Theory of the quantum Hall effect in a two-dimensional periodic potential

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The complete solution of the lifting of the degeneracy of a Landau level by a two-dimensional periodic potential of general form is given in the limit of a strong magnetic field for rational values of the magnetic flux Φ ($\Phi/\Phi_0 = r/p$, where Φ_0 is the quantum of magnetic flux) passing through a rectangular unit cell of a two-dimensional crystal. The Landau level is split into r sub-bands, and the number p determines the multiplicity of the energy degeneracy in the sub-bands. It is shown that for $r \geqslant 2$ the wavefunction of an electron in the magnetic sub-band, as a function of the wave vector $\mathbf{k}$, has characteristic singularities—branch points of the phase of the wavefunction, and the quantized value of the Hall conductivity of the sub-band is determined by the number and type of such singularities lying in the (r/p)th part of the magnetic Brillouin zone of the crystal, this quantized value being always an integer multiple of the conductivity quantum $e^2/2\pi\hbar$. A formula is obtained that makes it possible to determine the quantized value of the Hall conductivity of a magnetic sub-band from the given Fourier components of the potential that lifts the degeneracy of the Landau level. The general results are illustrated for a simple model with $r = 2$. It is shown that when sub-bands come into contact the quantized value of the Hall conductivity of a sub-band changes discontinuously. The Hall mobility of the electron states in the sub-bands can be anomalously large, owing to the presence of small (in comparison with the magnitude of the cyclotron quantum) gaps between the sub-bands of the Landau levels.

1. INTRODUCTION

The motion of an electron in a crystal lattice in the presence of an external uniform magnetic field is one of the classic problems of solid-state physics.[1] The overwhelming majority of papers in this field pertain to the case of rational values of the magnetic flux passing through the projection of the unit cell on to the plane perpendicular to the magnetic field, when the magnetic translations form a group and the classification of the electron states that leads to the magnetic band structure can be performed relatively simply.[2] Although the existence of a magnetic band structure in the case of rational fluxes has been proved rigorously, the magnetic band structure has, in fact, been derived only in a number of more or less simplified models. Usual formulations are the tight-binding approximation,[3–9] when the amplitude U_0 of the lattice potential is large in comparison with the cyclotron quantum $\hbar\omega_c$ ($U_0 \gg \hbar\omega_c$), and the almost-free-electron approximation, when $U_0 \ll \hbar\omega_c$ (Refs. 6, 9–16). A number of results not connected with approximations of this kind have also been obtained. In papers by Zak[10] and Wannier and coworkers[9,11–16] it has been shown that the abstract problem of a two-dimensional electron in a strong magnetic field and the field of a weak two-dimensional periodic potential is of interest as a very simple (but quite rich in content) model permitting the direct construction of a magnetic band structure. At the present time the interest in this formulation of the problem has become especially keen in connection with the discovery of the quantum Hall effect (see the review Ref. 20) and the prediction, made in Refs. 21 and 22, of nontrivial quantization of the Hall conductivity of a two-dimensional electron gas in magnetic sub-bands that are filled and separated from each other by gaps. Such sub-bands arise from the Landau levels of a two-dimensional electron in a quantizing magnetic field B on account of a weak two-dimensional periodic potential $U(\mathbf{r})$, when the magnetic flux passing through a unit cell is rational:

$$\eta = \Phi/\Phi_0 = a_1 a_2/2\pi\lambda^2 = r/p, \tag{1.1}$$

where $\Phi_0 = 2\pi\lambda^2 B$ is the quantum of magnetic flux, $\lambda = (c\hbar/eB)^{1/2}$ is the magnetic length, and a_1 and a_2 are the primitive vectors of the two-dimensional crystal lattice, which is assumed to be rectangular.

Novikov and co-workers[23–26] (see also the review Ref. 18) were the first to study in detail the general topological properties of the two-dimensional Schrödinger operator in a uniform (or periodic) magnetic field and a periodic electric field in the case of rational values of η (1.1). They showed that the formation of r magnetic sub-bands at a Landau level must be interpreted as a vector bundle of magnetic Bloch functions over a torus T^2—the magnetic Brillouin zone (MBZ). This vector bundle is a direct sum of one-dimensional-complex bundles ξ_l ($l = 0, 1, \ldots, r-1$), with a single condition on the first Chern number $C_1(\xi_l)$ characterizing the bundle:

$$\sum_{l=0}^{r-1} C_1(\xi_l) = 1. \tag{1.2}$$

In addition, it was shown that for perturbations of general type the different sub-bands do not touch each other anywhere in the MBZ.

Kohmoto showed[27] that the first Chern number $C_1(\xi_l)$ of a sub-band l is the negative of the quantized Hall conductivity σ_{xy}^{qu} of the sub-band in units of $e^2/2\pi\hbar$.

In the present paper we pose the problem of calculating explicitly the quantized value of the Hall conductivity of a sub-band from the given perturbation potential satisfying the condition (1.1). It is shown that, in the weak-binding approximation, for calculations of this quantity it is sufficient to determine the wavefunction of the magnetic sub-band in lowest order of perturbation theory in the parameter $U_0/\hbar\omega_c \ll 1$. The existence of singularities—branch points of the phase of the sub-band wavefunction was considered as a

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function of the wave vector **k**—was discovered. It was proved that the presence of the singularities is necessary for quantization of the sub-band conductivity σ_{xy}^{qu}. In the absence of these singularities of the sub-band wavefunction, the Hall conductivity of the sub-band would (in lowest order of perturbation theory) be equal to the corresponding ideal value. It is entirely natural, therefore, that, as shown in the present paper, the quantized value of the Hall conductivity of a sub-band is determined by the number and type of these singularities, which are contained in a certain **k**-space region of fixed area.

In the final analysis, the type of the singularities and their arrangement in **k**-space are determined by the relative values of the Fourier components of the potential that lifts the degeneracy of the Landau level. Therefore, the quantized value of the Hall conductivity of a sub-band is determined not only by the numbers r and p from (1.1); it also, generally speaking, depends implicitly and nontrivially on the relative values of the indicated Fourier components.

Earlier, Kohmoto[27] noted that the existence of singularities of the wavefunction of the magnetic sub-band is necessary for quantization of σ_{xy}. However, he did not give explicit formulas to calculate this quantity. From an analysis of magnetic translations of the sub-band wavefunction, the authors of Ref. 28 obtained a formula for σ_{xy}^{qu}, in which, however, an unknown integer appears. In Sec. 3 of the present paper it is shown that this integer is determined precisely by the contribution of the singularities, and a method of calculating this contribution is given.

