The Little–Hopfield model: recurrence relations for retrieval-pattern errors

V. A. Zagrebnov and A. S. Chvyrov

Joint Institute for Nuclear Research

(Submitted 18 June 1988)
Zh. Eksp. Teor. Fiz. **95**, 271–279 (January 1989)

Approximate recurrence relations for the overlap of retrieved and true patterns are derived for the case of parallel (synchronous) dynamics. The critical value of the neural-network saturation parameter $\alpha = M/N$ (M is the number of patterns and N the number of neurons), obtained from these relations, is $\alpha_c = 0.1398$ in the limit as $N \to \infty$. The result is compared with thermodynamic calculations and with a computer experiment for $N = 6000$ neurons.

1. It is known now that the complex structure of spin-glass states can be used to store and retrieve information. A spin model proposed in Refs. 1 and 2 operates as an associative memory system, i.e., is an analog of the brain's neural networks.

In this model, each neuron S_i, $i = 1, 2, ..., N$, is in two states: $S = \pm 1$ (excitation/inhibition states), the subscript i runs over a network of N neutrons. A neuron labeled i is acted upon by an electric field U_i whose potential is determined by a configuration of neurons coupled with i synaptic bonds V_{ij}:

$$U_i = \sum_{\substack{j=1\\(j\neq i)}}^{N} V_{ij} S_j. \qquad (1)$$

The synaptic bonds $\{V_{ij}\}$ can be excitations ($V_{ij} > 0$) or inhibitions ($V_{ij} < 0$). Therefore (at each given instant) the potentials $\{U_i\}_{i=1}^N$ are functions of the neuron configurations $S = \{S_i\}_{i=1}^N$ and of the complete set of synaptic bonds $V = \{V_{ij}\}_{i,j=1}^N$. It is known that a neuron S_i goes over into an excitation (inhibition) state if the potential U_i is higher (lower) than a certain threshold T_i i.e., the local-equilibrium conditions are of the form $(U_i \mp T_i^{\pm})S_i > 0$.

The Little–Hopfield model[2] corresponds to the assumption that $T_i = 0$ (no-threshold excitation), $V_{ij} = V_{ji}$ (symmetric synapses), and the evolution of the neutral network is determined by a relaxation dynamics for zero temperature:

$$S_i(t+1) = \begin{cases} \operatorname{sign} U_i(S(t)), & U_i \neq 0 \\ S_i(t), & U_i = 0 \end{cases}. \qquad (2)$$

Equation (2) corresponds then to a minimization of a Lyapunov function that takes in this case [see (1)] the form

$$H(S) = -\sum_{1\leqslant i<j\leqslant N} V_{ij} S_i S_j. \qquad (3)$$

The problem of remembering patterns [they are coded by N-bit "words" $\{\xi^{(p)}\}_{p=1}^M$ and correspond to fixed points of the $\{S_p^*\}_{p=1}^M$ dynamics (2)] reduces to solving the problem of organizing the synaptic bonds $V^{(M)}$ in such a way that the function (3) (the energy corresponding to a given configuration of neurons S) has not less than $2M$ global minima.

According to the Hebb–Little–Cooper hypothesis[1–3], the patterns entering the neutral network in the course of teaching modify the potentials of the synaptic bonds in accordance with the rule

$$V_{ij}^{(M)} = \frac{1}{N} \sum_{p=1}^{M} \xi_i^{(p)} \xi_j^{(p)}, \qquad i \neq j, \qquad (4)$$

and are random vectors with components $\{\xi_i^{(p)} = \pm 1\}_{i=1}^N$. This rule of teaching a neural network, together with the relaxation dynamics (2), actually solves the above problem of remembering an retrieving uncorrelated patterns $\{\xi^{(p)}\}_{p=1}^M$,

$$N^{-1} \sum_{1\leqslant i\leqslant N} \xi_i^{(p)} \xi_i^{(p')} \sim O(N^{-1/2}),$$

when their number M is not very large: $\alpha = M/N \lesssim 0.14$, and the temperature noise in the network does not exceed a definite level.[4,5]

