det\{(\mathscr{S}+1)\,i\partial_t\}=R\,\frac{\det_S\Delta_-}{V_-\det(1-A\bar A)}.\tag{21}$$

Recalling that the determinant $\det_S\Delta_-$ can be expressed in terms of the determinant $\det'_D\Delta$ by means of Eqs. (14) and (15), we obtain from Eq. (21) a relation between the determinants on closed surfaces of different genera: the genus of $\tilde S$ is equal to p_S, whereas the genus of D is equal to $p_D=2p_S+M$.

The simplest example is the case $p_S=0$, $p_D=M$. In this case the surface $\tilde S$ is found to be a sphere and $\det'_{\tilde S}\Delta$ reduces to the Liouville factor. In this situation we obtain from Eqs. (16) and (21), neglecting the Liouville factors:

$$\det_D\bar\partial\propto\frac{\det_S\Delta_-}{V_-}\propto\det\,(1-A\bar A).\tag{22}$$

This relation determines $\det\bar\partial$ only on the real subspace (of real dimension $3M-3$) of the modulus space $\mathscr{M}_M$—the subspace of doubles. However $\det\bar\partial$ on the whole modulus space $\mathscr{M}_M$ can be uniquely reconstructed by means of analytic continuation.

As an example we carry out the computation using Eq. (22) for the simplest case $p_D=M=1$. In this case the surface S is a cylinder which can be defined as the region $r<|z|<1$ on the sphere $\tilde S$. (The double of this cylinder or annulus is a torus $D=T$, for which the usual parameter $\tau=it$ is related to r by means of $r=e^{\pi t}$. We recall that the tori, being doubles, have $\operatorname{Re}\tau=0$.) In accord with the rules from Sec. 2 we choose the coordinate $\xi_\infty=1/z$ on the disk $d_\infty=\{|z|\geqslant1\}$ and $\xi_{(0)}=z/r$ on the second disk $d_0=\{|z|\leqslant r\}$. The basis functions have the form

$$f_n^{(\infty)}(z)=\xi_{(\infty)}^{-n}=z^n=r^n\xi_{(0)}^{n},$$

$$f_n^{(0)}(z)=\xi_{(0)}^{-n}=(z/r)^{-n}=r^n\xi_{(\infty)}^{n}.$$

Consequently the matrix A equals

$$A_{nm}^{(0,\infty)}=A_{nm}^{(\infty,0)}=r^n\delta_{nm},\tag{23}$$

with the other elements vanishing. Therefore

$$\det(1-A\bar A)=\Big[\prod_{n=1}^{\infty}(1-r^{2n})\Big]^2.$$

In order to obtain an exact formula for $\det_T\bar\partial$ on the double T of our cylinder it is necessary to reinsert the omitted Liouville factor in Eq. (22). If the torus and the cylinder are equipped with the flat metric, the only contribution to this factor is related to $\det'_{\tilde S}/\det N_0$ in Eq. (21). In order to obtain the exact formula for $\det_T\bar\partial$ on the double T of our cylinder, one must reinstate the omitted Liouville factor in Eq. (22). If the torus and the cylinder are equipped with the flat metric, the only contribution to this factor is related to $\det'_{\tilde S}\Delta/\det N_0$ in Eq. (21). In reality Eq. (21) is valid for the following choice of metric on the sphere $\tilde S$. For $|z|\leqslant r$ and for $|z|\geqslant1$ this is the standard metric of the unit hemisphere, and in the region $r\leqslant|z|\leqslant1$ which corresponds to the cylinder under consideration, it is the flat metric $|d\ln z|^2$. (The metrics agree for $|z|=r$ and for $|z|=1$.) Now $\det'_{\tilde S}\Delta/\det N_0$ reduces to the Liouville factor $\exp S_{\mathscr{L}}$ corresponding to the interpolation between the metric we have just described and the standard metric of the unit sphere. The factor is easily computed and equals $S_{\mathscr{L}}=\frac{1}{16}\ln r$ so that we obtain, finally