In the theory of the quantum Hall effect at the present time, the approach of Thouless *et al.*[21] is the most popular. It has been generalized to the case when impurities are present in the system,[29] and has also been applied to the description of the quantization of the Hall conductivity under the conditions of the fractional quantum Hall effect,[29–31] when the Coulomb electron-electron interaction is dominant in the system. This approach is also closely connected with the well known arguments of Laughlin,[32] based on gauge invariance. The existence, noted in the present paper, of singularities of the electron wavefunction as a function of the wave vector **k** in the MBZ leads in the present case to important changes of the specific calculations carried out in the scheme of Ref. 21. It is possible that this remark is also significant in a more general situation.[29–31]

Finally, in this problem it becomes possible to understand the nature of the anomalously large Hall mobility of the band states, which turns out to be connected with the presence of a small (in comparison with $\hbar\omega_c$) gap between the sub-bands of the Landau level. The properties of these states are interesting in connection with a possible analogy with the so-called delocalized states in the integer quantum Hall effect, which also possess an anomalously large mobility.[20]

2. PROPERTIES OF THE EIGENVECTORS OF THE MAGNETIC SUB-BANDS OF A LANDAU LEVEL

We shall consider the problem of a two-dimensional electron in a quantizing magnetic field and in the field of a weak two-dimensional periodic potential:

$$\hat{H}=\hat{H}_0+\hat{U}, \quad \hat{H}_0=\left(\mathbf{p}+\frac{|e|}{c}\mathbf{A}\right)^2/2m,$$

$$\mathbf{A}=[\mathbf{nr}]B/2, \quad U(\mathbf{r}+n_1\mathbf{a}_1+n_2\mathbf{a}_2)=U(\mathbf{r}), \quad n_i=0, \pm 1, \ldots \tag{2.1}$$

In the present paper all vectors are two-dimensional, with the exception of the vector **n** of the unit normal to the plane of motion of the electrons, which points along the external magnetic field. The primitive lattice vectors are orthogonal ($\mathbf{a}_1\cdot\mathbf{a}_2=0$); in other respects the potential $U(\mathbf{r})$ is arbitrary, and will be specified by its Fourier expansion in the reciprocal-lattice vectors of the crystal:

$$U(\mathbf{r})=\sum_{\mathbf{q}} U(\mathbf{q})\exp\{i\mathbf{qr}\}, \quad U^*(\mathbf{q})=U(-\mathbf{q}),$$
$$\mathbf{q}=n_x\mathbf{b}_1+n_y\mathbf{b}_2, \quad n_x, n_y=0, \pm 1, \ldots, \tag{2.2}$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the primitive vectors of the reciprocal lattice: $\mathbf{a}_i\cdot\mathbf{b}_j=2\pi\delta_{ij}$ $(i,j=1,2)$. The potential $U(\mathbf{r})$ is assumed to be weak:

$$|U(\mathbf{r})|\leqslant U_0\ll\hbar\omega_c, \tag{2.3}$$

so that the parameter $U_0/\hbar\omega_c$ is small and the problem (2.1) can be solved in perturbation theory (PT), while the quantum number N characterizing the levels of the unperturbed Hamiltonian $\hat{H}_0$ remains a good quantum number. With neglect of the perturbation $U(\mathbf{r})$ a Landau level is macroscopically degenerate, with multiplicity[33]

$$N_0=L_xL_y/2\pi\lambda^2, \tag{2.4}$$

where L_x and L_y are the dimensions of the fundamental region of periodicity of the two-dimensional crystal. The main problem of the theory is the removal of this degeneracy, i.e., the construction of the set of correct linear combinations of unperturbed states of the Landau level N that diagonalize the Hamiltonian (2.1) in the lowest approximation of PT. In this section we shall perform the formal diagonalization of the Landau level for an arbitrary rational $\eta=r/p$ [Eq. (1.1)]. We note that the complexity of specific calculations increases with r, since this number determines the order of the corresponding secular equation.

Because of the degeneracy of the states of the unperturbed Hamiltonian $\hat{H}_0$, a magnetic translation of some eigenfunction of $\hat{H}_0$ through an arbitrary vector R

$$\Psi(\mathbf{r})\to\Psi(\mathbf{r}+\mathbf{R})\exp\{i\mathbf{r}[\mathbf{nR}]/2\lambda^2\},$$

is also an eigenfunction of $\hat{H}_0$ (Ref. 2). Therefore, the periodic boundary conditions for the problem (2.1) have the form

$$\Psi(x+L_x, y)=\exp\{-iyL_x/2\lambda^2\}\Psi(x, y), \tag{2.5a}$$

$$\Psi(x, y+L_y)=\exp\{ixL_y/2\lambda^2\}\Psi(x, y). \tag{2.5b}$$

The physical meaning of the rationality condition (1.1) is that for rational values of η the magnetic "periodicity" [i.e., the condition that there is a quantized area $2\pi\lambda^2$ per quantum state in a Landau level; see (2.4)] is found to be commensurate with the periodicity of the two-dimensional crystal. To be precise, if in place of a unit cell of the crystal we take a so-called magnetic cell, whose primitive vectors we shall always choose in the form pa_1 and a_2, then in each magnetic cell, in accordance with (1.1), there will be exactly

r states of the Landau level. As is well known,[2] the choice of a magnetic cell is not unique, but this nonuniqueness manifests itself only in the fact that the magnetic Brillouin zone of a two-dimensional crystal can be chosen in different equivalent ways in the space of the reciprocal-lattice vectors. It is convenient to choose the dimensions of the fundamental region of periodicity of the two-dimensional crystal as follows:

$$L_x=pa_1M_x, \quad L_y=a_2M_y, \tag{2.6}$$

in order that an integer number M_xM_y of magnetic cells lie in the fundamental region. Then the total number of states contained in the fundamental region is equal to $N_0=rM_xM_y$.

We shall construct the following linear combinations from the states of the Nth Landau level[21]:

$$|t, N, k_x, X_0\rangle=M_x^{-1/2}\sum_{s=-\infty}^{\infty}\exp\{ik_xpa_1(s+t/r)\}(\lambda L_y)^{-1/2}$$
$$\times\exp\left\{i\frac{xy}{2\lambda^2}-i\frac{X_0+pa_1(s+t/r)}{\lambda^2}y\right\}\chi_N\left(\frac{x-X_0-pa_1(s+t/r)}{\lambda}\right), \tag{2.7}$$

where $\chi_N(x)$ is the normalized harmonic-oscillator eigenfunction of level N. The projection X_0 of the center of the cyclotron motion labels the initial unperturbed states of the Landau level. We shall restrict the variation of the quantum number X_0 in (2.7) to the range $0\leqslant X_0< pa_1/r$. From the boundary condition (2.5b) it follows that

$$X_0=2\pi\lambda^2l_y/L_y, \quad l_y=0, 1, \ldots, \bar{l}_y-1, \tag{2.8}$$

where the boundary value $\bar{l}_y$ is found from the condition $2\pi\lambda^2\bar{l}_y/L_y=pa_1/r$, which gives $\bar{l}_y=M_y$.

For fixed X_0 and t the wavefunction (2.7) is a linear combination of states lying at distances that are multiples of pa_1 from the given X_0 in the range (2.8). Performing the infinite summation over s in (2.7), we extend the wavefunction to the entire two-dimensional plane, this being convenient for application of the boundary conditions (2.5). At the same time, the scalar product of the wavefunctions (2.7), and all possible matrix elements constructed using these functions, will, as usual, be specified in the fundamental region of periodicity (2.6).