Under these conditions the neural network (3) functions as an associative memory. The process of recalling the pattern $\xi^{(q)}$ reduces to the evolution of the initial configuration $S(t=0)$ [see (2)] that relaxes to the nearest (global) minimum of the Hamiltonian (3) corresponding to this pattern. With increase of α, the interference between patterns fed to the neutron network causes the "energy landscape" corresponding (3), (4) to become strongly cut-up: for $\alpha = 1$ it corresponds to the Sherrington–Kirpatrick model.[6] In particular, the global minima which are separated by the barriers (plateaus) of height $O(N)$ and correspond to patterns $\{\xi^{(p)}\}_{p=1}^M$ are preserved only at small values of α and have a strongly cut-up basin.[7] The attractor for $S(t)$ will therefore be not the vector $\xi^{(q)}$ corresponding to the global minimum closest to $S(t=0)$, but a certain vicinity $A(\xi^{(q)})$ of this vector, determined by the structure of the basin. Consequently the overlap of the pattern and the retrieval $S(t)$

$$m_N^{(q)}(t) = \frac{1}{N} \sum_{i=1}^{N} \xi_i^{(q)} S_i(t), \qquad (5)$$

does not necessarily converge as $t \to \infty$ to 1 (or to -1).[2)] The "sticking" of the vector $S(t)$ depends also on the details of the neural-network dynamics.

The dynamics (2) is usually further defined by two methods[6,8]: (a) by consecutive (asynchronous) dynamics, in which the index i in (2) scans from 1 to N and the flipping of the next spin is carried out with allowance for all the preceding flips; (b) by parallel (synchronous) dynamics, when all the spins flip (do not flip) simultaneously according to (2). If $S(t=0)$ is in the attraction region of the attractor $A(\xi^{(q)})$, the first dynamics (Monte Carlo for zero temperature $t = 0$) corresponds to a monotonic relaxation

$$H(S(t+1)) < H(S(t)). \qquad (6)$$

 0038-5646/89/010153-05$04.00

The local minima are stationary points for the asynchronous dynamics. The retrieval of $S(t)$ can therefore "stick" far enough from $\xi^{(q)}$.

Synchronous dynamics can violate (6) for individual instants of time, and ensures relaxation to $A(\xi^{(q)})$ only over long time intervals. This makes the "sticking" of the vector $S(t)$ at local minima far from $A(\xi^{(q)})$ less probable.

It is not clear which dynamics is realized in neural networks. Much attention was paid in Refs. 2, 4, 6, and 8 to successive dynamics. In recent papers[6,8] an attempt was made to analyze parallel dynamics for the Little–Hopfield model. The results there, however, are based on rather rough approximations and contradict greatly the numérical experiment carried out in the present paper and in Ref. 4.

Our present purpose is: a) derivation of recurrence relations for the evolution of the overlap (5) in the Little–Hopfield model, when the retrieval of $S(t)$ obeys parallel dynamics; b) compare the results with a numerical experiment for parallel dynamics in this model.

2. We begin the derivation of recurrence equations for the overlaps $\{m_N^{(q)}(t)\}$ with the following remark. A meaningful theory of neutral networks can be developed only in the limit $N\to\infty$, $M\to\infty$, $M/N=\alpha$ (Refs. 4, 5, 9). Therefore the parallel dynamics (2) can be redefined to have $S_i(t+1)=0$ for $U_i(S(t))=0$, since this does not affect the quantities

$$m_\alpha^{(q)}(t)=\lim_{N\to\infty} m_N^{(q)}(t),$$

on which we shall in fact focus our attention. We obtain then with the aid of (4)[2)]

$$m_N^{(q)}(t+1)=\frac{1}{N}\sum_{i=1}^{N}\operatorname{sign}[m_N^{(q)}(t)+\xi_i^{(q)}U_i^{(q)}(S(t))]. \tag{7}$$

Here $\{U_i^{(q)}(S(t))\}_{i=1}^N$ is a sequence of (random) potentials of the form

$$U_i^{(q)}(S(t))=\pm\frac{1}{N}\sum_{p\neq q}^{M}\sum_{j\neq i}^{N}\xi_i^{(p)}\xi_j^{(p)}\xi_j^{(q)}+\frac{1}{N}\sum_{p\neq q}^{M}\sum_{j\neq i}^{N}\xi_i^{(p)}\xi_j^{(p)}(S_j(t)\mp\xi_j^{(q)}). \tag{8}$$

The upper signs correspond here to the case when $S(t=0)$ is in the attraction region of the attractor $A(\xi^{(q)})$, i.e., $m_N^{(q)}(t)>0$, and the lower signs correspond to the case when the initial retrieval is in the attraction region of the "negative," i.e., $A(-\xi^{(q)})$, when $m_N^{(q)}(t)<0$.