$$\det_T\bar\partial=r^{1/6}\prod_{n=1}^{\infty}(1-r^{2n})^2=[\eta^{24}(r)]^{1/12}$$

in remarkable agreement with a direct calculation of this determinant. Here $\eta^{24}(r)$ is the Dedekind function. The formula holds for the whole modulus space $\mathscr{M}_1$ if one considers it to be analytically continued to complex values of $r=e^{i\pi\tau}$. This example shows that Eq. (22) may be considered as a generalization of the representation of the one-loop determinant in the form of an infinite product. If α_n are the eigenvalues of the matrix A and $t_n=\pi^{-1}\ln|\alpha_n|$, then

$$\det_D\bar\partial\sim\det(1-A\bar A)\sim\prod_n(1-|\alpha_n|^2)\sim\prod_{n,m}(m+it_n).$$

Note, however, that at its best the expression (22) for $\det\bar\partial$ in terms of A is in good agreement with the global behavior of

$\det\bar\partial$ on the whole modulus space only for specially chosen coordinates. (It is clear that the matrix A introduced in Eq. (2) depends on the choice of coordinates ξ_μ near Γ_μ.) As an illustration of possible complications we consider a cylinder embedded in a sphere as the region

$$\{|z|\leqslant 1,\ |z-Q|\geqslant r_Q\}\ \text{c}\ 0<r_Q<1-|Q|.$$

(The case considered above corresponds to $Q=0$.) Then

$$\xi_{(\infty)}=1/z,\quad \xi_{(Q)}=(z-Q)/z_Q$$

and

$$f_n^{(\infty)}=\xi_{(\infty)}^{-n}=z^n=(Q+r_Q\xi_{(Q)})^n=\sum_{m=0}^{n}C_n{}^mQ^{n-m}r_Q{}^m\xi_{(Q)}^m,$$

$$f_n^{(Q)}=\xi_{(Q)}^{-n}=\left(\frac{r_Q}{z-Q}\right)^n=\sum_{m=n}^{\infty}C_{m-1}^{n-1}Q^{m-n}r_Q{}^n\xi_{(\infty)}^m,$$

where the binomial coefficents have been denoted by $C_l^k=l!/k!(l-k)!$. The nonzero elements of the matrix A are

$$A_{nm}^{(\infty,Q)}=C_n{}^mQ^{n-m}r_Q{}^m,\qquad 0\leqslant m\leqslant n,$$

$$A_{nm}^{(Q,\infty)}=C_{m-1}^{n-1}Q^{m-n}r_Q{}^n,\qquad m\geqslant n. \tag{24}$$

(As before, it is easy to verify the relation $mA_{nm}^{\infty,Q}=nA_{mn}^{Q,\infty}$.) We now have

$$\det(1-A\bar A)=\prod_{n=1}^{\infty}(1-r_Q^{2n})^2\Big\{1-|Q|^2\Big[\frac{1\cdot 2r^2}{1-r^2}+\frac{2\cdot 3r^4}{1-r^4}+\frac{3\cdot 4r^6}{1-r^6}$$
$$\ldots+\frac{1\cdot 2r^2}{1-r^4}+\frac{2\cdot 3r^4}{1-r^6}+\frac{3\cdot 4r^6}{1-r^8}\cdot$$
$$+\ldots+\frac{1\cdot 2r^4(1+r^2)}{(1-r^2)(1-r^4)}$$
$$+\frac{2\cdot 3r^8(1+r^2)}{(1-r^4)(1-r^6)}+\frac{3\cdot 4r^{12}(1+r^2)}{(1-r^6)(1-r^8)}+\ldots\Big]$$
$$+o(|Q|^4)\Big\}=\prod_{n=1}^{\infty}(1-\tilde r^{2n})^2. \tag{25a}$$

