Polarized solitons in three-level media

A. M. Basharov and A. I. Maĭmistov

Moscow Engineering-Physics Institute
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The propagation and interaction of ultrashort two-frequency light pulses under the conditions of double resonance in an optically thick three-level medium are discussed. The effects of field polarization and level degeneracy are taken into account for the first time. It is shown that the Maxwell-Bloch equations, suitably generalized, can be integrated by the inverse scattering method for (1) triply degenerate and nondegenerate and (2) doubly degenerate levels. The former case is a new example of an exactly integrable set of nonlinear evolution equations. The Lax representation, the Bäcklund transformation, the one-soliton solutions, and an infinite set of integrals of motion are found. The latter case provides a new physical interpretation of two independent sets of nonlinear equations whose integrability was established earlier. The collisions of differently polarized solitons are investigated, and it is shown that the behavior of the polarization of optical pulses depends on the degeneracy of the energy levels.

Exactly integrable models of the interaction of polarized waves in optically nonlinear media have been attracting increasing attention. The fact of exact integrability predetermines a number of remarkable properties of wave processes among which the most striking is the particle-like behavior of solitary waves. When such particles, now called solitons,[1] undergo an interaction with one another, they retain their shape and propagation velocity, and experience only a phase shift. A further degree of freedom is introduced into the theory when the polarization of these waves is taken into account. This enables us to examine differently polarized solitary waves and their collisions, which opens up new avenues for experimental investigation.

Studies of polarization properties have so far been confined to single-frequency solitary waves in nonresonant[2,3] and resonant[4–6] optically nonlinear media. The interaction between two-frequency waves in a nonresonant Kerr medium was considered in recent publications.[7,8] The exact integrability of the corresponding evolution equations was established subject to certain restrictions on the parameters of the medium, but for arbitrary polarizations of the waves.[7,8] In resonant media, the interaction between waves with different carrier frequencies ω_1 and ω_2 can occur under the conditions of double resonance

$$\omega_1 \approx |E_b - E_a| \hbar^{-1}, \quad \omega_2 \approx |E_b - E_c| \hbar^{-1}, \tag{1}$$

two-photon resonance

$$\omega_1 + \omega_2 \approx (E_c - E_a) \hbar^{-1} \tag{2}$$

and Raman resonance

$$\omega_1 - \omega_2 \approx (E_c \approx E_a) \hbar^{-1} \tag{3}$$

with energy levels E_a, E_b, and E_c between which optically allowed $E_b - E_a$, $E_b - E_c$ and optically forbidden $E_c - E_a$ quantum transitions take place. Polarization effects and exact integrability of the generalized evolution equations have not been analyzed. When polarization is ignored, i.e., if we consider waves that are linearly polarized along a given fixed axis, nondegenerate energy levels, and other additional assumptions, the simplified Maxwell-Bloch equations describing the resonant (1)–(3) propagation of ultrashort two-frequency optical pulses are found to be integrable by the inverse scattering method. This was demonstrated in Refs. 9 and 10 for double resonances and, in Refs. 11–13, for two-photon and Raman resonances. Under the conditions defined by (1)–(3), solitons are also ultrashort optical pulses with different carrier frequencies that propagate with equal velocities without change of shape or energy loss. In three-level media defined by (1), they are often referred to as simultons.

In this paper, we discuss the propagation and interaction of differently polarized two-frequency ultrashort optical pulses in a resonant medium consisting of three-level particles and defined by (1). We take into account the degeneracy of the resonant levels in the different orientations of the total angular momentum. This degeneracy is typical for real media, e.g., gases, and is necessary for the correct description of the interaction with arbitrarily polarized radiation. The integrability of the generalized Maxwell-Bloch equations (including arbitrary polarization) is demonstrated in the case of equal resonance absorption lengths at the two frequencies, the Λ ($E_b > E_c > E_a$) and V ($E_c > E_a > E_b$) configurations of the resonant energy levels (Fig. 1), threefold degenerate level E_b and nondegenerate levels E_a and E_c, and homogeneous broadening of the spectral lines. The demonstration is based on the inverse scattering method, found to give rise to a new 4×4 spectral problem [see (13a)] that had not previously been encountered in nonlinear optics. The Bäcklund transformation will be found for the generalized Maxwell-Bloch equations and the form of polarized simultons will be established for the V-configuration with an initially equilibrium population of the lower energy level E_b, and for the Λ-configuration with the initially fully populated energy levels E_a and E_c. The one-

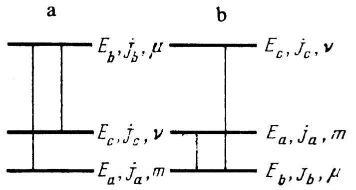

FIG. 1. (a) Λ- and (b) V-configurations of three-level systems. Vertical lines represent optically allowed transitions.

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soliton simulton consists of identically polarized ultrashort optical pulses; the collision of differently polarized simultons gives rise to a change in their polarization in accordance with a particular law that differs from that prevailing in the nonresonant case.[7,8]

The case of the Λ-configuration with a different population of the lower energy levels E_a and E_c has its own specificity that involves the transformation of the two-frequency ultrashort pulse into a one-frequency pulse. This effect was investigated in Ref. 14, but the polarization of the ultrashort optical pulses and the degeneracy of energy levels were ignored. The Lax representation (13) of our generalized Maxwell-Bloch equations can be used as a basis for studying the polarization properties of this effect, but this is a separate problem that will not be examined here.

1. EQUATIONS OF THE POLARIZATION MODEL OF DOUBLE RESONANCE AND THEIR LAX REPRESENTATION

The collinear propagation of two-frequency ultrashort pulses with electric fields

$$\vec{E}_j=\mathbf{E}_j \exp\,[i(k_jz-\omega_jt)]+\text{c.c.}, j=1,2 \tag{4}$$

under the conditions of the double resonance defined by (1), with energy levels E_a, E_b, E_c, which are additionally characterized by total angular momenta j_a, j_b, and j_c, and their components m, μ, and ν along the quantization axis, will be described by the classical Maxwell equations for the slowly-varying amplitudes $\mathbf{E}_1$ and $\mathbf{E}_2$

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{E}_j=i2\pi k_j\mathbf{P}_j, \quad j=1,2 \tag{5}$$

and the quantum-mechanical equations for the density matrix ρ. The latter can be written in the following form:

for the Λ-configuration:

$$\begin{aligned}
&i\hbar\frac{\partial}{\partial t}\rho^{(a)}_{mm'}=\tilde\rho^{(ba)*}_{\mu m}\mathbf{E}_1\mathbf{d}_{\mu m'}-\mathbf{E}_1^*\mathbf{d}_{m\mu}\tilde\rho^{(ba)}_{\mu m'},\\
&i\hbar\frac{\partial}{\partial t}\rho^{(c)}_{\nu\nu'}=\tilde\rho^{(bc)*}_{\mu\nu}\mathbf{E}_2\mathbf{d}_{\mu\nu'}-\mathbf{E}_2^*\mathbf{d}_{\nu\mu}\tilde\rho^{(bc)}_{\mu\nu'},\\
&i\hbar\frac{\partial}{\partial t}\rho^{(b)}_{\mu\mu'}=\tilde\rho^{(ba)}_{\mu m}\mathbf{E}_1^*\mathbf{d}_{m\mu'}-\mathbf{E}_1\mathbf{d}_{\mu m}\tilde\rho^{(ba)*}_{\mu' m}\\
&\qquad+\tilde\rho^{(bc)}_{\mu\nu}\mathbf{E}_2^*\mathbf{d}_{\nu\mu'}-\mathbf{E}_2\mathbf{d}_{\mu\nu}\tilde\rho^{(bc)*}_{\mu'\nu},
\end{aligned}\tag{6}$$