The quantum number k_x labels the superpositions (2.7) for fixed X_0 and t. The possible values of k_x are found from the boundary condition (2.5a), which is satisfied by the functions (2.7) with

$$\exp\{ik_xL_x\}=1, \quad k_x=2\pi l_x/L_x, \quad l_x=0, 1, \ldots, \bar{l}_x-1. \tag{2.9}$$

It follows from the form of the phase factor in (2.7) that, to within an unimportant constant phase, when k_x is varied we obtain, in all, $\bar{l}_x=L_x/pa_1=M_x$ different superpositions (2.7). Therefore, $k_x^{\max}=2\pi/pa_1=b_1/p$. This is the size of the MBZ along k_x. It depends on the number p, but not on r.

As follows from (2.7), the quantum number t is, in fact, defined only modulo r, and takes the values 0, 1,..., $r-1$. It is not difficult to check that the set of functions (2.7) is orthonormal:

$$\langle t, N, k_x, X_0|t', N', k_x', X_0'\rangle=\delta_{tt'}\delta_{NN'}\delta_{k_xk_x'}\delta_{X_0X_0'}. \tag{2.10}$$

It follows from (2.8) and (2.9) that the number of different pairs (k_x, X_0) is equal to M_xM_y, and since there are r different values of t the basis (2.7) constructed is complete.

The wavefunctions (2.7) are magnetic Bloch functions, but specified in the mixed representation (k_x, X_0). Below we shall use mainly the momentum representation (**k**-representation), the transformation to which is effected using the rule

$$X_0=-\lambda^2k_y, \quad k_y=2\pi l_y/L_y, \quad l_y=0, -1, \ldots, -M_y+1. \tag{2.11}$$

Our chosen standard MBZ $0\leqslant k_x\leqslant b_1/p$, $-b_2<k_y\leqslant 0$ contains exactly M_xM_y vectors **k**, i.e., the number of states in the MBZ is equal to the number of magnetic cells in the fundamental region of periodicity of the crystal. The area of the MBZ is smaller by a factor of p than the area of the Brillouin zone of the two-dimensional crystal in the absence of a magnetic field. Taking (2.11) into account, henceforth we shall denote the wavefunctions (2.7) by $|tN\mathbf{k}\rangle$. These wavefunctions are defined, of course, for all values of the wave vectors **k**, but, by virtue of the completeness of the basis (2.7), for values of **k** lying outside the limits of the chosen MBZ they should be linear combinations of the basis vectors. In fact, from (2.7) we have

$$|t, N, k_x+b_1/p, k_y\rangle=\exp\{i2\pi t/r\}|t, N, k_x, k_y\rangle, \tag{2.12a}$$

$$|t, N, k_x, k_y+b_2\rangle=\exp\{ik_xpa_1/r\}|t-1, N, k_x, k_y\rangle. \tag{2.12b}$$

In (2.12b) and everywhere below we must keep in mind the cyclic property of the number t: $|t+r, N, \mathbf{k}\rangle\equiv|t, N, \mathbf{k}\rangle$.

The significance of the introduction of the basis $|tN\mathbf{k}\rangle$ lies in the fact that in this basis the potential-energy operator U is diagonal in the quantum number **k**, as one can see by direct calculation, using the expansion (2.2) and calculating the matrix elements of the operator $\exp\{i\mathbf{q}\cdot\mathbf{r}\}$ in the basis $|tN\mathbf{k}\rangle$.

We turn now to the construction of the perturbation theory on an arbitrary Landau level N. Since the basis $|tN\mathbf{k}\rangle$ diagonalizes the operator $U(\mathbf{r})$ in the quantum number **k**, the PT can be constructed for each **k** in the MBZ independently. When the perturbation U is neglected the vectors $|tN\mathbf{k}\rangle$ ($t=0, 1,..., r-1$) are eigenfunctions of the Hamiltonian $\hat{H}_0$ that belong to the eigenvalue $\hbar\omega_c(N+\frac{1}{2})$ which is thus r-fold degenerate (for a fixed **k**). In the first approximation of PT we arrive at the secular equation

$$D(E)=\det\{\hat{\Delta}-E\hat{I}\}=0;$$
$$\hat{\Delta}=\begin{vmatrix} U_{00}, & U_{01}, & \ldots, & U_{0,r-1} \\ \ldots & \ldots & \ldots & \ldots \\ U_{r-1,0}, & U_{r-1,1}, & \ldots, & U_{r-1,r-1}\end{vmatrix}, \tag{2.13}$$

where $\hat{I}$ is the unit matrix. The dispersion equation (2.13) has r real solutions (magnetic sub-bands) $E_t(\mathbf{k})$ ($t=0, 1,..., r-1$), and the corresponding eigenvectors are linear combinations of the basis vectors (2.7):

$$|t, \mathbf{k}\rangle^{(1)}=\sum_l d_l[E_t(\mathbf{k})]|l, \mathbf{k}\rangle. \tag{2.14}$$

By virtue of (2.12a) the matrix elements of the operator $\hat{U}$ possess the following property:

$$U_{t_1t_2}(k_x+b_1/p, k_y)=U_{t_1t_2}(k_x, k_y)\exp\{i2\pi(t_2-t_1)/r\}. \tag{2.15}$$

From this it is not difficult to derive that $\hat{\Delta}(k_x + b_1/p, k_y) = \hat{\Delta}(k_x, k_y)$, and, consequently, the eigenvalues E_t are periodic in k_x with period equal to the size of the MBZ in this direction:

$$E_t(k_x+b_1/p,\ k_y)=E_t(k_x,\ k_y). \tag{2.16a}$$

We now note that for the basis vectors we can choose, instead of the vectors (2.7), the analogous linear combinations of functions belonging to the Nth Landau level, the oscillator factors of which depend on the coordinate y. In this way the initial symmetry of the problem is restored and it is verified that the energy eigenvalues possess period b_2/p in the k_y direction:

$$E_t(k_x,\ k_y+b_2/p)=E_t(k_x,\ k_y). \tag{2.16b}$$

By virtue of the fact that the size of the MBZ along k_y is equal to b_2, we see that the energy eigenvalues are p-fold degenerate in the MBZ. This fact is closely connected with the nonuniqueness of the choice of magnetic cell and, correspondingly, with the nonuniqueness of the shape of the MBZ in **k**-space.