Let us examine in greater detail the first term of (8). For each $p=1,2,\dots,M$ the pattern $\xi^{(p)}$ is a realization of a sequence of independent uniformly distributed random quantities

$$\xi(\omega)=\{\xi_i=\pm1\}_{i=1}^N,\qquad \Pr\{\xi_i=\pm1\}=1/2,$$

i.e., $\xi^{(p)}=\xi(\omega_p)$. Therefore, by virtue of the central limit theorem (see, e.g., Ref. 10), the sums

$$\frac{1}{N^{1/2}}\sum_{\substack{j=1\\(j\neq i)}}^{N}\xi_j^{(p)}\xi_j^{(q)}=\zeta_N^{(p)}$$

converge as $N\to\infty$ to a sequence of independent (q is fixed and $p\neq q$) Gaussian random quantities $\zeta^{(p)}$ with $M\zeta^{(p)}=0$ and $D\delta^{(p)}=1$. For the same reasons (with allowance for the independence of $\xi_i^{(p)}$ and $\zeta_N^{(p)}$) the sums

$$\frac{1}{M^{1/2}}\sum_{\substack{p=1\\(p\neq q)}}^{N}\xi_i^{(p)}\zeta_N^{(p)}=\delta_i^{(M)}$$

converge as $M\to\infty$ to sequences of Gaussian random quantities δ_i with $M\delta_i=0$ and $D\delta_i=1$. Therefore in the limit $N\to\infty$, $M\to\infty$, and $M/N=\alpha$ we have for the first term of (8)

$$\lim_{N\to\infty}\pm\frac{1}{N}\sum_{\substack{p=1\\(p\neq q)}}^{M}\xi_i^{(p)}\sum_{\substack{j=1\\(j\neq i)}}^{N}\xi_j^{(p)}\xi_j^{(q)}=\alpha^{1/2}\delta_i^{\pm}, \tag{9}$$

where $\{\delta_i\}_{i\geq1}$ is a sequence of equally distributed independent Gaussian random quantities with $M\delta_i^{\pm}=0$ and $D\delta_i^{\pm}=\sigma_2^2=1$.

Similar arguments are valid also for the second term of (8). It must only be noted that the sequence of $\eta_j^{\pm}=S_j\mp\xi_j^{(q)}$ takes on values $\{0,\pm2\}$, with $M\eta_j^{\pm}(t)=0$, and

$$D\eta_j^{\pm}(t)=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}(\eta_i^{\pm}(t))^2=2(1-|m_\alpha^{(q)}(t)|).$$

To calculate the variance we have used here the ergodicity of the sequence $\{\eta_j^{\pm}(t)\}_{j\geq1}$. Next, using the independence of $\xi^{(p)}$ and $\eta^{\pm}(t)$ and the central limit theorem, we obtain

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{p=1\\(p\neq q)}}^{M}\xi_i^{(p)}\sum_{\substack{j=1\\(j\neq i)}}^{N}\xi_j^{(p)}\eta_j^{\pm}(t)=\alpha^{1/2}\mu_i^{\pm}(t). \tag{10}$$

Here, $\{\mu_i^{\pm}(t)\}_{i\geq1}$ is a sequence of independent equally distributed Gaussian random quantities: $M\mu_i^{\pm}(t)=0$ and $D\mu_i^{\pm}(t)=\sigma_\mu^2=2(1-|m_\alpha^{(q)}(t)|)$.

Thus, in the limit indicated above, the sequence $\{U_i^{(q)}(S(t))\}_{i\geq1}$ reduces to a sum of the Gaussian random sequences (9) and (10), the correlation coefficient between which can be calculated in the manner used for $D\eta_j^{\pm}$:

$$r=(\sigma_\delta\sigma_\mu)^{-1}M\delta_i^{\pm}\mu_i^{\pm}(t)=\lim_{N\to\infty}(N\sigma_\delta\sigma_\mu)^{-1}\sum_{i=1}^{N}\delta_i^{\pm}\mu_i^{\pm}(t)\approx(\alpha\sigma_\delta\sigma_\mu)^{-1}(1-|m_\alpha^{(q)}(t)|)(1-\alpha). \tag{11}$$

Assume that the sequences $\{\xi_i^{(q)}\}_{i\geq1}$ and $\{U_i^{(q)}(S(t))\}_{i\geq1}$ can be regarded as independent. Then, by virtue of the Birkhoff-Khintchine ergodic theorem[10] and of (9) and (10), the recurrence relations (7) take in the limit $N\to\infty$, $M/N=\alpha$ the form

$$m^{(q)}(t+1)=\int dx\,\frac{1}{2}\{\delta(x-1)+\delta(x+1)\}\iint du\,dv\,p_{\delta\mu}(u,v)\times\operatorname{sign}[m^{(q)}(t)+x\alpha^{1/2}(u+v)]. \tag{12}$$