Of course, it is clear from the outset that $\det(1-A\bar A)$ can be written as an infinite product

$$\det(1-A\bar A)=\prod_{n=1}^{\infty}(1-\tilde r^{2n})^2, \tag{25b}$$

where the relation between $\tilde r$ and r_Q can be defined in the following manner. One must make the change of variables $z\mapsto\tilde z$ on the sphere, transforming the circles $|z|=1$ and $|z-Q|=r_Q$ respectively into $|\tilde z|=1$ and $|\tilde z|=\tilde r$ for some $\tilde r$. The general transformation of the sphere which leaves the circle $|z|=1$ in place has the form

$$\tilde z=e^{i\alpha}\frac{z-P}{\bar Pz-1}, \tag{26a}$$

and the general transformation which maps the circle $|z-Q|=r_Q$ onto $|\tilde z|=\tilde r$ is

$$\tilde z=\tilde re^{i\beta}\frac{z-(Q+Sr_Q)}{\bar Sz-r_Q-\bar SQ} \tag{26b}$$

with some complex P and S. Equating (26a) and (26b) one can determine P, S and $\tilde r$ in terms of r_Q. More precisely, one obtains a quadratic equation for $\tilde r$:

$$\tilde r^2r_Q+\tilde r(|Q|^2-1-r_Q{}^2)+r_Q=0, \tag{26c}$$

from which it follows that

$$\tilde r^2=r_Q{}^2+|Q|^22r_Q{}^2/(1-r_Q{}^2)+O(|Q|^4) \tag{27}$$

in agreement with Eq. (25a).

3.3. The sewing of open Riemann surfaces

Until now we have considered only two examples where the sewing of open Riemann surfaces leads to a closed surface: (*i*) $D=S+S^*$ and (*ii*) $\tilde S=S+d_0+\cdots+d_M$. Of course, one can use the same method to analyze any sewing together of surfaces and one can derive relations among different determinants.

We note that the process of sewing together a closed Riemann surface out of one, two, or more open ones adds free parameters. Specifically, for each pair of identified components of the boundary, Γ, $\tilde\Gamma$ there appears one twist. In terms of the boundary conditions $\phi_{(\gamma)}(\xi)$ and $\phi_{(\tilde\Gamma)}(\tilde\xi)$ a twist means that one can set

$$\phi_{(\tilde\Gamma)}(\tilde\xi)=\phi_{(\Gamma)}(e^{-i\theta}\tilde\xi).$$

In terms of the Fourier components

$$\phi_{(\tilde\Gamma)}=\tilde c_n\tilde\xi^{-n}+\tilde d_n\bar{\tilde\xi}^{-n}$$

on $\tilde\Gamma$ and

$$\phi_{(\Gamma)}=c_n\xi^{-n}+d_n\bar\xi^{-n}$$

on Γ, we have

$$\tilde c_n=c_ne^{in\theta},\quad \tilde d_n=d_ne^{-in\theta}.$$

Below we shall use the notation

$$\phi_{(\tilde\Gamma)}=U_\theta\phi_{(\Gamma)}.$$

We give another illustrative example. We consider two different cylinders C_1 and C_2 parametrized by $r_1=e^{-\pi t_1}$ and $r_2=e^{-\pi t_2}$ sew them together into a torus T. Then

$$I_T=\int|D\phi_{(0)}|^2|D\phi_{(\infty)}|^2I_{C_1}\{\phi_{(0)},\phi_{(\infty)}\}I_{C_2}\{\phi_{(0)},U_\theta\phi_{(\infty)}\}.$$

We obtain the relation

$$I_T\propto\det{}_{C_1}\Delta_-\det{}_{C_2}\Delta_-\det(\mathscr{P}_{C_1}+U_\theta{}^+\mathscr{P}_{C_2}U_\theta).$$