We proceed to the analysis of the eigenvectors (2.14). It follows from basic properties of determinants[34] that the coefficients $d_l(E_t)$ in (2.14) for the eigenvector corresponding to the eigenvalue E_t can be chosen to be proportional to the cofactors D_{jl} of any (e.g., the jth) row of the secular determinant $D(E)$ for $E = E_t$:

$$d_l(E_t) \propto D_{jl}[E_t(\mathbf{k})], \quad l=0,\ 1,...,r-1. \tag{2.17}$$

We next consider the situation when the different magnetic sub-bands do not intersect, and, in particular, for each **k** the values of $E_t(\mathbf{k})$ for all the branches t are different. Consequently, to each t and each **k** from the MBZ there corresponds a single eigenvector of the problem. On the other hand, this eigenvector can be specified by the relations (2.17) for arbitrary $j = 0, 1,..., r-1$. It follows from this that under these conditions different representations (2.17) lead to one and the same eigenvector, i.e.,

$$\frac{D_{i0}(E_t)}{D_{j0}(E_t)}=\frac{D_{i1}(E_t)}{D_{j1}(E_t)}=\ldots=\frac{D_{i,r-1}(E_t)}{D_{j,r-1}(E_t)} \tag{2.18}$$

for arbitrary i and j. We note also that, by virtue of the hermiticity of the matrix $\hat{\Delta}$ in (2.13), we have $D^*_{ij}(E_t) = D_{ji}(E_t)$. Then from (2.18), in particular, it follows that

$$|D_{ij}(E_t)|^2=D_{ii}(E_t)D_{jj}(E_t). \tag{2.19}$$

Thus, for a given branch all the principal cofactors have a fixed sign throughout the MBZ.

For the normalization of the eigenvector specified by the relations (2.17), taking (2.19) into account we find

$$N_j^2(E_t)=D_{jj}(E_t)\sum_{l=0}^{r-1}D_{ll}(E_t). \tag{2.20}$$

As is well known, the sum of the principal cofactors is an invariant of the matrix under similarity transformations. It is easily calculated in the diagonal representation of the matrix Δ; namely,

$$\sum_{l=0}^{r-1}D_{ll}(E_t)=\prod_{l=0;\ l\neq t}^{r-1}[E_l(\mathbf{k})-E_t(\mathbf{k})]. \tag{2.21}$$

Since we have assumed that for each **k** in the MBZ all the eigenvalues $E_t(\mathbf{k})$ are different, the above sum is nonzero everywhere. On the other hand, as shown by the examples given below (see Sec. 4), the quantities $D_{jj}(E_t)$ themselves can vanish at certain points of the MBZ. By virtue of (2.19) this implies that at such a singular point all the cofactors D_{jl} of the jth row of the secular determinant also vanish simultaneously.[1)] In such a case this simply means that at a singular point rows $l \neq j$ of the secular determinant are linearly dependent. In this case the rank of the matrix $\Delta(E_t)$ remains equal to $r-1$, since, by virtue of (2.21), not all the $D_{ll}(E_t)$ can vanish simultaneously.

Suppose that, at a certain point $\mathbf{k} = \mathbf{k}_0$, we have $D_{jj}(k_0) = 0$ and, e.g., $D_{ll}(k_0) \neq 0$. We shall assume that D_{jl} near $\mathbf{k}_0$ has the form

$$\begin{aligned} D_{jl}'&=(\partial D_{jl}'/\partial k_x)(k_x-k_{0x})+(\partial D_{jl}'/\partial k_y)(k_y-k_{0y}),\\ D_{jl}''&=(\partial D_{jl}''/\partial k_x)(k_x-k_{0x})+(\partial D_{jl}''/\partial k_y)(k_y-k_{0y}), \end{aligned} \tag{2.22}$$

where we have set $D'_{jl} = \mathrm{Re}D_{jl}$ and $D''_{jl} = \mathrm{Im}D_{jl}$, and have assumed that at least one of the expansion coefficients in (2.22) is nonzero. We shall say that in such a case D_{jl} has at the point $\mathbf{k}_0$ a first-order zero. Then, by virtue of (2.19) $D_{jj}(E_t)$ has at this point a second-order zero. It follows from this that at a singular point the eigenvector component satisfies $d_j(E_t) = 0$. In this case it turns out that the components $d_l(E_t)$ with $l \neq j$ do not have a definite limit as $\mathbf{k} \to \mathbf{k}_0$. These facts are also obvious from the following formulas for the coefficients d_l, which follow from (2.19) and (2.20):

$$\begin{aligned} d_l^{(j)}&=\Big(D_{ll}\Big/\sum_s D_{ss}\Big)^{1/2}\frac{D_{jl}}{|D_{jl}|}, \quad l\neq j,\\ d_j^{(j)}&=\Big(D_{jj}\Big/\sum_s D_{ss}\Big)^{1/2} \end{aligned} \tag{2.23}$$

Here the superscript indicates the representation that has originated from the jth row of the determinant $\Delta(E_t)$. Thus, upon variation of **k** in the MBZ an eigenvector of the sub-band is rotated continuously with respect to the basis axes specified at each point in **k**-space. Singularities of the components of the normalized eigenvector (2.23) arise in those cases when, as a result of such a rotation, the eigenvector becomes orthogonal to one (or more) basis vectors. We stress that, for the subsequent applications in Sec. 3, we need to have a continuous and differentiable representation for the eigenvector. The representation (2.23) that we have constructed is differentiable throughout the MBZ except at the singular points indicated.

Of course, when the representation is changed the positions and number of the singular points in the MBZ change. In particular, it is sometimes possible to choose for the eigenvector a representation in which, for a certain branch $E_t(\mathbf{k})$, the corresponding minor satisfies $D_{jj}(E_t) \neq 0$ throughout the MBZ. In this case, however, it turns out that the given minor necessarily has zeros at other points of **k**-space, outside the MBZ. Moreover, the different representations for the eigenvector are closely related to each other. In fact, from (2.12b) it follows that

$$U_{t_1,t_2}(k_x,\ k_y+b_2)=U_{t_1-1,t_2-1}(k_x,\ k_y); \tag{2.24}$$

whence, taking into account the cyclic character of the in-

dices $t_{1,2}$ and the periodicity (2.16b) of the energy, we find the rule for transformation of the cofactors of the secular determinant (2.13) for a shift along k_y by the size of the MBZ:

$$D_{jl}(k_x,\ k_y+b_2)=D_{j-1,l-1}(k_x,\ k_y). \tag{2.25}$$

Therefore, apart from a relabeling of the basis vectors after such a shift, the representation $\{D_{jl}\}$ goes over into the representation $\{D_{j-1,l}\}$ $(l=0,\ 1,\ldots,\ r-1)$. Furthermore, if a minor $D_{jj}(E_t)$ has a zero at the point (k_x,k_y), the minor $D_{j-1,j-1}(E_t)$ necessarily vanishes at the point (k_x,k_y-b_2). This obviously gives a relationship between the singularities of different representations.