$$\begin{aligned}
&i\hbar\left(\frac{\partial}{\partial t}-i\delta\right)\tilde\rho^{(ba)}_{\mu m}=-\mathbf{E}_2\mathbf{d}_{\mu\nu}\tilde\rho^{(ca)}_{\nu m}+\rho^{(b)}_{\mu\mu'}\mathbf{E}_1\mathbf{d}_{\mu' m}-\mathbf{E}_1\mathbf{d}_{\mu m'}\rho^{(a)}_{m'm},\\
&i\hbar\left(\frac{\partial}{\partial t}-i\delta\right)\tilde\rho^{(bc)}_{\mu\nu}=-\mathbf{E}_1\mathbf{d}_{\mu m}\tilde\rho^{(ca)*}_{\nu m}+\rho^{(b)}_{\mu\mu'}\mathbf{E}_2\mathbf{d}_{\mu'\nu}-\mathbf{E}_2\mathbf{d}_{\mu\nu'}\rho^{(c)}_{\nu'\nu},\\
&i\hbar\frac{\partial}{\partial t}\tilde\rho^{(ca)}_{\nu m}=\tilde\rho^{(bc)*}_{\mu\nu}\mathbf{E}_1\mathbf{d}_{\mu m}-\mathbf{E}_2^*\mathbf{d}_{\nu\mu}\tilde\rho^{(ba)}_{\mu m},\\
&\delta=\omega_1-(E_b-E_a)\hbar^{-1}=\omega_2-(E_b-E_c)\hbar^{-1};
\end{aligned}$$

for the V-configuration:

$$\begin{aligned}
&i\hbar\frac{\partial}{\partial t}\rho^{(a)}_{mm'}=\tilde\rho^{(ab)}_{m\mu}\mathbf{E}_1^*\mathbf{d}_{\mu m'}-\mathbf{E}_1\mathbf{d}_{m\mu}\tilde\rho^{(ab)*}_{m'\mu},\\
&i\hbar\frac{\partial}{\partial t}\rho^{(c)}_{\nu\nu'}=\tilde\rho^{(cb)}_{\nu\mu}\mathbf{E}_2^*\mathbf{d}_{\mu\nu'}-\mathbf{E}_2\mathbf{d}_{\nu\mu}\tilde\rho^{(cb)*}_{\nu'\mu},\\
&i\hbar\frac{\partial}{\partial t}\rho^{(b)}_{\mu\mu'}=\tilde\rho^{(ab)*}_{m\mu}\mathbf{E}_1\mathbf{d}_{m\mu'}-\mathbf{E}_1^*\mathbf{d}_{\mu m}\tilde\rho^{(ab)}_{m\mu'}\\
&\qquad+\tilde\rho^{(cb)*}_{\nu\mu}\mathbf{E}_2\mathbf{d}_{\nu\mu'}-\mathbf{E}_2^*\mathbf{d}_{\mu\nu}\tilde\rho^{(cb)}_{\nu\mu'},
\end{aligned}\tag{7}$$

$$\begin{aligned}
&i\hbar\left(\frac{\partial}{\partial t}-i\delta\right)\tilde\rho^{(ab)}_{m\mu}=-\mathbf{E}_1\mathbf{d}_{m\mu'}\rho^{(b)}_{\mu'\mu}+\rho^{(a)}_{mm'}\mathbf{E}_1\mathbf{d}_{m'\mu}+\tilde\rho^{(ca)*}_{\nu m}\mathbf{E}_2\mathbf{d}_{\nu\mu},\\
&i\hbar\left(\frac{\partial}{\partial t}-i\delta\right)\tilde\rho^{(cb)}_{\nu\mu}=-\mathbf{E}_2\mathbf{d}_{\nu\mu'}\rho^{(b)}_{\mu'\mu}+\rho^{(c)}_{\nu\nu'}\mathbf{E}_2\mathbf{d}_{\nu'\mu}+\tilde\rho^{(ca)}_{\nu m}\mathbf{E}_1\mathbf{d}_{m\mu},\\
&i\hbar\frac{\partial}{\partial t}\tilde\rho^{(ca)}_{\nu m}=-\mathbf{E}_2\mathbf{d}_{\nu\mu}\tilde\rho^{(ab)*}_{m\mu}+\tilde\rho^{(cb)}_{\nu\mu}\mathbf{E}_1^*\mathbf{d}_{\mu m},\\
&\delta=\omega_1-(E_a-E_b)\hbar^{-1}=\omega_2-(E_c-E_b)\hbar^{-1}.
\end{aligned}$$

The matrix elements $\rho^{(a)}_{mm'}$, $\rho^{(b)}_{\mu\mu'}$, and $\rho^{(c)}_{\nu\nu'}$ characterize the states of the atoms in the corresponding energy levels, and the optical coherence matrices $\rho^{(ba)}_{\mu m}$ $(\rho^{(ab)}_{m\mu})$, $\rho^{(bc)}_{\mu\nu}$ $(\rho^{(cb)}_{\nu\mu})$ and $\rho^{(ca)}_{\nu m}$ describe transitions between the Zeeman sublevels of different levels and determine the polarization of the medium. The slowly varying amplitudes in the matrix describing the optical coherence and the polarizaton of the medium are related as follows

Λ-configuration:

$$\mathbf{P}_1=\tilde\rho^{(ba)}_{\mu m}\mathbf{d}_{m\mu},\quad \mathbf{P}_2=\tilde\rho^{(bc)}_{\mu\nu}\mathbf{d}_{\nu\mu},\quad \rho^{(ba)}_{\mu m}=\tilde\rho^{(ba)}_{\mu m}\exp[i(k_1z-\omega_1t)],$$

$$\rho^{(bc)}_{\mu\nu}=\tilde\rho^{(bc)}_{\mu\nu}\exp[i(k_2z-\omega_2t)],$$

$$\rho^{(ca)}_{\nu m}=\tilde\rho^{(ca)}_{\nu m}\exp\{i[(k_1-k_2)z-(\omega_1-\omega_2)t]\};$$

V-configuration:

$$\mathbf{P}_1=\tilde\rho^{(ab)}_{m\mu}\mathbf{d}_{\mu m},\quad \mathbf{P}_2=\tilde\rho^{(cb)}_{\nu\mu}\mathbf{d}_{\mu\nu},$$

$$\rho^{(ab)}_{m\mu}=\tilde\rho^{(ab)}_{m\mu}\exp[i(k_1z-\omega_1t)],\quad \rho^{(cb)}_{\nu\mu}=\tilde\rho^{(cb)}_{\nu\mu}\exp[i(k_2z-\omega_2t)],$$

$$\rho^{(ca)}_{\nu m}=\tilde\rho^{(ca)}_{\nu m}\exp\{i[(k_2-k_1)z-(\omega_2-\omega_1)t]\}.$$

Summation over repeated matrix indices is implied throughout.

We shall adopt the following relationship between the matrix elements of the dipole moment operator $\mathbf{d}$ and the reduced dipole moment d_{ba}, etc. of the corresponding optically allowed transitions:

$$\begin{aligned}
&d_{m\mu}{}^q=(-1)^{j_b-m}\begin{pmatrix} j_a & 1 & j_b\\ -m & q & \mu\end{pmatrix} d_{ba}^*,\\
&d_{\mu m}{}^q=(-1)^{j_b-\mu}\begin{pmatrix} j_b & 1 & j_a\\ -\mu & q & m\end{pmatrix} d_{ba},
\end{aligned}\tag{8}$$

where the $q=0\pm1$ labels the spherical component of the vector. When the quantization axis lies along the z direction, we then have $d^0=d_z$, $d\pm1=\mp 2^{-1/2}(d_x\pm id_y)$. The formulas given by (8) refer to the Λ-configuration; for the V-configuration, we must introduce the replacements $a\rightleftarrows b$ and $m\rightleftarrows\mu$ in (8).

Equations (5)–(7) ignore relaxation, the difference between the velocities of weak ultrashort optical pulses, and the inhomogeneous broadening of spectral lines.