Here $p_{\delta\mu}(u,v)$ is the probability density of the distribution of the random potential $U_i^{(q)}$:

$$p_{\delta\mu}(u,v)=[2\pi\sigma_\delta\sigma_\mu(1-r^2)^{1/2}]^{-1}\times\exp\left\{-\left[\frac{u^2}{\sigma_\delta^2}-2r\frac{uv}{\sigma_\delta\sigma_\mu}+\frac{v^2}{\sigma_\mu^2}\right]\Big/2(1-r^2)\right\}. \tag{13}$$

For the recurrence relation that determines the overlap of the recollection and of the pattern in parallel dynamics we obtain therefore ultimately in the limit $N\to\infty$, $M/N=\alpha$

$$m_\alpha^{(q)}(t+1)=\mathscr{F}_\alpha[m_\alpha^{(q)}(t)],\quad q=1,2,\ldots. \tag{14}$$

Here

$$\mathscr{F}_\alpha[m]=\Phi\left(\frac{m}{[\alpha+2(1-|m|)]^{1/2}}\right),$$

$$\Phi(z)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^z dt\exp\left(-\frac{t^2}{2}\right).$$

The fact that the function $\mathscr{F}_\alpha[m]$ in the right-hand side of (14) is odd reflects the aforementioned global-minima degeneracy corresponding to the patterns $\{\xi^{(q)}\}_{q\geqslant1}$ and their "negatives" $\{-\xi^{(q)}\}_{q\geqslant1}$. We shall therefore consider below only the convergence of the recollections $S(t)$ to the patterns, i.e., $m_\alpha^{(q)}(t)\geqslant0$.

3. It follows from (14) that the quality of the memory depends essentially on the saturation parameter α of the neural network. At $\alpha<\alpha_c=0.1398$ the recurrence relation (14) has three fixed points, two stable ($\bar{m}_\alpha=0$, $m_\alpha^*>0$) and one ($\tilde{m}_\alpha$) unstable: $\bar{m}_\alpha<\tilde{m}_\alpha<m_\alpha^*$ (see Figs. 1 and 2). The points $\bar{m}_\alpha$ and m_α^* are attractors of the image (14), with $m_\alpha^*\leqslant1$ corresponding precisely to the attractor $A(\xi^{(q)})$ which degenerates into the vector $\xi^{(q)}$ only as $\alpha\to0$, when $m_\alpha^*\to1$.[3)] The point $\tilde{m}_\alpha$ is the separatrix of the pattern (14). The presence of separatrices means that for the recollection $S(t)$ to converge in the vicinity $A(\xi^{(q)})$ of the pattern $\xi^{(q)}$ it is necessary that the initial overlap $m^{(q)}(t=0)$ exceed the threshold $\tilde{m}_\alpha>0$. Note that the attraction region of the attractor $A(\xi^{(q)})$ is quite narrow, and the threshold does not vanish even as $\alpha\to0$: $\tilde{m}_{\alpha\to0}=0.08$, see Fig. 2. (The limit $\alpha\to0$ corresponds as before to an infinite number of patterns, only $N/N\to0$ as $N\to\infty$).

If the initial retrieval of $S(t=0)$ is such that $m^{(q)}(t=0)>\tilde{m}_\alpha$ and $m^{(p)}(t=0)\lesssim\tilde{m}_\alpha$ for $p\neq q$, we have

$$\lim_{t\to\infty} m_\alpha^{(p)}(t)=\begin{cases} m_\alpha^*, & p=q\\ 0, & p\neq q\end{cases}. \tag{15}$$

For $\alpha>\alpha_c$ the recurrent relations (14) have no nontrivial fixed points: m_α^* vanishes jumpwise, see Fig. 2. This means that the memory is overfilled with patterns and is a state of "chaos," i.e., it is incapable of reconstructing the pattern with a nonzero finite overlap, even if the initial retrieval of $S(t=0)$ is close to the original.