3. CALCULATION OF THE QUANTIZED VALUE OF THE HALL CONDUCTIVITY OF A MAGNETIC SUB-BAND

For the contribution of an individual filled magnetic sub-band of a Landau level to the Hall conductivity of a system the authors of Ref. 21 obtained the formula (see also Ref. 20)

$$\sigma_{xy}{}^{t}=i\frac{e^2}{S\hbar}\sum_{\mathbf{k}\in\mathrm{MBZ}}\left\{\left\langle\frac{\partial u_t}{\partial k_y}\middle|\frac{\partial u_t}{\partial k_x}\right\rangle-\text{c.c.}\right\}, \tag{3.1}$$

where S is the area of the region (2.6). The exact Bloch factors $u_{tN\mathbf{k}}=\exp\{-i\mathbf{k}\cdot\mathbf{r}\}\Psi_{tN\mathbf{k}}$ (where $\Psi_{tN\mathbf{k}}$ is the exact electron wavefunction in sub-band t of Landau level N) can be understood as the corresponding PT series in the small parameter $U_0/\hbar\omega_c$. It is obvious that when such a series is substituted into (3.1) the quantized value $\sigma_{xy}^{\mathrm{qu}}$ of the Hall conductivity should be obtained in zeroth order in this parameter. Therefore, it is sufficient to substitute into (3.1) the wavefunction (2.14):

$$u_t^{(0)}=\exp\{-i\mathbf{kr}\}\,|t,\mathbf{k}\rangle^{(1)}=\sum_{l=0}^{r-1}d_l(E_t)f_{l\mathbf{k}}(\mathbf{r}),$$
$$f_{l\mathbf{k}}(\mathbf{r})=\exp\{-i\mathbf{kr}\}\,|l,\mathbf{k}\rangle. \tag{3.2}$$

Substituting (3.2) into (3.1), we can transform the expression for $\sigma_{xy}^{\mathrm{qu}}$ into a form more convenient for calculations:

$$\sigma_{xy}^{\mathrm{qu}}=i\frac{e^2}{S\hbar}\sum_{\mathbf{k}}\left\{i\lambda^2+\sum_{l=0}^{r-1}\left[\frac{\partial d_l^{*}}{\partial k_y}\frac{\partial d_l}{\partial k_x}-\text{c.c.}\right]\right\}. \tag{3.3}$$

In deriving (3.3) we used the relations

$$\left\langle f_{l_1}\middle|\frac{\partial f_{l_2}}{\partial k_x}\right\rangle=i\lambda^2k_y\delta_{l_1l_2},\qquad\left\langle f_{l_1}\middle|\frac{\partial f_{l_2}}{\partial k_y}\right\rangle=0,$$
$$\left\langle\frac{\partial f_{l_1}}{\partial k_y}\middle|\frac{\partial f_{l_2}}{\partial k_x}\right\rangle=i\frac{\lambda^2}{2}\delta_{l_1l_2}$$

and took into account that the vector $\mathbf{d}$ (2.23) is normalized.

It is clear that the first term in the braces in (3.3) gives the ideal Hall current of the magnetic sub-band, since in the conditions (1.1) the number of points in the MBZ is equal to N_0/r:

$$\sigma_{xy,1}^{\mathrm{qu}}=-\frac{e^2}{2\pi\hbar}\frac{1}{r}. \tag{3.4}$$

Following Ref. 21, we introduce vectors $\mathbf{J}_l$ with components

$$J_l^{\alpha}=\frac{1}{2}\left\{d_l^{*}\frac{\partial d_l}{\partial k_\alpha}-\text{c.c.}\right\},\qquad\alpha=x,y, \tag{3.5}$$

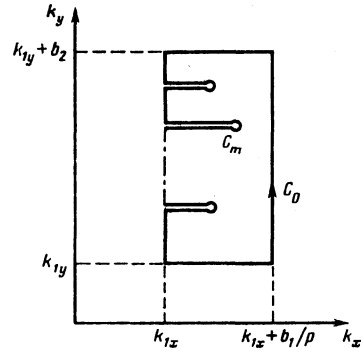

FIG. 1.

and, going over to integration over the MBZ, write the second term in the braces in (3.3) in the form

$$\sigma_{xy,2}^{\mathrm{qu}}=-i\frac{e^2}{(2\pi)^2\hbar}\int_{\mathbf{k}\in\mathrm{MBZ}}\sum_{l=0}^{r-1}\left[\frac{\partial J_l^{y}}{\partial k_x}-\frac{\partial J_l^{x}}{\partial k_y}\right]dk_x\,dk_y. \tag{3.6}$$

We note that, because the coefficients (2.23) may have singularities in the MBZ, the components of the vectors $\mathbf{J}_l$ do not have everywhere-continuous derivatives in the MBZ, and it is not admissible to use Green's theorem to transform (3.6) directly into a contour integral over the perimeter of the MBZ.[35] On the other hand, the singularities of the integrand in (3.6) are integrable and the given integral exists. To calculate it we proceed as follows. We take as the MBZ the rectangle $k_{1x}\leqslant k_x<k_{1x}+b_1/p$, $k_{1y}\leqslant k_y<k_{1y}+b_2$ (see the figure), and place it in such a way that all the singularities of the coefficients d_l lie inside it, at the points[2)] $\mathbf{k}_m$. We draw the contour C in the positive direction, enclosing all the singularities $\mathbf{k}_m$, as shown in the figure. The region lying inside the contour C is simply-connected and no longer contains singularities of the vector $\mathbf{d}$; therefore, for this region, Green's theorem is valid. The contribution to the integral (3.6) from the parts of the MBZ lying outside the contour C can, by virtue of the integrability of the singularities, be made negligibly small. Consequently, we have

$$\sigma_{xy,2}^{\mathrm{qu}}=-i\frac{e^2}{(2\pi)^2\hbar}\oint_C\sum_{l=0}^{r-1}\mathbf{J}_l\,d\mathbf{k}. \tag{3.7}$$

For the components of the vector $\mathbf{d}$ we use the representation (2.23) for a certain arbitrary j. We set

$$d_l^{(j)}=R_l^{(j)}\exp\{i\varphi_l^{(j)}\},\qquad R_l^{(j)}=\left(D_{ll}\Big/\sum_{s=0}^{r-1}D_{ss}\right)^{1/2}, \tag{3.8a}$$

and define the phase function $\varphi_l^{(j)}(k)$ by the relations

$$\cos\varphi_l^{(j)}=\frac{D_{jl}'}{|D_{jl}|},\qquad\sin\varphi_l^{(j)}=\frac{D_{jl}''}{|D_{jl}|},\qquad l\neq j. \tag{3.8b}$$

We note that the coefficient $d_l^{(j)}$ is real, so that the contribution of the term with $l=j$ to (3.6), (3.7) vanishes. In the region inside the contour C, for each l we can distinguish a unique continuous branch of the phase function, and, below, by $\varphi_l(k)$ we shall mean precisely such unique branches. Each of these functions is defined, of course, only to within $2\pi n$, but the corresponding vector $\mathbf{J}_l$ is defined uniquely, since

$$J_l^\alpha = iR_l^2 \partial\varphi_l/\partial k_\alpha, \quad \alpha = x, y. \tag{3.9}$$