Next, we assume equal oscillator strengths for the E_b-E_a and E_c-E_b, transitions, i.e., we demand that

$$k_1|d_{ab}|^2=k_2|d_{cb}|^2 \tag{9}$$

which is necessary for the existence of simultons. We shall confine our attention to low angular momenta, i.e.,

$$j_a=j_c=0,\quad j_b=1. \tag{10}$$

We then obtain the following dimensionless equations from (5)–(7):

$$
\begin{gathered}
\frac{\partial}{\partial \zeta} \varepsilon_j{}^q = -ip_j{}^q, \quad \frac{\partial}{\partial \tau} n_j = -i\sum_q (\varepsilon_j{}^q p_j{}^{q*} - \varepsilon_j{}^{q*} p_j{}^q), \quad j=1,2, \\
\left(\frac{\partial}{\partial \tau} - i\Delta\right) p_1{}^q = -i\left(\sum_{q'} \varepsilon_1{}^{q'} m_{q'q} - \varepsilon_1{}^q n_1 - \varepsilon_2{}^q r\right), \\
\left(\frac{\partial}{\partial \tau} - i\Delta\right) p_2{}^q = -i\left(\sum_{q'} \varepsilon_2{}^{q'} m_{q'q} - \varepsilon_2{}^q n_2 - \varepsilon_1{}^q r^*\right), \\
\frac{\partial}{\partial \tau} m_{qq'} = -i\sum_{j=1,2} (\varepsilon_j{}^{q*} p_j{}^{q'} - p_j{}^{q*} \varepsilon_j{}^{q'}), \\
\frac{\partial}{\partial \tau} r = -i\sum_q (\varepsilon_1{}^q p_2{}^{q*} - \varepsilon_2{}^{q*} p_1{}^q),
\end{gathered}
\tag{11}
$$

which describe both the V-configuration

$$
\begin{gathered}
\boldsymbol{\varepsilon}_1 = \mathbf{E}_1 t_0 d_{ab}/3^{1/2}\hbar, \ \boldsymbol{\varepsilon}_2 = \mathbf{E}_2 t_0 d_{cb}/3^{1/2}\hbar, \\
p_1{}^q = \tilde{\rho}_{0q}^{(ab)}/N_0, \quad p_2{}^q = \tilde{\rho}_{0q}^{(cb)}/N_0, \\
r = \tilde{\rho}_{00}^{(ac)}/N_0, \quad n_1 = \rho_{00}^{(a)}/N_0, \quad n_2 = \rho_{00}^{(c)}/N_0, \\
m_{qq'} = \rho_{qq'}^{(b)}/N_0, \quad N_0 = N_b/3,
\end{gathered}
$$

and the Λ-configuration

$$
\begin{gathered}
\boldsymbol{\varepsilon}_1 = \mathbf{E}_1 t_0 d_{ba}/3^{1/2}\hbar, \quad \boldsymbol{\varepsilon}_2 = \mathbf{E}_2 t_0 d_{bc}/3^{1/2}\hbar \\
p_1{}^q = \tilde{\rho}_{-q0}^{(ba)}/N_0, \quad p_2{}^q = \tilde{\rho}_{-q0}^{(bc)}/N_0, \\
r = -\tilde{\rho}_{00}^{(ca)}/N_0, \quad n_1 = -\rho_{00}^{(a)}/N_0, \quad n_2 = -\rho_{00}^{(c)}/N_0, \\
m_{qq'} = -\rho_{-q'-q}^{(b)}/N_0, \quad N_0 = N_a
\end{gathered}
$$

where $t_0 = (2\pi\omega_1 |d_{ab}|^2 N_0/3\hbar)^{-1/2}$ is a constant with the dimensions of time, N_b (N_a) are the densities of atoms populating the lower level in the $V(\Lambda)$-configuration in thermodynamic equilibrium, and the indices q and q' assume the values ± 1 and label the spherical components of the vector perpendicular to the quantization axis ζ:

$$
\zeta = z/ct_0, \quad \tau = (t - z/c)/t_0, \quad \Delta = \delta t_0.
$$

When the medium to be excited is initially in thermodynamic equilibrium, the initial and boundary conditions for (11) are

$$
\begin{gathered}
\boldsymbol{\varepsilon}_j|_{\tau=-\infty} = \mathbf{p}_j|_{\tau=-\infty} = r|_{\tau=-\infty} = 0, \quad n_j|_{\tau=-\infty} = n_{0j}, \\
m_{qq'}|_{\tau=-\infty} = m_0 \delta_{qq'}, \quad \boldsymbol{\varepsilon}_j|_{\zeta=0} = \boldsymbol{\varepsilon}_{0j}(\tau), \quad j=1,2.
\end{gathered}
\tag{12}
$$

The point $\zeta = 0$ corresponds to the entry of the two-frequency pump pulse into the resonant medium. The amplitude profiles of the pulse are described by the functions $\boldsymbol{\varepsilon}_{01}(\tau)$ and $\boldsymbol{\varepsilon}_{02}(\tau)$. The constants n_{01}, n_{02}, and m_0 characterize the Boltzmann population of the resonance levels.

When one of the fields $\boldsymbol{\varepsilon}_1$ or $\boldsymbol{\varepsilon}_2$ is absent and there is no coherence in the $E_c - E_a$ transition (i.e., $r = 0$), Eqs. (11) become identical with the Maxwell-Bloch equations for the two-level system with transitions involving the $1 \rightleftarrows 0$ change in the total angular momentum.[4] When the polarization of all the fields is the same and is left-handed, e.g., $\varepsilon_1^1 = \varepsilon_2^1 = 0$, then (11) again becomes identical with the Maxwell-Bloch equations for the $1 \rightleftarrows 0$ transition. This is also valid for the right-handed polarization or fixed and equal linear polarization of the fields ε_1 and ε_2. These properties enable us to find the Lax representation of (11). Direct verification will show that the equations in (11) constitute the condition for the compatibility of the following sets of linear equations:

$$
\frac{\partial}{\partial \tau}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} -i\lambda & 0 & -i\varepsilon_1^{-1} & -i\varepsilon_2^{-1} \\ 0 & -i\lambda & -i\varepsilon_1{}^1 & -i\varepsilon_2{}^1 \\ -i\varepsilon_1^{-1*} & -i\varepsilon_1^{1*} & i\lambda & 0 \\ -i\varepsilon_2^{-1*} & -i\varepsilon_2^{1*} & 0 & i\lambda \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix},
\tag{13a}
$$

$$
\frac{\partial}{\partial \zeta}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{i}{2\lambda + \Delta}\begin{pmatrix} -m_{-1-1} & -m_{1-1} & p_1^{-1} & p_2^{-1} \\ -m_{-11} & -m_{11} & p_1{}^1 & p_2{}^1 \\ p_1^{-1*} & p_1^{1*} & -n_1 & -r^* \\ p_2^{-1*} & p_2^{1*} & -r & -n_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}.
\tag{13b}
$$

We thus see that, when the arbitrariness in the polarization of the two-frequency ultrashort pulses and the simple degeneracy (10) of the energy level are taken into account, this results in a new model of nonlinear optics, defined by (11), that is integrable by the inverse scattering method. Existing models[4,9,10] are special cases of (11). The spectral problem defined by (13a) can be looked upon as a nontrivial generalization of the 3×3 spectral problem of Manakov.[2] Another view of (13a) is provided by the following version of (13):

$$
\frac{\partial}{\partial \tau}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} -i\lambda\hat{I} & -i\hat{\varepsilon} \\ -i\hat{\varepsilon}^+ & i\lambda\hat{I} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}
\tag{14a}
$$

$$
\frac{\partial}{\partial \zeta}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{i}{2\lambda + \Delta}\begin{pmatrix} -\hat{M} & \hat{P} \\ \hat{P}^+ & -\hat{N} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix},
\tag{14b}
$$

where Q_1 and Q_2 are two-component column vectors and the other quantities are 2×2 matrices of the form

$$
\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\varepsilon} = \begin{pmatrix} \varepsilon_1{}^{-1} & \varepsilon_2{}^{-1} \\ \varepsilon_1{}^1 & \varepsilon_2{}^1 \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} m_{-1-1} & m_{1-1} \\ m_{-11} & m_{11} \end{pmatrix},
$$

$$
\hat{N} = \begin{pmatrix} n_1 & r^* \\ r & n_2 \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} p_1{}^{-1} & p_2{}^{-1} \\ p_1{}^1 & p_2{}^1 \end{pmatrix}.
$$

The spectral problem defined by (13a) can therefore also be looked upon as a matrix generalization of the Zakharov-Shabat problem[1] [in the sense of (13a)]. This suggests a way toward a solution of the inverse scattering problem for (13a). However, in this paper, we shall not perform a direct inverse scattering solution for (13a).