The factors $\det_{C_i}\Delta_-$ in the right-hand side of this relation have already been considered in Section 3.2, so that we can use the result

$$\det{}_{C_i}\Delta\propto\prod_{n=1}^{\infty}(1-r_i{}^{2n})^2.$$

Making use of the matrixes A defined by Eq. (23) we find for the cylinders C_1 and C_2

$$\det(\mathscr{P}_{C_1}+U_\theta{}^+\mathscr{P}_{C_2}U_\theta)=\prod_{n=1}^{\infty}\det(\mathscr{P}_{C_1(n)}+U^+_{\theta(n)}\mathscr{P}_{C_2(n)}U_{\theta(n)})$$
$$=\prod_{n=1}^{\infty}\frac{|1-r_1{}^nr_2{}^ne^{in\theta}|^4}{(1-r_1{}^{2n})^2(1-r_2{}^{2n})^2}, \tag{28}$$

where S_{C_i} and U_θ are block-diagonal matrices consisting of 4×4 blocks corresponding to the variables $(c_n^{(0)}, c_n^{(\infty)}, d_n^{(0)}, d_n^{(\infty)})$. These blocks have the following form

$$
\mathscr{L}C_{i\,(n)}=\begin{pmatrix}
\dfrac{1+r_i^{2n}}{1-r_i^{2n}} & 0 & 0 & -\dfrac{2r_i^{\,n}}{1-r_i^{2n}}\\
0 & \dfrac{1+r_i^{2n}}{1-r_i^{2n}} & -\dfrac{2r_i^{\,n}}{1-r_i^{2n}} & 0\\
0 & -\dfrac{2r_i^{\,n}}{1-r_i^{2n}} & \dfrac{1+r_i^{2n}}{1-r_i^{2n}} & 0\\
-\dfrac{2r_i^{\,n}}{1-r_i^{2n}} & 0 & 0 & \dfrac{1+r_i^{2n}}{1-r_i^{2n}}
\end{pmatrix}, \tag{29}
$$

$$
U_{\theta(n)}=\begin{pmatrix}
1 & 0 & 0 & 0\\
0 & e^{in\theta} & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & e^{-in\theta}
\end{pmatrix}. \tag{30}
$$

We thus have

$$
I_T\sim\prod_{n=1}^{\infty}|1-r_1^{\,n}r_2^{\,n}e^{in\theta}|^4,
$$

in agreement with the well-known result for the determinant of the Laplace operator on a torus with the τ-parameter determined by the relation $e^{2\pi i\tau}=r_1r_2e^{i\theta}$. (In fact one must still restore the Liouville contribution, leading to the missing factor $|\exp 2\pi i\tau|^{1/6}$, see above, Sec. 3.2.)

This method of calculation can, of course, be used to sew any Riemann surface from elementary 3-string diagrams (the "pants" diagrams). In this way the contribution of an arbitrary Riemann surface can be represented as aa combination of fundamental 3-string correlators, integrated over the boundary conditions. The fundamental block will be an open surface S without handles ($p_S=0$) and with three boundaries ($M=2$), i.e., a sphere with three holes. It is easy to determine the corresponding matrix A. Let the holes on the sphere be obtained by throwing away the following three disks:

$$
d_0=\{|z|<r\},\quad d_Q=\{|z-Q|<r_Q\}\ \text{и}\ d_\infty=\{|z|>1\}.
$$

Then the matrix A is obtained as a direct generalization of Eqs. (23) and (24):

$$
\begin{aligned}
A_{nm}^{(\infty,0)}&=A_{nm}^{(0,\infty)}=r^n\delta_{nm}, && m,\ n\geqslant1,\\
A_{nm}^{(\infty,Q)}&=C_n^{\,m}Q^{n-m}r_Q^{\,m}, && m\leqslant n,\\
A_{nm}^{(Q,\infty)}&=C_{m-1}^{n-1}Q^{m-n}r_Q^{\,n}, && m\geqslant n,\\
A_{nm}^{(0,Q)}&=(-)^mC_{m+n-1}^{\,m}Q^{-(m+n)}r_Q^{\,m}r^n, && m,\ n\geqslant1,\\
A_{nm}^{(Q,0)}&=(-)^nC_{m+n-1}^{\,m}Q^{-(m+n)}r_Q^{\,n}r^m, && m,\ n\geqslant1
\end{aligned} \tag{31}
$$

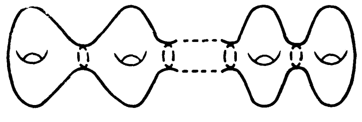

FIG. 1.