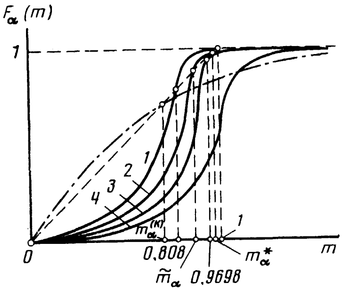

FIG. 1. Graphic representation of the recurrence relation (14) for different values of the parameter α: curve 1—$\alpha=0$; 2—$\alpha_1<\alpha_c=0.1398$; 3—$\alpha_c=0.1398$; 4—$\alpha_2>\alpha_c$. The points $\bar{m}_\alpha=0$ and m_α^* are attractors, the point $\tilde{m}_\alpha$ is a separatrix. The dash-dot line and $m_\alpha^{(K)}$ correspond to the recurrence relation (16) for $\alpha<\alpha_c^{(K)}$.

All these results differ substantially from those of Kinzel,[6] who obtained for the recurrence relation in parallel dynamics the expression

$$m_\alpha^{(q)}(t+1)=\Phi\left(\frac{m_\alpha^{(q)}(t)}{\alpha^{1/2}}\right). \tag{16}$$

It follows from (16) that for $\alpha<\alpha_c^{(K)}=2/\pi\approx0.6366$ there are only two fixed points; unstable $m_\alpha=0$ and stable $m^{(K)}$ for which $m^{(K)}_{\alpha\to\alpha_c^{(K)}}\to0$ continuously (see Fig. 2). For $\alpha\geqslant\alpha_c^{(K)}$, just as for (14), there is only one fixed point $\bar{m}_\alpha=0$.

At the same time, thermodynamic calculations independent of the dynamics[4,11] for $T=0$ yield $\alpha_c^{(A)}\approx0.138$ and a jump of $m_\alpha^{(A)}$ at this point: $m^{(A)}_{\alpha_c^{(A)}-0}=0.967$, $m^{(A)}_{\alpha_c^{(A)}+0}$ (cf. m_α^* in Fig. 2). The computer experiment described in Fig. 4 (but for sequential dynamics) confirms these conclusions. Starting with $N=3000$ it is clearly seen that a jumpwise decrease of m_α from $m_{\alpha=0,14}\approx0.972$ to $m_{\alpha=0,16}\approx0.35$ takes place between $\alpha=0.14$ and $\alpha=0.16$. Finite-dimensional scaling for $N=500$, 1000, 2000, and 3000 yields $\alpha_c^{(A)}\approx0.145$ (Ref. 4).

Kinzel[6] also gives results of computer experiments. They do not agree with the dynamics (16). For example, they indicate the presence of a threshold and yield for its value at $\alpha\approx0.075$ the estimate $\tilde{m}_\alpha\approx0.4$, which differs from our result $\tilde{m}_{\alpha=0}=0.808$. It must be noted, however, that Kinzel[6] considered also sequential dynamics[4)] and, more importantly, for a small system with $N\approx400$. We have therefore performed computer experiments with parallel dynamics, aimed primarily at verifying the recurrence relation (14).[5)]

4. Before we present the results of the computer experiment, let us compare the equation that follows for fixed points from the recurrence relation (14):

$$m=\Phi\left(\frac{m}{[\alpha+2(1-|m|)]^{1/2}}\right), \tag{17}$$

with the equation for the order parameter of the overlap of Mattis states ($m^{(p)}=m\delta_{p,q}$)

$$m=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\xi_i^{(q)}\langle S_i\rangle$$

[here $\langle\ \rangle$ stands for a thermodynamic mean value with the Hamiltonian (3) for zero temperature (see Refs. 4, 9, and 11)];

$$\begin{gathered} m=\Phi(m/(\alpha R)^{1/2}),\\ R^{1/2}=1+(2/\pi\alpha)^{1/2}\exp(-m^2/2\alpha R). \end{gathered} \tag{18}$$

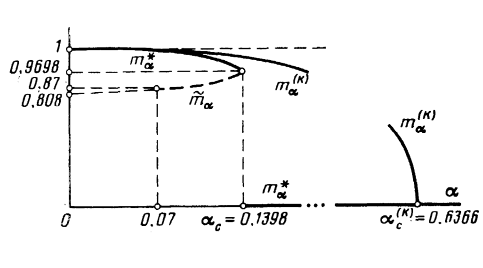

FIG. 2. Behavior of the fixed points of the reproduction (14) as functions of $m^*_{\alpha_c}=0.96978$, $\alpha_c=0.1398$. The same for the reproduction (16): $\alpha_c^{(K)}=0.6366$.