2. BÄCKLUND TRANSFORMATION. POLARIZED SIMULTONS

Let us write (13) in the Riccati form. We shall use the pseudopotentials

$$
u_1 = \frac{q_3}{q_1}, \quad u_2 = \frac{q_4}{q_1}, \quad w = \frac{q_2}{q_1}
$$

and, for compactness, we shall introduce the following notation:

$$
E^\pm \equiv (\varepsilon_1{}^{\pm 1}, \varepsilon_2{}^{\pm 1}), \quad P^\pm \equiv (p_1{}^{\pm 1}, p_2{}^{\pm 1}),
$$

$$U\equiv(u_1,u_2),\qquad (UP)\equiv\sum_{j=1,2}u_jp_j,$$
$$V\equiv((m_{-1-1}-n_1)u_1-r^*u_2,\quad (m_{-1-1}-n_2)u_2-ru_1).$$

We then obtain the following equations (13):

$$\frac{\partial U}{\partial\tau}=2i\lambda U-iE^{-*}+iU(E^-U)-iE^{+*}w,$$
$$\frac{\partial w}{\partial\tau}=-i(E^+U)+i(E^-U)w;\tag{15}$$
$$\frac{\partial U}{\partial\zeta}=\frac{i}{2\lambda+\Delta}\{P^{-*}-U(P^-U)+wP^{+*}+wm_{1-1}U+V\},$$
$$\frac{\partial w}{\partial\zeta}=\frac{i}{2\lambda+\Delta}\{-m_{-11}+(m_{-1-1}-m_{11})w+m_{1-1}w^2$$
$$+(P^+U)-w(P^-U)\}.\tag{16}$$

We must now find a transformation for the quantities

$U\to U'$, $w\to w'$, $\lambda\to\lambda'$, $E^\pm\to E^{\pm\prime}$ and so on,

that appear in (15) and (16), which does not alter the form of (15) and (16). If $E^\pm\to E^{\pm\prime}$ nonidentically, it is called the Bäcklund transformation (in the pseudopotential form[15,16]). The simplest transformation of this type can be obtained by putting

$$U'=U,\quad w'=w,\quad \lambda'=\lambda^*$$

and seeking $E^{\pm\prime}$ in the form

$$E^{\pm\prime}=E^\pm+(\lambda-\lambda^*)\beta^\pm U^*,\tag{17}$$

where $\beta^\pm$ are unknown parameters. It is readily shown that

$$\beta^+=w\beta^-=\frac{2w}{1+|w|^2+(U^*U)}.$$

Similar transformations can also be found for $P^\pm$, $m_{qq'}$, and V. Together with (17), these transformations define the required Bäcklund transformation: we use the known solution of the initial equations (11) to find the pseudopotentials from (15) and (16), and then use (17) to find the potentials $E^\pm$, etc., which become the new solutions of (11).

To be specific, let us consider a three-level system in the V-configuration, with only the bottom energy level populated in thermodynamic equilibrium. The trivial solution

$$\varepsilon_j^{\pm1}=p_j^{\pm1}=n_j=r=0,\quad m_{qq'}=\delta_{qq'}$$

of (11) corresponds to the following pseudopotentials:

$$u_j=c_j\exp\left(2i\lambda\tau+\frac{i}{2\lambda+\Delta}\zeta\right),\qquad w=c_0,\qquad j=1,2,$$

where c_0, c_1, and c_2 are the integration constants. Assuming, for simplicity, that $\lambda=i\sigma$, $\sigma=\sigma^*$, we find from (17) a new solution of (11), generated by the trivial solution

$$\varepsilon_j=\mathbf{l}\theta_j\exp\left(-\frac{i\Delta\zeta}{4\sigma^2+\Delta^2}\right)2\sigma\operatorname{sech}2\sigma\left(\tau-\frac{\zeta}{4\sigma^2+\Delta^2}-\tau_0\right),\ j=1,2,\tag{18}$$

where the unit vector $\mathbf{l}$ has the spherical components

$$l^{-1}=i(1+|c_0|^2)^{-1/2},\quad l^1=ic_0(1+|c_0|^2)^{-1/2},\quad l^0=0,$$

and the other parameters are related to c_0, c_1, and c_2 by

$$c_j=(1+|c_0|^2)^{1/2}\theta_j^*e^{2\sigma\tau_0},\qquad |\theta_1|^2+|\theta_2|^2=1.$$

The formulas obtained above are also valid for the three-level system in the Λ-configuration with initially equally populated energy levels E_a and E_c. The corresponding trivial (seed) solution of (11) is

$$\varepsilon_j^{\pm1}=p_j^{\pm1}=m_{qq'}=r=0,\quad n_1=n_2=-1.$$

The expression given by (18) is the one-soliton solution of (11) that describes the envelope of the two-frequency ultrashort pulse (4). This pulse propagates resonantly in the three-level medium with velocity $(4\sigma^2+\Delta^2)/(4\sigma^2+\Delta^2+1)$ (in units of c) without change of shape or energy loss. An ultrashort pulse with this type of envelope is, in fact, a polarized simulton. It is important to emphasize that pulses constituting the one-soliton simulton (18) have the same polarization $\mathbf{l}$; their polarizations can differ only in the two-soliton case. The formula given by (18) describes a 2π-pulse of single-frequency self-induced transparency[4] if we assume that one of the fields, ε_1 or ε_2, is absent. On the other hand, if we ignore the polarization $\mathbf{l}$ and the detuning Δ, the expressions given by (18) become identical with the results obtained in Refs. 9 and 10 for the nondegenerate three-level system. In the latter case, θ_1 and θ_2 appear as the "Cartesian" coordinates of some vector $\boldsymbol{\theta}=(\theta_1,\theta_2)$ that formally plays the same role as the vector $\mathbf{l}$ of the solitons of self-induced transparency in two-level media in transitions with the $1\rightleftarrows0$ and $1\rightleftarrows1$ changes in total angular momentum (Refs. 4–6). The physical significance of $\boldsymbol{\theta}$ is different: it shows the distribution of energy over the pulses making up the simulton.

The simultons and the conclusions associated with them can be naturally generalized to the inhomogeneous (but not Doppler) broadening of spectral lines by analogy with the procedure adopted for nondegenerate media in Ref. 17.

3. SCATTERING MATRTIX FOR SIMULTONS

Let us now determine two linearly dependent sets of Jost functions $\varphi^{(k)}(\tau,\lambda)$ and $\psi^{(k)}(\tau,\lambda)$, $k=1,2,3,4$, which are solutions of (13a) with real λ and the following asymptotic behavior:

$$\varphi^{(1)}\to g^{(1)}e^{-i\lambda\tau},\quad \varphi^{(2)}\to g^{(2)}e^{-i\lambda\tau},$$
$$\varphi^{(3)}\to g^{(3)}e^{i\lambda\tau},\quad \varphi^{(4)}\to g^{(4)}e^{i\lambda\tau},\quad \tau\to-\infty,$$
$$\psi^{(1)}\to g^{(1)}e^{-i\lambda\tau},\quad \psi^{(2)}\to g^{(2)}e^{-i\lambda\tau},$$
$$\psi^{(3)}\to g^{(3)}e^{i\lambda\tau},\quad \psi^{(4)}\to g^{(4)}e^{i\lambda\tau},$$
$$\tau\to+\infty,$$

where $g^{(k)}$ is a four-component column vector with elements $g_j^{(k)}=\delta_{kj}$. Since $d\operatorname{sech}x/dx=-\operatorname{th}x\operatorname{sech}x$ and $d\operatorname{th}x/dx=\operatorname{sech}^2x$, it is natural to seek a solution of (13) that corresponds to the spectral problems for the simulton (18) in the form

$$q_k=\{\alpha_k\operatorname{sech}2\sigma(\tau-\tau_0)+\beta_k\operatorname{th}2\sigma(\tau-\tau_0)+\gamma_k\}e^{i\delta\lambda\tau},$$

where each Jost function has its own constants δ, α_k, β_k and γ_k.