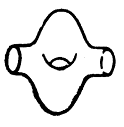

FIG. 2.

(all other matrix elements are equal to zero). Thus, the problem of calculating the characteristics of sewn Riemann surfaces can be reduced to the combinatorial problem of inversion with infinite-dimensional matrices and their determinants. Apparently it is not very simple to relate this representation to other representations which differ in the parametrization of the modulus space, e.g., in terms of the period matrices.[2)] We assume, however, that the representation we have described may turn out to be useful for various problems, independently of its relation to the more usual parametrizations.

The 3-string tree correlator is, of course, not the only possible building block. For instance, the following possibility seems appealing. Each closed Riemann surface of genus $p>1$ (Fig. 1) can be represented as a chain of $p-2$ one-loop propagators (Fig. 2) and two one-loop caps (Fig. 3) at the ends of the chain. We denote the functional integrals $I\{\phi\}$ for such caps and propagators respectively by $I_{\rm cap}\{\phi\}\equiv C$ and $I_{\rm prop}\{\phi\}\equiv P$. These are functionals depending on the modulus and the boundary conditions, and they can be expressed in terms of the corresponding det Δ_- and the matrices A. We retain the same notations C and P for analogous functionals which take into account the contributions of 13 commuting scalars and anticommuting ghosts. It is better to write them as $C\,d^3m$ and $P\,d^6m$, since these quantities are defined as measures on the modulus spaces of the caps and propagators. (Each of our caps has three real moduli, and each propagator has six, so that the total number of parameters $3\times2+6(p-2)=6p-6$ is exactly the dimension of the modulus space of closed surfaces of genus p. If we had used spherical caps in place of the one-loop caps, the number of parameters would have been incorrect and in the process of gluing we would have had to eliminate six redundant parameters. Hemisphere-caps may turn out to be convenient in the analysis of amplitudes for surfaces with distinguished points.)

The p-loop contribution to the cosmological constant is given by the integral

$$
Z_p=g^p\frac{1}{N_p}\int C\,d^3m*\underbrace{\int P\,d^6m*\ldots*\int P\,d^6m}_{p-2}*\int C\,d^3m. \tag{32}
$$

Here $*$ denotes the integration over the coinciding boundary conditions for adjacent propagators and/or caps

FIG. 3

and with respect to the twists. As is known from Ref. 2, C and P are exponentials of quadratic forms in the boundary conditions and therefore * is just a trivial Gaussian integration; in Eq. (32) g is just the string coupling constant and N_p is an (infinite) combinatorial factor equal to the number of times a combination of modulus spaces for two caps and $p-2$ propagators covers the modulus space of closed Riemann surfaces of genus p. We define

$$H \equiv \int C\, d^3m, \quad G \equiv \int P\, d^6m$$

which are functionals of the boundary conditions which do not depend on the moduli. They can be determined by means of the methods of the present paper. In fact it is rather complicated to carry out the calculation, but such a calculation is of manifest interest. Let us assume that H and G are known; then

$$Z_p = \frac{g^p}{N_p} H G^{p-2} H, \tag{33}$$

where the multiplication is to be understood in the sense of *. This representation is very good for a summation of the perturbation theory series. If, for instance, $g^p/N_p = \tilde{g}^p$ for some (renormalized?) constant $\tilde{g}$ (independent of p), we would obtain

$$\sum_p^\infty Z_p \sim H \frac{1}{1-\tilde{g}G} H.$$

This expression expands equally well for small $\tilde{g}$ (reproducing the series in terms of the genera) and for small $1/\tilde{g}$, leading this time to the strong coupling expansion.