The solutions m_α of Eq. (17) and $m_\alpha^{(A)}$ of Eq. (18) are very close: $\alpha_c = 0.1398$, $\alpha_c^{(A)} = 0.138$, a $m^*_{\alpha_c} = 0.9698$, $m^{(A)}_{\alpha_c^{(A)}} = 0.967$. Therefore, using the smallness $\alpha(<\alpha_c)$ and the proximity of the nontrivial solution m_α to unity, we obtain from (18)

$$m_\alpha = \Phi\left(\frac{m_\alpha}{\{\alpha[1+2(2/\pi\alpha)^{1/2}\exp(-m_\alpha^2/2\alpha)]\}^{1/2}}\right). \quad (19)$$

On the other hand in this region of variation of the parameter α we have

$$1-|m_\alpha| \approx 1-\Phi(|m_\alpha|/\alpha^{1/2}) \approx (2\alpha/\pi)^{1/2}\exp(-m_\alpha^2/2\alpha). \quad (20)$$

From (19) and (20) we therefore obtain (17), and this explains the similarity of the solutions m^*_α and $m_\alpha^{(A)}$, including the branch of separatrices $\tilde{m}_\alpha$.

An experiment with the CDC-6500 computer (JINR, Dubna) was carried out for parallel dynamics and $N = 6000$. Such a large N was chosen to refine the dependence of the critical parameters on the system dimensions. It was noted in Ref. 4 that the singularities of the behavior of the overlap m_α, which are typical, for example, for a transition of memory from Mattis states, when the error in the pattern reproduction is small, into the spin-glass phase (chaotic behavior of the memory when the memory is overfilled), can be observed only starting with $N = 2000$.

On the whole, our experiment confirmed the conclusions of Ref. 4 (see Sec. 3), and clarified the status of Eqs. (14) and (17) as well as certain singularities of the dynamics:

a) It was confirmed that $0.14 \leqslant \alpha_c < 0.16$. For $\alpha = 0.14$ the recollection of $S(t=0)$ with $m(t=0) = 1$ converged already after 16 steps to an attractor which consisted in our case of two vectors with $m_\alpha^{(1)} = 0.9790$ and $m_\alpha^{(2)} = 0.9797$. Starting with $t = 16$ the vector $S(t)$ becomes a periodic function: $m_\alpha(t=16) = 0.9797$, $m_\alpha(t=17) = 0.9790$, $m_\alpha(t=18) = 0.9797$ etc. On going from the first vector of the attractor to the second, 8 spins flipped. A total of 26 iterations were performed, see Fig. 3.

b) For $\alpha = 0.16$ the overlap m_α, in contrast to Eqs. (17) and (18), is not zero. In our experiment $m_\alpha = 0.3457$. This value was reached after 198 iterations: $m_\alpha(t = 198\text{–}206) = 0.3457$. To elucidate the structure of the attractor we calculated the specific energy $E(t) = H(S(t))/N$. It was found that the attractor again consists of two vectors: $E(t=198) = -0.5843$, $E(t=199) = -0.58425$, $E(t=200) = -0.5843$, etc. and the transition is accompanied by 14 spin flips.

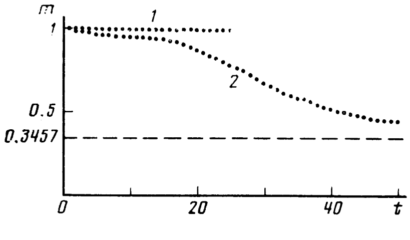

FIG. 3. Evolution of overlap $m_\alpha(t)$ [for $\alpha = 0.14$ (curve 1) and $\alpha = 0.16$ (curve 2)] under the initial condition $m_\alpha(t=0) = 1$.

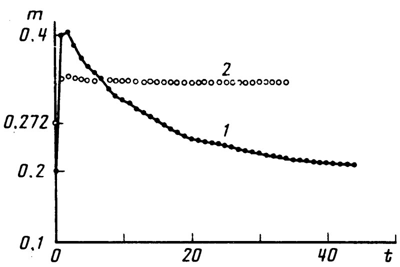

FIG. 4. Evolution of noise-perturbed vectors for $\alpha = 0.14$ (curve 1) and 0.16 (curve 2), when the initial overlap $m_\alpha(t=0)$ is lower than the $\tilde{m}_\alpha$ threshold.