For $\varphi^{(1)}$: $\delta=-1$, $\beta_1=\gamma_1-1=-i\sigma|l^{-1}|^2(\lambda+i\sigma)^{-1}$,

$\beta_2=\gamma_2=-i\sigma l^1l^{-1*}(\lambda+i\sigma)^{-1}$, $\alpha_3=\sigma\theta_1^*l^{-1*}(\lambda+i\sigma)^{-1}$,

$\alpha_4=\sigma\theta_2^*l^{-1*}(\lambda+i\sigma)^{-1}$;

for $\varphi^{(2)}$: $\delta=-1$, $\beta_1=\gamma_1=-i\sigma l^{-1}l^{1*}(\lambda+i\sigma)^{-1}$,

$\beta_2=\gamma_2-1=-i\sigma|l^1|^2(\lambda+i\sigma)^{-1},\quad \alpha_3=\sigma\theta_1{}^*l^{1*}(\lambda+i\sigma)^{-1},$

$\alpha_4=\sigma\theta_2{}^*l^{1*}(\lambda+i\sigma)^{-1};$

for $\varphi^{(3)}$: $\delta=1,\quad \alpha_1=-\sigma\theta_1 l^{-1}(\lambda-i\sigma)^{-1},$

$\alpha_2=-\sigma\theta_1 l^1(\lambda-i\sigma)^{-1},\quad \beta_3=\gamma_3-1=i\sigma|\theta_1|^2(\lambda-i\sigma)^{-1},$

$\beta_4=\gamma_4=i\sigma\theta_1\theta_2{}^*(\lambda-i\sigma)^{-1};$

for $\varphi^{(4)}$: $\delta=1,\quad \alpha_1=-\sigma l^{-1}\theta_2(\lambda-i\sigma)^{-1},$

$\alpha_2=-\sigma l^1\theta_2(\lambda-i\sigma)^{-1},\quad \beta_3=\gamma_3=i\sigma\theta_1{}^*\theta_2(\lambda-i\sigma)^{-1},$

$\beta_4=\gamma_4-1=i\sigma|\theta_2|^2(\lambda-i\sigma)^{-1};$

for $\psi^{(1)}$: $\delta=-1,\quad \beta_1=1-\gamma_1=-i\sigma|l^{-1}|^2(\lambda-i\sigma)^{-1},$

$\beta_2=-\gamma_2=-i\sigma l^1 l^{-1*}(\lambda-i\sigma)^{-1},\quad \alpha_3=\sigma\theta_1{}^*l^{-1*}(\lambda-i\sigma)^{-1},$

$\alpha_4=\sigma\theta_2{}^*l^{-1*}(\lambda-i\sigma)^{-1};$

for $\psi^{(2)}$: $\delta=-1,\quad \beta_1=-\gamma_1=-i\sigma l^{-1}l^{1*}(\lambda-i\sigma)^{-1},$

$\beta_2=1-\gamma_2=-i\sigma|l^1|^2(\lambda-i\sigma)^{-1},$

$\alpha_3=\sigma\theta_1{}^*l^{1*}(\lambda-i\sigma)^{-1},\quad \alpha_4=\sigma\theta_2{}^*l^{1*}(\lambda-i\sigma)^{-1};$

for $\psi^{(3)}$: $\delta=1,\ \alpha_1=-\sigma\theta_1 l^{-1}(\lambda+i\sigma)^{-1},$

$\alpha_2=-\sigma\theta_1 l^1(\lambda+i\sigma)^{-1},\quad \beta_3=1-\gamma_3=i\sigma|\theta_1|^2(\lambda+i\sigma)^{-1},$

$\beta_4=-\gamma_4=i\sigma\theta_1\theta_2{}^*(\lambda+i\sigma)^{-1};$

for $\psi^{(4)}$: $\delta=1,\ \alpha_1=-\sigma l^{-1}\theta_2(\lambda+i\sigma)^{-1},$

$\alpha_2=-\sigma l^1\theta_2(\lambda+i\sigma)^{-1},\ \beta_3=-\gamma_3=i\sigma\theta_1{}^*\theta_2(\lambda+i\sigma)^{-1},$

$\beta_4=1-\gamma_4=i\sigma|\theta_2|^2(\lambda+i\sigma)^{-1}.$

All the constants that have not been written out explicitly are equal to zero. These formulas clearly demonstrate the analyticity of the Jost functions and define the scattering matrix $S_{ij}(\lambda)=\psi^{(j)+}(\tau,\lambda)\varphi^{(1)}(\tau,\lambda)$ (19) for the simulton (18):

$$S_{ij}(\lambda)=\begin{pmatrix} 1-2i\sigma|l^{-1}|^2(\lambda+i\sigma)^{-1} & -2i\sigma l^1 l^{-1*}(\lambda+i\sigma)^{-1} & 0 & 0\\ -2i\sigma l^{-1}l^{1*}(\lambda+i\sigma)^{-1} & 1-2i\sigma|l^1|^2(\lambda+i\sigma)^{-1} & 0 & 0\\ 0 & 0 & 1+2i\sigma|\theta_1|^2(\lambda-i\sigma)^{-1} & 2i\sigma\theta_1\theta_2{}^*(\lambda-i\sigma)^{-1}\\ 0 & 0 & 2i\sigma\theta_1{}^*\theta_2(\lambda-i\sigma)^{-1} & 1+2i\sigma|\theta_2|^2(\lambda-i\sigma)^{-1} \end{pmatrix}. \tag{19}$$

As $\tau\to-\infty$, only the functions $\varphi^{(1)}$ and $\varphi^{(2)}$, are bounded at the point $\lambda=i\sigma$; as $\tau\to+\infty$, only the functions $\psi^{(3)}$ and $\psi^{(4)}$ are so bounded. A bounded solution Φ can therefore be written in the form

$$\Phi=l^{-1}\varphi^{(1)}(\tau,\ i\sigma)+l^1\varphi^{(2)}(\tau,\ i\sigma)$$
$$=-\{\theta_1{}^*\psi^{(3)}(\tau,\ i\sigma)+\theta_2{}^*\psi^{(4)}(\tau,\ i\sigma)\}e^{2\sigma\tau_0} \tag{20}$$

where $\lambda=i\sigma$ is a root of the equation

$$\det\begin{pmatrix} S_{11}(\lambda) & S_{12}(\lambda)\\ S_{21}(\lambda) & S_{22}(\lambda)\end{pmatrix}=0. \tag{21}$$

For arbitrary potentials ε_j^q, roots of (21) determine, in general, a discrete spectrum of the problem defined by (13a). When (21) is satisfied, we can eliminate the singularities $\psi^{(1)}$ and $\psi^{(2)}$ from

$$\varphi^{(1)}=S_{11}\psi^{(1)}+S_{12}\psi^{(2)}+S_{13}\psi^{(3)}+S_{14}\psi^{(4)},$$

$$\varphi^{(2)}=S_{21}\psi^{(1)}+S_{22}\psi^{(2)}+S_{23}\psi^{(3)}+S_{24}\psi^{(4)}$$

and construct a bounded solution Φ as a linear combination of $\varphi^{(1)}$ and $\varphi^{(2)}$ ($\psi^{(3)}$ and $\psi^{(4)}$), e.g.,

$$\Phi=-S_{21}\varphi^{(1)}+S_{11}\varphi^{(2)},$$

which differs from (20) by the factor l^{1*} in the case of the simulton (18).