We now consider the contour integrals (3.7). From the continuity of the quantities D_{jl} it is obvious that the values of the phase functions at points corresponding to each other on the upper and lower edges of the cuts in the figure can differ only by an integer multiple of 2π. Then it follows from (3.9) that the contributions to (3.7) from the upper and lower edges of the cuts (contributions taken in opposite directions) cancel each other, so that

$$\oint_c \mathbf{J}_l\, d\mathbf{k} = \oint_{c_0} \mathbf{J}_l\, d\mathbf{k} - \sum_m \oint_{c_m} \mathbf{J}_l\, d\mathbf{k}, \tag{3.10}$$

where the first term is the contour integral over the perimeter of the MBZ and the second term is the sum of the integrals over the infinitesimal contours surrounding the singularities of the chosen representation for the eigenvector. In Sec. 2 it was noted that the quantities $R_l(\mathbf{k}_m)$ exist, and, therefore, for the second term in (3.10) we have

$$\sum_m \oint_{c_m} \mathbf{J}_l\, d\mathbf{k} = i \sum_m R_l^2(\mathbf{k}_m) 2\pi S_l(\mathbf{k}_m), \tag{3.11}$$

where the integers $S_l(\mathbf{k}_m)$ are determined by the advance of the phase functions $\varphi_l^{(j)}$ in one circuit in the positive direction around the singular point $\mathbf{k}_m$. A simple argument shows that for a given singular point $\mathbf{k}_m$ the numbers $S_l(\mathbf{k}_m)$ ($l \neq j$, $l = 0, 1, \ldots, r-1$) that must be taken into account in (3.11) are equal. In fact, as indicated in Sec. 2 there exists a j_1 such that $D_{jj_1}(\mathbf{k}_m) \neq 0$. Consequently, at the point $\mathbf{k}_m$ we can always choose a nonsingular representation for the eigenvector by taking the coefficients d_l, at this point, to be proportional to the row j_1 of the determinant (2.13). By virtue of the continuity of the quantities $D_{j_1 l}$ the vector $\mathbf{d}$ at points on the infinitesimal contour C_m will be parallel to its value at the center of this contour. On the other hand, at points of the contour C_m the relations (2.18) are fulfilled, and, therefore, upon complete passage round the contour C_m, the quantities D_{jl} can be multiplied only by a common phase, so that the numbers S_l are the same for all l. (Below, we omit the subscript from these numbers.) Exceptions to this rule are possible only for those components D_{jl} for which $R_l(\mathbf{k}_m) = 0$. But such components certainly do not give a contribution to (3.11). Since the vector $\mathbf{d}$ is normalized, and, by definition, $R_j^{(j)}(\mathbf{k}_m) = 0$, we find from (3.11) that the total contribution of the singular points to (3.7) is

$$i\frac{e^2}{(2\pi)^2\hbar} \sum_{l=0}^{r-1} \sum_m \oint_{c_m} \mathbf{J}_l\, d\mathbf{k} = -\frac{e^2}{2\pi\hbar} \sum_{\mathbf{k}_m \in \Omega_0} S^{(j)}(\mathbf{k}_m). \tag{3.12}$$

The summation in (3.12) runs over all the singular points of the representation $\{D_{jl}\}$ that lie inside the MBZ (Ω_0 is the area of the MBZ). For the most frequently encountered case in which D_{jl} has a first-order zero at a point $\mathbf{k}_m$ [see (2.22)], the corresponding advance of the phase is given by the formula

$$S^{(j)}(\mathbf{k}_m) = \operatorname{sign} \begin{vmatrix} \partial D_{jl}'/\partial k_x, & \partial D_{jl}'/\partial k_y \\ \partial D_{jl}''/\partial k_x, & \partial D_{jl}''/\partial k_y \end{vmatrix}. \tag{3.13}$$

By virtue of what has been said above, to use this formula for calculation one must choose a quantity D_{jl} such that $D_{ll}(\mathbf{k}_m) \neq 0$. In the general case it can be shown that if the maximum order of the zeros of D_{jl}' and D_{jl}'' at the point $\mathbf{k}_m$ is equal to n, the absolute value of the advance of the phase at this point does not exceed $2\pi n$. From (3.12) we see that the contribution of the singular points to the integral (3.7) in units of $e^2/2\pi\hbar$ is quantized.

We now consider the contribution to (3.7) from the outer contour C_0. Expanding the determinant (2.13) on an arbitrary row j and taking into account the properties (2.15) of the matrix elements, we find the following transformation rule for the quantities D_{jl}:

$$D_{jl}(k_x + b_1/p,\ k_y) = D_{jl}(k_x,\ k_y) \exp\{i2\pi(j-l)/r\}. \tag{3.14}$$

It follows from this that the quantities R_l have period b_1/p in k_x, while the difference of the phases $\varphi_l^{(j)}$ between points corresponding to each other on vertical segments of the contour C_0 is constant:

$$R_l(k_x + b_1/p,\ k_y) = R_l(k_x,\ k_y), \tag{3.15a}$$

$$\varphi_l^{(j)}(k_{1x} + b_1/p, k_y) - \varphi_l^{(j)}(k_{1x}, k_y) = \frac{2\pi}{r}(j-l) + 2\pi n_l^{(j)}. \tag{3.15b}$$

It follows from (3.9) and (3.15) that the contributions to the integral (3.7) from the vertical segments of the contour C_0 cancel each other. Analogously, from (2.25) we find the rule for the transformation of the coefficients R_l under displacements by the size of the MBZ along k_y:

$$R_l(k_x,\ k_y + b_2) = R_{l-1}(k_x,\ k_y). \tag{3.16a}$$

Using (2.18), we can represent the transformation law (2.25) in the following form:

$$D_{jl}(k_x, k_y + b_2) = D_{j,l-1}(k_x, k_y) D_{j,j-1}^*(k_x, k_y)/D_{jj}(k_x, k_y). \tag{3.17}$$

Here we have used the hermiticity of the quantities D_{jl}. We note also that on the contour C_0 we must have $D_{jj} \neq 0$. From (3.17) there follows a relationship between values of the phase functions on the horizontal segments of the contour C_0:

$$\varphi_l^{(j)}(k_x, k_{1y} + b_2) = (1 - \delta_{l,j+1})\varphi_{l-1}^{(j)}(k_x, k_y) - \varphi_{j-1}^{(j)}(k_x, k_y) + \text{const}. \tag{3.16b}$$

In this relation it is understood that $\varphi_{l+r} \equiv \varphi_l$, and δ_{ij} is the Kronecker symbol. Using the formulas (3.16), we can derive the transformation rule for the sum of the components J_l^α ($\alpha = x, y$) of the vectors:

$$\sum_{l \neq j} J_l^\alpha(k_x, k_{1y} + b_2) = \sum_{l \neq j} J_l^\alpha(k_x, k_{1y}) - i\frac{\partial\varphi_{j-1}^{(j)}(k_x, k_{1y})}{\partial k_\alpha}. \tag{3.18}$$