It seems likely that with some efforts one can find all the quantities entering Eq. (33). Similar representations exist also for scattering amplitudes. In this case all external ends can only be connected to caps, without changing the propagators. Now there arises a covering of the modulus space with distinguished points (punctures) and the coefficients N_p can change. We are not in a position to discuss this more fully here. However, we think that it deserves to be investigated. It is not excluded that this construction has some advantages over the generally adopted approach to second quantization.

3.4 Grassmannians and the universal modulus space

Another method of dealing with the whole perturbation theory series, which is possible in the case of strings, is less well known to those active in ordinary field theory. The idea consists in expressing the determinants on Riemann surfaces of arbitrary genus in terms of a unique infinite-dimensional universal modulus space. One of the constructions of such a space is based on the so-called Krichever mapping. We consider a closed Riemann surface of genus p and remove a disk from it. Then we obtain an open surface S with $p_S = p$ and $M = 0$. The matrix A corresponding to S (after the introduction of a coordinate ξ near the boundary of the removed disk), can be considered as an element of an infinite-dimensional matrix space, or rather as a point of an infinite-dimensional Grassmannian (Grassmann manifold) G. In order to sum the perturbation theory series it is necessary, at a minimum, (*i*) to determine the subspace $\hat{G}$ in G consisting of all the matrices A which indeed correspond to some Riemann surfaces, and (*ii*) to relate the Mumford measures on the modulus subspaces of Reimann surfaces of finite genera with an appropriate measure $\hat{G}$. Until now it was only possible to express in terms of Grassmannians simple factors distinguishing the correlators of fields on Riemann surfaces from the determinants (Ref. 6). It was much more complicated to express the Mumford measures, i.e., the determinants themselves, in terms of the matrices A (see Ref. 7 for some simplifications arising in the super-case). In the preceding subsections we have succeeded in constructing such expressions, e.g., Eq. (22), but in terms of somewhat different matrices A. The A matrices used there define a mapping of the Krichever type for a sphere with several (versus one) removed disks. Utilizing Eq. (22) (and analytic continuation) one can write the determinant on a surface of genus p in terms of the A matrix corresponding to a sphere with $p+1$ puctures.[3)] These matrices can then be embedded in the Grassmannian G. We do not discuss this possibility in more detail. We only note that the structure of the determinant det $(1-A\bar{A})$ which appears in our calculations is very similar to the exponential of the usual Kähler potential ln det $(1+A\bar{A})$ on the standard Grassmannian. The geometrical meaning of det $(1-A\bar{A})$, particularly the consideration of the "minus" sign, may deserve clarification.

The collection of ideas described in this paper may find applications to various problems in ongoing string theory. Of course, for this it is necessary to solve certain technical complications, but in itself the representation of different objects in terms of explicitly known infinite-dimensional matrices seems to be useful for further work.

[1)]In the general case on a Riemann surface of genus p_S there exist only meromorphic functions for which the total order of the poles exceeds p_S. Therefore in Eqs. (1) and (2) n takes values in the interval from p_S+1 to $+\infty$, and the sum with respect to m in Eq. (2) must extend over all integers from $-p_S$ to $+\infty$. Moreover, if $M>0$, i.e., the number of boundaries exceeds one, the system of basis functions must be supplemented by functions with principal parts $\xi_{(\nu)}^{-k_\nu}$ with $\Sigma_\nu k_\nu > p_S$ and $k_\nu < p_S + 1$ for all ν. We will, however, omit all these technical details below. Equation (2) and several other relations involving the matrix A are literally true in the the case $P_S = 0$ for any M, but their qualitative structure is the same in the general case.

[2)]For this purpose a different method of cutting a surface into elementary blocks is more suitable. It corresponds to the usual second-quantized picture.[5] The blocks are cylinders and degenerate pants, the first corresponding to string propagators, the latter ones to 3-string vertices. This representation may be analyzed by means of the same method.

[3)]In this case an arbitrary closed surface of genus p can be transformed by means of cutting along a $p+1$-cycle into two pieces, each of which is a sphere with $p+1$ punctures. Instead, one could cut the surface along p nonintersecting noncontractible cycles. This yields one sphere with $2p$ punctures.

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