It is important to emphasize that this definition does not depend on the coordinate ζ because S_{21} and S_{11} (like S_{12}, S_{22}, S_{33}, S_{34}, S_{43}, and S_{44}) do not depend on ζ if we assume that the resonant medium is in the state of thermodynamic equilibrium as $\tau\to+\infty$. The ζ evolution of the other scattering data is determined by (13b) and has the form

$$S_{nm}=S_{nm}(\zeta=0)\exp[i\zeta(2\lambda+\Delta)^{-1}],$$

$$S_{mn}=S_{mn}(\zeta=0)\exp[-i\zeta(2\lambda+\Delta)^{-1}],$$

$$n=1,\ 2,\qquad m=3,\ 4.$$

In the ensuing analysis, we shall need the two-simulton scattering matrix for the "potential"

$$\varepsilon_j=\mathbf{l}^{(1)}\theta_j{}^{(1)}2\sigma_1\ \mathrm{sech}\ 2\sigma_1(\tau-\tau_1)+\mathbf{l}^{(2)}\theta_j{}^{(2)}2\sigma_2\ \mathrm{sech}\ 2\sigma_2(\tau-\tau_2)$$

with $\sigma_1\neq\sigma_2$. Instead of directly solving the equations of the spectral problem, we proceed as follows. We assume that the simultons are well separated from one another: $\tau_2-\tau_1\gg 2\delta$, $1/2\sigma_2$ (Fig. 2a). Consider the Jost functions $\varphi^{(i)}(\tau,\lambda)$, $i=1,2$. As $\tau\to-\infty$, we have $\varphi^{(i)}(\tau,\lambda)=g^{(i)}e^{-i\lambda\tau}$, and, for $\tau_1\ll\tau\ll\tau_2$, we have $\varphi^{(2)}(\tau,\lambda)=\sum_{k=1,2}S_{ik}^{(1)}(\lambda)g^{(k)}e^{-i\lambda\tau}$, so that, for $\tau\to+\infty$, we obtain

$$\varphi^{(i)}(\tau,\lambda)=\sum_{k,j=1,2}S_{ik}^{(1)}(\lambda)S_{kj}^{(2)}(\lambda)g^{(j)}e^{-i\lambda\tau}.$$

The superscripts 1 and 2 on S label the one-simulton scattering matrix (19) for $\mathbf{l}^{(1)}$, $\theta^{(1)}$, σ_1 and $\mathbf{l}^{(2)}$, $\theta^{(2)}$, σ_2, respectively.

The two-simulton scattering matrix can therefore be expressed in terms of the one-simulton matrices

$$S_{ij}(\lambda)=\sum_{k=1,2}S_{ik}^{(1)}(\lambda)S_{kj}^{(2)}(\lambda),$$

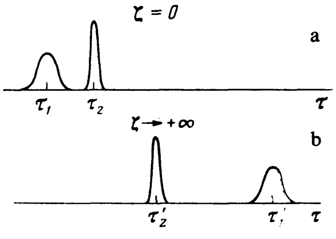

FIG. 2. Disposition of simultons along the τ axis before (a) and after (b) collision.

and, for $\lambda = i\sigma_{1,2}$, it can be transformed into the following form:

$$S_{11}(i\sigma_1)=l^{(1)1}L^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},\qquad S_{11}(i\sigma_2)=l^{(2)1*}L^{(2)1}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2},$$
$$S_{21}(i\sigma_1)=-l^{(1)-1}L^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},\qquad S_{21}(i\sigma_2)=-l^{(2)1*}L^{(2)-1}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2},\tag{22}$$

where $L^{(j)\pm 1}$ are the spherical components of the vector $\mathbf{L}^{(j)}$:

$$\mathbf{L}^{(1)}=\mathbf{l}^{(1)}-\frac{2\sigma_2}{\sigma_2-\sigma_1}\mathbf{l}^{(2)}(\mathbf{l}^{(1)}\mathbf{l}^{(2)*}),$$
$$\mathbf{L}^{(2)}=\mathbf{l}^{(2)}+\frac{2\sigma_1}{\sigma_2-\sigma_1}\mathbf{l}^{(1)}(\mathbf{l}^{(1)*}\mathbf{l}^{(2)}).\tag{23}$$

4. COLLISION OF DIFFERENTLY POLARIZED SIMULTONS

Suppose that two different polarized simultons $\boldsymbol{\varepsilon}_j^{(1)}$ and $\boldsymbol{\varepsilon}_j^{(2)}$: are incident on a three-level medium:

$$\boldsymbol{\varepsilon}_{0j}(\tau)=\boldsymbol{\varepsilon}_j^{(1)}(\tau,\ \zeta=0)+\boldsymbol{\varepsilon}_j^{(2)}(\tau,\ \zeta=0),$$
$$\boldsymbol{\varepsilon}_j^{(n)}(\tau,\ \zeta=0)=l^{(n)}\theta_j^{(n)}2\sigma_n\operatorname{sech}2\sigma_n(\tau-\tau_n),\ n=1,\ 2,$$

where the second simulton $\boldsymbol{\varepsilon}_j^{(2)}$, which lies on the τ axis to the right the first $\boldsymbol{\varepsilon}_j^{(1)}$, $\tau_2>\tau_1$ (in Fig. 2a), has a higher propagation velocity than the first simulton $\sigma_2>\sigma_1$ and, as it penetrates the medium, it interacts with and overtakes the first simulton. This means that, as $\zeta\to+\infty$, the first simulton is found to the right of the second simulton on the τ axis (Fig. 2b). Each simulton will be characterized, in general, by the polarization vectors

$$\boldsymbol{\varepsilon}_j^{(n)}(\tau,\zeta)=\mathbf{l}^{(n)\prime}\theta_j^{(n)\prime}$$
$$\exp\left(-\frac{i\Delta\zeta}{4\sigma_n^2+\Delta^2}\right)2\sigma_n\operatorname{sech}2\sigma_n\left(\tau-\frac{\zeta}{4\sigma_n^2+\Delta^2}-\tau_n'\right),$$
$$n=1,2,\qquad j=1,2.\tag{24}$$

The unit vectors $\mathbf{l}^{(j)\prime}$ and $\boldsymbol{\theta}^{(j)\prime}$ and the shift $\tau_j'-\tau_j$ of the center of the jth simulton can be found for $\zeta\to+\infty$ by a method analogous to that proposed in Ref. 18.

We shall assume that, for both $\zeta=0$ and $\zeta\to+\infty$, the simultons are sufficiently distant from one another, i.e., $\tau_2-\tau_1\gg 1/2\sigma_1,\ 1/2\sigma_2,\ \tau_1'-\tau_2'\gg 1/2\sigma_1,\ 1/2\sigma_2$. Consider the linear combination of Jost functions $\varphi^{(1)}$ and $\varphi^{(2)}$:

$$\Phi(\tau,\ \zeta,\ \lambda)=-S_{21}(\lambda)\varphi^{(1)}(\tau,\ \zeta,\ \lambda)+S_{11}(\lambda)\varphi^{(2)}(\tau,\ \zeta,\ \lambda).$$

As $\tau\to-\infty$, the function $\Phi(\tau,\zeta,\lambda)$ is independent of ζ:

$$\Phi(\tau,\ \zeta,\ \lambda)=-S_{21}(\lambda)g^{(1)}e^{-i\lambda\tau}+S_{11}(\lambda)g^{(2)}e^{-i\lambda\tau}.$$

Let us now calculate $\Phi(\tau,\zeta,\lambda)$ and ($\tau\to+\infty$) and two values of λ, namely, $\lambda_j=i\sigma_j$, by "transporting" the Jost functions through the simultons. The results for $\zeta=0$ and $\zeta\to+\infty$ must be the same if we take into account the ζ evolution of the scattering data. Using (20), (22), and (23), we find that, at $\zeta=0$ (Fig. 2a):

for $\tau\to-\infty$

$$\Phi(\lambda_1)=\{l^{(1)-1}g^{(1)}e^{-i\lambda_1\tau}+l^{(1)1}g^{(2)}e^{-i\lambda_1\tau}\}L^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},$$

$$\Phi(\lambda_2)=\{L^{(2)-1}g^{(1)}e^{-i\lambda_2\tau}+L^{(2)1}g^{(2)}e^{-i\lambda_2\tau}\}l^{(2)1*}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2};$$

for $\tau_1\ll\tau\ll\tau_2$

$$\Phi(\lambda_1)=-\{\theta_1^{(1)*}g^{(3)}e^{i\lambda_1\tau}+\theta_2^{(1)*}g^{(4)}e^{i\lambda_1\tau}\}e^{2\sigma_1\tau_1}L^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},$$

$$\Phi(\lambda_2)=\{l^{(2)-1}g^{(1)}e^{-i\lambda_2\tau}+l^{(2)1}g^{(2)}e^{-i\lambda_2\tau}\}l^{(2)1*}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2};$$

for $\tau\to+\infty$

$$\Phi(\lambda_1)=-\{[\theta_1^{(1)*}S_{33}^{(2)}(\lambda_1)+\theta_2^{(1)*}S_{43}^{(2)}(\lambda_1)]g^{(3)}e^{i\lambda_1\tau}$$
$$+[\theta_1^{(1)*}S_{43}^{(2)}(\lambda_1)$$
$$+\theta_2^{(1)*}S_{44}^{(2)}(\lambda_1)]g^{(4)}e^{i\lambda_1\tau}\}e^{2\sigma_1\tau_1}L^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},$$

$$\Phi(\lambda_2)=-\{\theta_1^{(2)*}g^{(3)}e^{i\lambda_2\tau}+\theta_2^{(2)*}g^{(4)}e^{i\lambda_2\tau}\}e^{2\sigma_2\tau_2}l^{(2)1*}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}$$