From this it is now easy to show that the integral (3.7) over the horizontal segments of the contour C_0 reduces to

$$i\int_{k_{1x}}^{k_{1x}+b_1/p} \frac{\partial\varphi_{j-1}^{(j)}(k_x, k_{1y})}{\partial k_x} dk_x = i\left[\frac{2\pi}{r} + 2\pi n_{j-1}^{(j)}\right], \tag{3.19}$$

where in the right-hand side of this equality we have used the formula (3.15b). Thus, the integral (3.7) over the outer contour C_0 is equal to

$$-i\frac{e^2}{(2\pi)^2\hbar}\oint_{C_0}\sum_{l\neq j}\mathbf{J}_l\,d\mathbf{k}=\frac{e^2}{2\pi\hbar}\left(\frac{1}{r}+n_{j-1}^{(j)}\right). \tag{3.20}$$

We see that the first term in (3.20) exactly cancels the ideal contribution (3.4) to the Hall conductivity of the sub-band, while the remaining terms, i.e., the contribution of the singularities (3.12) and the second term in (3.20), are integer multiples of $e^2/2\pi\hbar$, i.e., lead to quantization of the Hall conductivity of the sub-band. However, for practical application these results are useless, since the determination of the number $n_{j-1}^{(j)}$ in (3.20) is a difficult problem. In addition, it is not clear whether the result obtained for σ_{xy}^{qu} is invariant under the choice of representation $\{D_{jl}\}$. We shall show how to circumvent these difficulties. We increase the size of the region of integration on the figure by a factor of r along k_y and consider the corresponding contour C', analogous to the contour C and enclosing all the singularities of the eigenvector that are contained in the expanded region of integration, of area $r\Omega_0$. It is clear that for the contribution of the singularities inside the contour C' we obtain precisely the formula (3.12), which should now take into account the contributions of all the singularities (of the eigenvector taken in a certain fixed representation) contained in the region of area $r\Omega_0$. The contribution from the outer contour C_0' to the corresponding integral now vanishes. In fact, the contribution from the vertical segments of this contour is equal to zero as a consequence of the relations (3.15) extended to the area $r\Omega_0$. The contribution from the horizontal segments of the contour C_0' vanishes because, by virtue of (2.25), upon displacement by rb_2 along k_y each given representation of the eigenvector is transformed into itself, having passed through each of the intermediate representations once, so that

$$R_l(k_x, k_y+rb_2)=R_l(k_x, k_y),$$

$$\varphi_l^{(j)}(k_x, k_y+rb_2)=\varphi_l^{(j)}(k_x, k_y)+\text{const}.$$

From this, incidentally, follows the fact that the value of σ_{xy}^{qu} calculated in this way is independent of the label j of the representation used. Finally, we note that it follows from (3.18) that the integrand of the original integral (3.6) is periodic in k_y with period b_2. As a result, we obtain the following formula for the expression of σ_{xy}^{qu}:

$$\sigma_{xy}^{qu}=-\frac{e^2}{2\pi\hbar}\left[\frac{1}{r}+\frac{1}{r}\sum_{\mathbf{k}_m\in r\Omega_0}S(\mathbf{k}_m)\right]. \tag{3.21}$$

The result (3.21) can be strengthened. It follows from (2.15) that the period in k_x of the matrix elements $U_{t_1t_2}$ is the number rb_1/p. Analogously, the period in k_y of these elements is the number rb_2/p:

$$U_{t_1t_2}(k_x,\ k_y+rb_2/p)=U_{t_1t_2}(k_x,\ k_y). \tag{3.22}$$

In fact, a shift in k_y by rb_2/p corresponds to a shift of the quantum number X_0 in (2.7) by an amount $-\lambda^2 rb_2/p=-a_1$ equal to the period of the potential U in the coordinate x. Such a transformation of the wavefunctions does not affect the magnitude of the matrix elements of the potential. From (3.22) and (2.16b) it then follows that the true period of the quantities D_{jl} in k_y is not the number rb_2 but the number rb_2/p. In this case the expanded region of integration, of length rb_2 along k_y, contains p periods of any representation, and, instead of (3.21), we obtain, finally,

$$\sigma_{xy}^{qu}=-\frac{e^2}{2\pi\hbar}\left[\frac{1}{r}+\frac{p}{r}\sum_{\mathbf{k}_m\in r\Omega_0/p}S(\mathbf{k}_m)\right]. \tag{3.23}$$

The summation in the second term of this formula is over the singularities of any of the representations lying in the (r/p)th part of the area of the MBZ.

First of all, we see that the (r/p)th part of the MBZ should contain at least one of the singularities of any of the representations of the eigenvector. Otherwise, the Hall conductivity of the sub-band would be given in the lowest approximation of PT by its ideal value (3.4). As can be seen from (3.13), the contribution of the singularities to σ_{xy}^{qu}, as a rule, depends implicitly and nontrivially on the relative values of the bare Fourier components of the potential $U(\mathbf{r})$. Above, we have shown that the entire expression in brackets in (3.23) is an integer. It is known from the work of Kohmoto[27] that this integer is the first Chern number for the corresponding sub-band. Then the result (1.2) of Novikov implies that the sum of the contributions from the singularities, summed itself over all sub-bands of the Landau level, is equal to zero. In other words, the Hall conductivity of a completely filled Landau level is equal to $-e^2/2\pi\hbar$. This fact is well known in the theory of the ordinary integer quantum Hall effect.[20]

From (3.22) it also follows that the period in k_y of the integrand in (3.6) is the number rb_2/p. But earlier we showed that it has period b_2 in k_y. Since the numbers r and p are relatively prime, the true period in k_y of this integrand will be the number b_2/p. This implies that we can write an expression of the type (3.23) in which the singularities of all r representations in the $(1/p)$th part of the MBZ would be taken into account.

We now give a prescription for calculating σ_{xy}^{qu} by Eq. (3.23). Having chosen a certain representation $\{D_{jl}\}$ of the eigenvector (2.23) it is necessary to determine all its singular points lying inside the (r/p)th part of an MBZ arbitrarily chosen in k-space. These singular points $\mathbf{k}_m$ can be sought as the simultaneous solution of the equations $D_{jl}(\mathbf{k})=0$ for $l=0, 1,..., r-1$ and fixed j. It is then necessary to determine, for each singular point $\mathbf{k}_m$, the numbers $(S\mathbf{k}_m,)$ which are proportional to the advance of the phase at this singular point for each of those quantities D_{jl} for which $D_{ll}(\mathbf{k}_m)\neq 0$, and substitute the result into the formula (3.23). The formula (3.13) given above for the calculation of the advance of the phase at singularities of the simplest type (2.22) is applicable only in the case when the determinant in this formula is nonzero at the singular point; otherwise, it is necessary to take into account the next terms of the expansion (2.22).