c) An interesting singularity of the dynamics was observed after perturbing the vectors of the attractor for $\alpha = 0.14$ with the aid of noise (40%). This led to a decrease of the everlap of the noise-perturbed vector $S_n(t=0)$ with the pattern to $m_\alpha(t=0) = 0.1977$. The evolution of $m(t)$ for this case is shown in Fig. 4. It confirms the presence of a separatrix (threshold) with $\tilde{m}_{\alpha=0.14} \gtrsim 0.4$, but differs from the evolution predicted by Eq. (14). The main differences are the nonmonotonic $m_\alpha(t)$ dependences and the fact that $\lim_{t\to\infty} m_\alpha(t) > 0$. The vector $S_n(t)$ first tends to the original pattern (retrieval takes place), and only later (after $t = 2,3$) it begins to forget it, but not completely, with a nonzero limiting overlap $E(t=45) = -0.5935$.

d) A similar behavior was observed for the evolution, at $\alpha = 0.16$, of the vector $S_n(t)$ obtained from $S(t=206)$ by using a 10% noise perturbation, which decreased the overlap of the pattern to $m_\alpha(t=0) = 0.272$, see Fig. 4. The number of iterations was 44, and $m_\alpha(t=44) = 0.3297$, $E(t=44) = -0.5865$ [cf. item b)].

e) To check on the Gaussian character and on the independence of the random quantities that appear in the derivation of (14) (see Sec. 2), pertinent histograms were constructed for each constant t. If $\alpha = 0.14$, the histograms for $\{U_i^{(q)}(S(t=0\text{–}25))\}$ and $\{\xi_i^{(q)}U_i^{(q)}(S(t=0\text{–}25))\}$ are indeed close to Gaussian with parameters close to those predicted in Sec. 2, see Fig. 5. For $\alpha = 0.16$ they are close to Gaussian only during the first few steps, and then deviate from Gaussian substantially, see Fig. 6.

5. The approximate recurrence relation (14) describes well the evolution of the Little–Hopfield model only for $\alpha \leqslant \alpha_c$, and furthermore only in the vicinity of attractors. The deviation of the image (14) from the numerical experiment (especially for $\alpha > \alpha_c$, see the histogram of Fig. 6) is the consequence of neglecting in (14) correlation effects, above all between $\{\xi_i^{(q)}\}$ and $\{U_i^{(q)}\}$ (see Sec. 2 and the discussion in Refs. 6 and 8). Equation (17) for fixed points duplicates with high accuracy the solutions of the system (18) obtained from thermodynamic considerations for $T = 0$. The value $\alpha_c = 0.1398$ obtained from (17) is closer to the result of the computer experiment ($\alpha_c = 0.145$) than $\alpha_c^{(A)} = 0.138$ which follows from Eqs. (18). It is important that both the recurrence relation (14) and the numerical experiment of the present paper predict the presence of a threshold $\tilde{m}_\alpha > 0$ starting with which the retrieval $S(t)$ is "captured" by the attractor of the corresponding pattern (see Figs. 1 and 2). The value $\tilde{m}_{\alpha=0} = 0.808$ obtained from (14) and (17) is apparently too high. The computer experiment shows (see Fig. 4) that for $\alpha = 0.14$ the threshold is

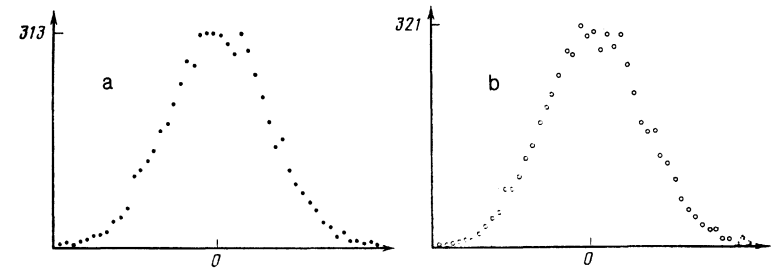

FIG. 5. a—Histogram of $\{U_i^{(q)}(S(t=25))\}$ for $\alpha=0.14$ ($M=0.0041$, $\sigma=0.4546$); b—histogram of $\{\xi_i^{(q)}U_i^{(q)}(S(t=25))\}$ for $\alpha=0.14$ ($M=0.0382$, $\sigma=0.4530$).