For $\zeta\to+\infty$, the situation after the collision of the simultons is as follows (Fig. 2b):

for $\tau\to-\infty$

$$\Phi(\lambda_1)=\{L^{(1)-1}g^{(1)}e^{-i\lambda_1\tau}+L^{(1)1}g^{(2)}e^{-i\lambda_1\tau}\}l^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},$$

$$\Phi(\lambda_2)=\{l^{(2)-1}g^{(1)}e^{-i\lambda_2\tau}+l^{(2)1}g^{(2)}e^{-i\lambda_2\tau}\}L^{(2)1*}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2};$$

for $\tau_2'\ll\tau\ll\tau_1'$

$$\Phi(\lambda_1)=\{l^{(1)-1}g^{(1)}e^{-i\lambda_1\tau}+l^{(1)1}g^{(2)}e^{-i\lambda_1\tau}\}l^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},$$

$$\Phi(\lambda_2)=-\{\theta_1^{(2)*}g^{(3)}e^{i\lambda_2\tau}+\theta_2^{(2)*}g^{(4)}e^{i\lambda_2\tau}\}e^{2\sigma_2\tau_2'}L^{(2)1*}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2};$$

for $\tau\to+\infty$

$$\Phi(\lambda_1)=-\{\theta_1^{(1)*}g^{(3)}e^{i\lambda_1\tau}+\theta_2^{(1)*}g^{(4)}e^{i\lambda_1\tau}\}e^{2\sigma_1\tau_1'}l^{(1)1*}\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2},$$

$$\Phi(\lambda_2)=-\{[\theta_1^{(2)*}S_{33}^{(1)}(\lambda_2)+\theta_2^{(2)*}S_{43}^{(1)}(\lambda_2)]g^{(3)}e^{i\lambda_2\tau}$$
$$+[\theta_1^{(2)*}S_{34}^{(1)}(\lambda_2)$$
$$+\theta_2^{(2)*}S_{44}^{(1)}(\lambda_2)]g^{(4)}e^{i\lambda_2\tau}\}e^{2\sigma_2\tau_2'}L^{(2)1*}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}.$$

For simplicity, and to avoid unwieldy expressions, we have omitted from these formulas the evolution factors (which subsequently cancel out) and primes on the quantities $\theta^{(j)}$, $\mathbf{l}^{(j)}$, $\mathbf{L}^{(j)}$, and $S^{(j)}(\lambda_2)$, so that it must be remembered that these latter refer to the simultons defined by (24).

We note that the equations for the spectral problem (13a) are invariant under the replacements

$$q_3^*\to q_1,\quad q_4^*\to q_2,\quad q_1^*\to q_3,\quad q_2^*\to q_4,$$
$$-\varepsilon_1^{-1}\to\varepsilon_1^{-1},\quad -\varepsilon_1^{1}\to\varepsilon_2^{-1},\quad -\varepsilon_2^{-1}\to\varepsilon_1^{1},\quad -\varepsilon_2^{1}\to\varepsilon_2^{1}.$$

Bearing this in mind, and comparing the formulas for $\Phi(\lambda_j)$ as $\tau\to+\infty$ for $\zeta=0$ and $\zeta\to+\infty$, we obtain the following transformation laws for the vectors $\mathbf{l}^{(j)}$ and $\boldsymbol{\theta}^{(j)}$, which characterize polarized simultons:

$$\mathbf{b}^{(1)\prime}=B^{-1}\{-\mathbf{b}^{(1)}+2\sigma_2\mathbf{b}^{(2)}(\mathbf{b}^{(1)}\mathbf{b}^{(2)*})(\sigma_2-\sigma_1)^{-1}\},$$
$$\mathbf{b}^{(2)\prime}=B^{-1}\{\mathbf{b}^{(2)}+2\sigma_1\mathbf{b}^{(1)}(\mathbf{b}^{(1)*}\mathbf{b}^{(2)})(\sigma_2-\sigma_1)^{-1}\},\tag{25}$$
$$B=\{1+4\sigma_1\sigma_2|\mathbf{b}^{(1)}\mathbf{b}^{(2)*}|^2(\sigma_2-\sigma_1)^{-2}\}^{1/2}$$

where the vectors $\mathbf{b}^{(j)}$ represent either $\mathbf{l}^{(j)}$, or $\boldsymbol{\theta}^{(j)}$.

It follows that the collision of polarized simultons is accompanied by the rotation of the polarization vectors independently of one another and by the redistribution of energy between the ultrashort pulses forming the simulton. These processes are governed by (25), i.e., a law identical to the transformation law for the polarization vector of pulses of single-frequency self-induced transparency involving the $1 \rightleftarrows 0$ and $1 \rightleftarrows 1$ transitions,[4-6] but differs from the properties of the interaction of two-frequency waves in a nonresonant Kerr medium investigated in Refs. 7 and 8.

The phase shifts of simultons that accompany their collisions are found to depend both on the polarization of the ultrashort pulses and on the energy distribution between them:

$$\tau_1'-\tau_1=\frac{1}{2\sigma_1}\ln\beta,\qquad \tau_2'-\tau_2=-\frac{1}{2\sigma_2}\ln\beta,$$

$$\beta=\left\{\left(1+\frac{4\sigma_1\sigma_2}{(\sigma_2-\sigma_1)^2}|\mathbf{l}^{(1)}\mathbf{l}^{(2)*}|^2\right)\left(1+\frac{4\sigma_1\sigma_2}{(\sigma_2-\sigma_1)^2}|\theta^{(1)}\theta^{(2)*}|^2\right)\right\}^{1/2}.$$

Whatever the polarization, the simultons behave as if they repel one another: the overtaken simulton receives a positive increase in its coordinate $\tau_1' - \tau_1 > 0$, and the overtaking simulton receives a negative increase $\tau_2' - \tau_2 < 0$.

5. HIGHER-ORDER CONSERVATION LAWS

To find the higher-order conservation laws and the infinite sequence of integrals of motion for the equations given by (11), it is convenient to rewrite the linear equation (13) of the inverse scattering method in the form of the Riccati matrix equations, e.g.,

$$\frac{\partial}{\partial\tau}\hat{\Gamma}=-2i\lambda\hat{\Gamma}-i\hat{\varepsilon}+i\hat{\Gamma}\hat{\varepsilon}^+\hat{\Gamma}, \tag{26a}$$

$$\frac{\partial}{\partial\zeta}\hat{\Gamma}=\frac{i}{2\lambda+\Delta}(\hat{P}+\hat{\Gamma}\hat{N}-\hat{M}\hat{\Gamma}-\hat{\Gamma}\hat{P}^+\hat{\Gamma}) \tag{26b}$$

where $\hat{\Gamma}$ is a 2×2 matrix. We can verify that (26) is a suitable pair of equations for the inverse scattering method by checking that the condition $\partial^2\hat{\Gamma}/\partial\zeta\partial\tau = \partial^2\hat{\Gamma}/\partial\tau\partial\zeta$ leads to (11).