We shall make a further important comment concerning the PT that we are using. From the outset, we assumed the presence of gaps between all sub-bands of the Landau level, and also used the smallness of the parameter $U_0/\hbar\omega_c$. Let us suppose that all independent Fourier components of the potential (2.2) are proportional to the same amplitude U_0, so that the ratio of different Fourier components to each other remains fixed. Then the matrix elements of the potential in (2.13) and, consequently, the energy eigenvalues (2.16) will be proportional to U_0. Upon variation of U_0 within certain limits that preserve the smallness of the parameter $U_0/\hbar\omega_c$ the band structure will be determined by the lowest approximation of PT and will be expanded or compressed

along the energy axis essentially in proportion to U_0. If for certain relative values of the components $U(\mathbf{q})$ there existed gaps between all sub-bands of the Landau level, it is clear that when U_0 varies within the indicated limits none of these gaps can close. Furthermore, the quantities D_{jl} contain U_0 only in the factor U_0^{r-1}, and, therefore, upon variation of U_0 neither the type nor the number of the singular points of these quantities in any region of **k**-space can change. Consequently, neither can the quantized value (3.23), of the Hall conductivity of the subband change upon variation of U_0. On the other hand, the terms of the next orders of PT, which were discarded in the expansion of the wavefunction (3.2), are proportional to nonzero powers of the parameter $U_0/\hbar\omega_c$, and are certainly not universal constants. Thus, if the sub-band conductivity is quantized, all these terms must necessarily vanish.

These arguments become inapplicable if, at any point of the MBZ, two neighboring sub-bands of the Landau level come into contact. First of all, the eigenvector (2.14) acquires new singularities, associated with the vanishing, at a certain point, of the sum (2.21) of the principal cofactors. Still more important is the fact that at such a point, strictly speaking, the PT is inapplicable, since even in the second approximation of PT in the expression for the wavefunction of a degenerate level small denominators appear containing differences of the eigenvalues (2.16) (Ref. 33).

From the physical point of view the cases of touching sub-bands are especially interesting because, as shown in Ref. 25, they change the "regime" of the quantization in the sub-band, i.e., discontinuously change the value of σ_{xy}^{qu}. In Sec. 4 we shall analyze a simple example of such behavior of σ_{xy}^{qu}.

4. A SIMPLE EXAMPLE

We shall consider the case of two magnetic sub-bands of a Landau level, when $r=2$, p is an odd number, and in (2.2) we retain, for simplicity, only three independent Fourier components:

$$U(\mathbf{r}) = U_1^* \cos \mathbf{r}\mathbf{b}_1 + U_2 \cos \mathbf{r}\mathbf{b}_2 + U_3 \cos \mathbf{r}(\mathbf{b}_1+\mathbf{b}_2). \qquad (4.1)$$

Calculating the matrix elements of this operator in the corresponding basis (2.7), we arrive at the secular determinant

$$D(E) = \begin{vmatrix} \overline{U}_1 \cos y - E, & \overline{U}_2 \cos x + i(-1)^{(p-1)/2}\overline{U}_3 \cos(x-y) \\ \overline{U}_2 \cos x - i(-1)^{(p-1)/2}\overline{U}_3 \cos(x-y), & -\overline{U}_1 \cos y - E \end{vmatrix} = 0, \qquad (4.2)$$

where $\overline{U}_i$ are the renormalized Fourier components, proportional to the bare values of U_i; $x = \pi p k_x/b_1$ and $y = \pi p k_y/b_2$. In the variables (x,y) the MBZ is specified by the conditions $\Delta x = \pi$ and $\Delta y = \pi p$. In accordance with (2.15) and (3.22) the matrix elements in (4.2) have period 2π in both variables, and the energy eigenvalues are periodic in x and y with period π:

$$E_\pm = \pm\{\overline{U}_1^2 \cos^2 y + \overline{U}_2^2 \cos^2 x + \overline{U}_3^2 \cos^2(x-y)\}^{1/2}, \qquad (4.3)$$

so that they are p-fold degenerate in the MBZ. If $\overline{U}_i \neq 0$, then between the sub-bands there is a gap of width equal in order of magnitude to the width of the sub-bands (4.3). For $\overline{U}_1 = 0$ the gap closes at the points $(\pi/2 + \pi n, \pi m)$, for $\overline{U}_2$ it closes at the points $(\pi n, \pi/2 + \pi m)$, and for $\overline{U}_3 = 0$ it closes at the points $(\pi/2 + \pi n, \pi/2 + \pi m)$. We note that the simultaneous vanishing of any two quantities $\overline{U}_i$ would contradict the condition (1.1) expressing the conservation of the magnetic flux through a cell.

In accordance with the prescription of Sec. 3, we shall find the singularities of the eigenvector representation given, e.g., by the cofactors of the first row of (4.2) in the rectangle $0 \leqslant x < \pi$, $0 \leqslant y < 2\pi$. The cofactor D_{01} in this rectangle vanishes only at the points $(\pi/2, 0)$ and $(\pi/2, \pi)$. If $\overline{U}_1 > 0$, the first point corresponds to a singularity of this representation lying in the lower zone, while the second point leads to a singularity in the upper zone. For $\overline{U}_1 < 0$ the singular points exchange positions. Taking this circumstance into account, with the aid of (3.13) and (3.23) we obtain for the quantized values of the Hall conductivities of the sub-bands

$$\sigma_{xy,\pm}^{qu} = -\frac{e^2}{2\pi\hbar}\left[\frac{1}{2} \mp \frac{p}{2}(-1)^{(p-i)/2} \operatorname{sign}(\overline{U}_1\overline{U}_2\overline{U}_3)\right]. \qquad (4.4)$$

Thus, the sum of the conductivities of the sub-bands is equal to $-e^2/2\pi\hbar$, and a change of the quantization "regime" occurs only when any of the quantities $\overline{U}_i$ passes through its zero value. As we have seen, at this point the gap between the sub-bands closes.

It follows from (4.4) that for large values of p the absolute value of the second term in the brackets can be many times greater than the first term, corresponding to the ideal Hall conductivity of the sub-band. Thus, the states in the sub-bands, like the familiar delocalized states in the ordinary integer quantum Hall effect, possess an anomalously large Hall mobility. For the characteristic electron velocity in the sub-band we can obtain the estimate

$$\bar{v} \sim \frac{U_0}{\hbar/a} p \exp\{-\alpha p\}; \quad \alpha \sim 1. \qquad (4.5)$$

The proportionality of this quantity to the number p follows from the fact that the period of the oscillations of the energy eigenvalues in the MBZ is inversely proportional to p. But for $p \gg 1$ the electron wavelength λ is large in comparison with the lattice constant a ($\lambda/a \sim p^{1/2}$), and, therefore, the width of the magnetic sub-band decreases exponentially with the number p. This gives the exponential factor in (4.5). Simultaneously with the decrease of the width of the magnetic sub-band there is also an exponential decrease of the characteristic magnitude of the gaps between the sub-bands, and it is this which leads, in the given case, to the anomalously large response of the system to an external electric field.

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[1)]Since the quantities D_{jl} are complex, they can vanish, generally speaking, only at isolated points.

[2)]We recall that, by virtue of (2.19), a point $\mathbf{k}_m$ is singular for all d_l ($l = 0, 1, \ldots, r-1$) simultaneously.

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