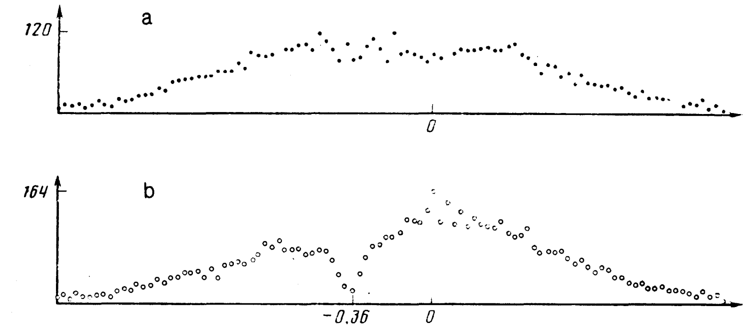

FIG. 6. a—Histogram of $\{U_i^{(q)}(S(t=206))\}$ for $\alpha=0.16$ ($M=-0.0155$, $\sigma=1.2970$); b—histogram of $\{\xi_i^{(q)}U_i^{(q)}(S(t=206))\}$ for $\alpha=0.16$ ($M=0.1754$, $\sigma=1.2852$).

$\tilde{m}_\alpha \gtrsim 0.4$. All these results differ from those of the dynamics (16) proposed by Kinzel: there is no jump of the parameter m_α^*, there is no threshold ($\tilde{m}_\alpha=0$), etc. Recall that the computer experiment of Ref. 6 for $N=400$ in sequential dynamics yields $\tilde{m}_{\alpha=0.075}\approx 0.4$.

Finally, for $\alpha=0.16>\alpha_c$ both Eqs. (17) and (18) yield $m_\alpha^*=0$, which differs from the numerical experiment. A zero value of $m_{\alpha>\alpha_c}$ would mean that when the memory is overfilled with patterns at $\alpha>\alpha_c$ the stable state is that of spin glass and not any Mattis state. A discussion of the physical meaning of this residual overlap can be found in Refs. 4–7. The presence of a residual overlap for $\alpha<\alpha_c$ for the evolution of the vector $S(t)$ with initial condition below the threshold $\tilde{m}_\alpha$ was first observed here. It coresponds to "sticking" of the vector $S(t)$ to a local minimum on a cut-up plateau of the "energy landscape" far from the attractors corresponding to the patterns stored in the memory. The latter constitute a system of global minima $\{A(\xi^{(q)})\}$ separated by plateaus (see Secs. 1 and 2).

[1)]Note that since the Hamiltonian (3), (4) is even its global minima are at least twofold degenerate. Therefore $S(t)$ can converge in the vicinity of the "negative" of the pattern, i.e., of the vector $-\xi^{(q)}$, if the initial overlap $m_N^{(q)}(t=0)<0$.

[2)]More accurately, Eq. (7) should contain in place of $m_N^{(q)}(t)$ the sum (5) without the ith term; this introduces an error $O(1/N)$.

[3)]This means that $A(\xi^{(q)})\to\xi^{(q)}$ as $\alpha\to 0$, only in the sense that $m_\alpha^*\to 1$: the vectors $\{\hat{\xi}^{(s)}\}$ for which $m^{(q)}(\xi^{(s)})=1$, will be assumed to be equivalent to $\xi^{(q)}$.

[4)]It is suggested in a recent paper[8] that the difference between the dynamics is not significant for the properties of the Little–Hopfield model.

[5)]It may be specially interesting to perform a computer experiment to compare parallel dynamics with the relaxation dynamics considered in a recent paper.[11]

[1]W. A. Little, Math. Biosci. **19**, 101 (1974); **39**, 281 (1974).
[2]J. J. Hopfield, Proc. Nat. Acad. Sci. USA, **79**, 2554 (1982); **81**, 3088 (1984).
[3]D. O. Hebb, *The Organization of Behavior*, Wiley, 1949.
[4]D. J. Amit, H. Gutfreund, and H. Sompolinsky, Phys. Rev. Lett. **55**, 1530 (1985). Ann. Phys. (NY) **173**, 30 (1987).
[5]L. B. Ioffe and M. V. Feigel'man, Europhys. Lett. **1**, 197 (1986).
[6]W. Kinzel, Z. Phys. **B60**, 205 (1985).
[7]Ch. M. Newman, Preprint, 1987.
[8]A. D. Bruce, E. J. Gardner, and D. J. Wallace, J. Phys. **A20**, 2909 (1987).
[9]J. L. Van Hemmen and V. A. Zagrebnov, J. Phys. **A20**, 3989 (1987).
[10]B. V. Gneden, *Course of Probability Theory* [in Russian], Nauka, 1988.
[11]M. V. Feĭgel'man and L. B. Ioffe, Int. of Mod. Phys. **B1**, 51 (1987).

Translated by J. G. Adashko