If we multiply (26a) by $\hat{\varepsilon}^+$, and (26b) by $\hat{P}^+$, we obtain the following expression after some simple algebraic manipulation:

$$\frac{\partial}{\partial\zeta}\mathrm{Sp}(\hat{\varepsilon}^+\hat{\Gamma})=\frac{1}{2\lambda+\Delta}\frac{\partial}{\partial\tau}\mathrm{Sp}(\hat{M}-\hat{P}^+\hat{\Gamma}), \tag{27}$$

which takes the form of a conservation law, i.e., the divergence of the current

$$(\mathrm{Sp}(\hat{\varepsilon}^+\hat{\Gamma}),\ (2\lambda+\Delta)^{-1}\,\mathrm{Sp}(\hat{P}^+\hat{\Gamma}-\hat{M}))$$

is zero. If the boundary condition for the problem corresponds to thermodynamic equilibrium, i.e., as $|\tau|\to\infty$, the matrix $\hat{M}=\hat{M}_0$ is constant and $\hat{P}=0$, it follows from (27) that

$$\int_{-\infty}^{\infty}\mathrm{Sp}(\hat{\varepsilon}^+\hat{\Gamma})\,d\tau=\mathrm{const}. \tag{28}$$

As usual,[1] the representation of $\hat{\Gamma}(\lambda)$ by a series in powers of $1/i\lambda$, i.e.,

$$\hat{\Gamma}(\lambda)=\sum_{n=1}^{\infty}(1/i\lambda)^n\hat{\Gamma}^{(n)} \tag{29}$$

leads to an infinite sequence of higher-order conservation laws and integrals of motion:

$$I_n=\int_{-\infty}^{\infty}\mathrm{Sp}(\hat{\varepsilon}^+\hat{\Gamma}^{(n)})\,d\tau. \tag{30}$$

The quantities $\Gamma^{(n)}$ in (30) are determined by substituting (29) in (26a) and equating to zero the coefficients of equal powers of $1/i\lambda$. This procedure results in the following recurrence relation:

$$\hat{\Gamma}^{(n+1)}=-\frac{1}{2}\left\{\frac{\partial}{\partial\tau}\hat{\Gamma}^{(n)}-i\sum_{l=1}^{n-1}\hat{\Gamma}^{(l)}\hat{\varepsilon}^+\hat{\Gamma}^{(n-l)}\right\},\qquad n\geqslant 2,$$

$$\hat{\Gamma}^{(1)}=-\frac{1}{2}i\hat{\varepsilon},\qquad \hat{\Gamma}^{(2)}=\frac{i}{4}\frac{\partial}{\partial\tau}\hat{\varepsilon},$$

which is the matrix generalization of the well-known result for all nonlinear evolution equations that can be integrated by the inverse scattering method for the Zakharov-Shabat spectral problem.[1] As an illustration, we reproduce the first three terms in the sequence $\{I_n\}$:

$$I_1=-\frac{i}{2}\int\mathrm{Sp}(\hat{\varepsilon}\hat{\varepsilon}^+)\,d\tau,\qquad I_2=\frac{i}{4}\int\mathrm{Sp}\left(\hat{\varepsilon}^+\frac{\partial}{\partial\tau}\hat{\varepsilon}\right)d\tau,$$

$$I_3=-\frac{i}{8}\int\mathrm{Sp}\left\{\hat{\varepsilon}^+\frac{\partial^2}{\partial\tau^2}\hat{\varepsilon}+\hat{\varepsilon}^+\hat{\varepsilon}\hat{\varepsilon}^+\hat{\varepsilon}\right\}d\tau.$$

6. CONCLUSIONS

The special case (10) of total angular momenta of resonant energy levels is not unique, but it does lead to an exactly integrable set of nonlinear evolution equations. Another example is provided by the three-level system of the Λ- or V-configuration with $j_a = j_b = j_c = 1/2$. According to the general equations given by (5)–(7), the left- and right-polarized circular components of ultrashort pulses with the same carrier frequency propagate independently of one another, so that the interaction between arbitrarily polarized fields in this type of medium is described by two independent sets of nonlinear evolution equations that reduce to those investigated previously.[9,10] It is readily shown that, in contrast to (25), linearly polarized simultons preserve their polarization in a collision in the three-level medium with $j_a = j_b = j_c = 1/2$, but the collision between a linearly polarized simulton and a circularly polarized simulton results in the formation of three circularly polarized simultons. The latter result is a consequence of the separation of the left- and right-polarized components of the linearly polarized simulton due to the interaction between one of them and the circularly polarized simulton.

One further example of an exactly integrable set of evolution equations is provided by the three-level system of the Λ- or V-configuration with angular momenta $j_a = j_c = 1$ and $j_b = 0$. In contrast to the case defined by (10), here we have another spectral problem of dimensions 5×5, which gives rise to new laws for the rotation of the polarization vectors and for the redistribution of the energy of the colliding simultons. The angle of rotation of the polarization vector is also found to depend on the energy distribution between the ultrashort pulses constituting the simulton, and the redistribution of energy between the ultrashort pulses depends on their polarization. These results will be published separately. Here, we merely note that, from the formal point of view, the generalized Maxwell-Bloch equations given by (5)–(7) for $j_a = j_c = 1$ and $j_b = 0$ are equivalent to the equations describing the propagation of a four-frequency ultrashort pulse in a nondegerate five-level medium[19] in the case where the oscillator strengths corresponding to the resonance conditions are equal.

The Lax representation (13) and the infinite series of integrals of motion (30) are of interest for the further investigation of the problem, including (1) the solution of the quantum field theory model of emission and interaction of quanta in a three-level medium with a degenerate energy level, (2) the investigation of the polarization properties of different types of three-level echo in optically thick media, (3) the analysis of the dynamics of ultrashort pulses with randomly modulated polarization, and (4) the numerical simulation of the propagation and interaction of differently polarized ultrashort pulses when contrary to (9), the oscillator strengths of optically allowed transitions are not equal.

[1] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, *Theory of Solitons: The Inverse Problem Method* [in Russian], Nauka, Moscow, 1980; M. Ablowitz and H. Segur, *Solitons and the Inverse Problem Method*, SIAM, Philadelphia, 1981 [Russ. transl., Mir, Moscow, 1987].

[2] S. V. Manakov, Zh. Eksp. Teor. Fiz. **65**, 505 (1973) [Sov. Phys. JETP **38**, 248 (1973)].

[3] G. B. Al'tshuler, V. B. Karasev, S. A. Kozlov, *et al.*, Opt. Spektrosk. **61**, 359 (1986) [Opt. Spectrosc. (USSR) **61**, 228 (1986)].

[4] A. M. Basharov and A. I. Maĭmistov, Zh. Eksp. Teor. Fiz. **87**, 1594 (1984) [Sov. Phys. JETP **60**, 913 (1984)].

[5] V. Ya. Chernyak and V. I. Rupasov, Phys. Lett. **A108**, 434 (1985).

[6] A. M. Basharov, A. I. Maĭmistov, and Yu. M. Sklyarov, Opt. Spektrosk. **63**, 707 (1987) [Opt. Spectrosc. (USSR) **62**, 418 (1987)].

[7] V. E. Zakharov and A. V. Mikhaĭlov, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 279 (1987) [JETP Lett. **45**, 353 (1987)].

[8] M. V. Tratnik and J. E. Sipe, Phys. Rev. Lett. **58**, 1104 (1987).

[9] L. A. Bol'shov, V. V. Likhanskiĭ, and M. I. Persiantsev, Zh. Eksp. Teor. Fiz. **84**, 903 (1983) [Sov. Phys. JETP **57**, 524 (1983)].

[10] A. I. Maĭmistov, Opt. Spektrosk. **57**, 564 (1986) [Opt. Spectrosc. (USSR) **57**, 340 (1984)].

[11] H. Steudel, Physica (Utrecht) D **6**, 155 (1983).

[12] D. J. Kaup, Physica (Utrecht) D **6**, 143 (1983).

[13] A. I. Maĭmistov and E. A. Manykin, Phys. Lett. **A95**, 216 (1983).

[14] L. A. Bol'shov, N. N. Elkin, V. V. Likhanskiĭ, and M. I. Persiantsev, Zh. Eksp. Teor. Fiz. **88**, 47 (1985) [Sov. Phys. JETP **61**, 27 (1985)].

[15] H.-H. Chen, Lect. Notes Meth. **515**, 241 (1976).

[16] H. D. Wahlquist and F. B. Estabrook, J. Math. Phys. **16**, 1 (1975).

[17] A. I. Maĭmistov and Yu. M. Sklyarov, Opt. Spektrosk. **59**, 760 (1985) [Opt. Spectrosc. (USSR) **59**, 459 (1985)].

[18] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys. JETP **37**, 823 (1973)].

[19] L. A. Bol'shov and V. V. Likhanskiĭ, Kvantovaya Elektron. (Moscow) **12**, 1339 (1985) [Sov. J. Quantum Electron. **12**, 889 (1985)].

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