Multiwave coherent interaction and nonlinear frequency shift

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(Submitted 5 April 1988)
Zh. Eksp. Teor. Fiz. **94**, 33–45 (November 1988)

Models are considered of one- and two-photon interaction in a two-level nondegenerate medium and of four-photon interaction in a medium with cubic nonlinearity and a resonant two-level transition. A model of three-wave interaction in a medium with quadratic nonlinearity is also considered. All the models are characterized by the presence of a nonlinear frequency shift and can be described within the framework of a new integrable equation set. A suitable formalism of the inverse scattering problem method is developed for solving the Cauchy problem. Soliton and nonsoliton quasi-self-similar asymptotic solutions are found. It is shown that a nonlinear frequency shift leads not only to qualitative but also to quantitative changes in the behavior of the solutions. It is demonstrated that the theory is applicable in a broad range of parameters which can readily be attained in experiment.

1. INTRODUCTION

Over the last several years intensive research of the coherent interaction of ultrashort impulses (USP) in a nonlinear medium has been ongoing. This research is of interest because such an interaction may be utilized for the transmission of information, for frequency conversion, and for the description of processes which proceed in more intensive fields and at times shorter than to relaxation time[1]. It is known that a number of models of the coherent USP interaction are integrable by the inverse scattering transform method (ISTM). At present, the value of such models is that the ISTM is actually the only method which can solve the Cauchy problem for the evolution of USP. Specifically, by utilizing the ISTM it is possible to answer a question of practical importance: which initial conditions are necessary to get a specific asymptotic state of the system.

There are not many integrable models describing the resonant interaction of light in a two-level nonlinear medium. These are the reduced Maxwell-Bloch equations describing one-photon resonance interaction[3] and the two-photon interaction (sum or difference of the carrier frequency close to the transition frequency).[4] The first model was investigated in the framework of the ISTM in Ref. 5, and the second in Refs. 6 and 7 under zero detuning conditions. These models have a well-known reduction, namely the sine-Gordon equation.[2–5] In a recent paper[8] Zakharov and Mikhaĭlov have shown that interaction of two polarized wave packets propagating in a cubic nonlinear medium and having carrier frequencies such that their difference is close to the frequency of the two-level transition, is described by equations that formally agree with the non-isotropic-chiral-field equations in the 0(3) group. This model originated in the theory of elementary particles and is integrable in the framework of the ISTM. Several new integrable versions of four-field interactions in an analogous medium are shown in Ref. 10. These models, and also a combination of the Maxwell-Bloch and the nonlinear Schrödinger equations, which is integrable under certain values of the wave detuning, account for all the known integrable models of the USP interaction in a two-level nondegenerate medium.

In the present paper are given several variants of the USP interaction, USI, which are described in the framework of a new integrable set of equations. The common trait of these physical models is the presence of a nonlinear frequency shift and of a linear detuning, the influence of which can lead to qualitative changes of the dynamics of UPS propagation. The ISTM has yielded soliton solutions that can be associated with self-induced transparency, and nonsoliton solutions that describe the decay of the unstable initial state of the system after the action of a weak perturbation.

In the next section are UPS interaction schemes that are described in the framework of a common set of equations. In Sec. 3, a suitable ISTM formalism is developed. The solutions derived with the aid of this method are given in Sec. 4. In the last section it is shown that the models and the corresponding solutions "work" in a wide range of fully attainable values of parameters of the medium and of field intensities.

2. SCHEMES OF COHERENT USP INTERACTION WITH A TWO-LEVEL NONLINEAR MEDIUM AND FORMULATIONS OF THE PROBLEMS

1. The four wave interaction. Four schemes of coherent USP interaction in a cubic nonlinear medium are given in Ref. 10. In these schemes two pairs of fields propagate toward each other. The carrier frequencies of the waves are subject to the conditions

$$\Omega_i+\gamma_i\omega_i=\omega_0+\nu_0,\quad \gamma_i=\pm 1,\quad i=1,\ 2; \tag{1}$$

here Ω_i and ω_i-are the carrier frequencies of wave packets with respective envelopes P_i and S_i and with wave vectors k_i and q_i:

$$\mathscr{E}(z,t)=\sum_{i=1}^{2}\{P_i(z,t)\exp[i(k_iz-\Omega_it)]$$
$$+S_i(z,t)\exp[i(q_iz-\omega_it)]+\text{ c.c.}\}. \tag{2}$$

Different schemes of four-wave USP interaction are realized, depending on the signs of phase velocities V_i and U_i of the respective fields P_i and S_i, and on the sign of γ_i. These equations were derived in the constant population-level approximation level (see Ref. 10 for details). The differences between Eqs. (3) below and the similar equations in Ref. 1 (page 245) is that allowance is made for time dependence on the envelopes and that the substitution $i\nu_0\to\Gamma$ is made. The equations in the case $\gamma_i=1$ are:

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$$
\begin{aligned}
&(\partial_z+V_1^{-1}\partial_t)P_1=i[\alpha_{11}P_1|S_1|^2+\alpha_{12}P_2(S_2^*S_1)e^{i\Delta z}],\\
&(\partial_z+V_2^{-1}\partial_t)P_2=i[\beta_{11}P_2|S_2|^2+\beta_{12}P_1(S_2S_1^*)e^{-i\Delta z}],\\
&(\partial_z+U_1^{-1}\partial_t)S_1=i[\alpha_{22}S_1|P_1|^2+\alpha_{21}S_2(P_2^*P_1)e^{-i\Delta z}],\\
&(\partial_z+U_2^{-1}\partial_t)S_2=i[\beta_{22}S_2|P_2|^2+\beta_{21}S_1(P_2P_1^*)e^{i\Delta z}];
\end{aligned}
\tag{3}
$$

here

$$
\begin{aligned}
&\alpha_{11}=\frac{2\pi\Omega_1^2nN_0}{k_1\hbar\nu_0}|\varkappa(\Omega_1)|^2, \qquad \alpha_{12}=\frac{2\pi\Omega_1^2nN_0}{k_1\hbar\nu_0}\varkappa^*(\Omega_1)\varkappa(\Omega_2),\\
&\alpha_{21}=\frac{2\pi\omega_1^2nN_0}{q_1\hbar\nu_0}\varkappa^*(\Omega_1)\varkappa(\Omega_2), \qquad \alpha_{22}=\frac{2\pi\omega_1^2nN_0}{q_1\hbar\nu_0}|\varkappa(\Omega_2)|^2,
\end{aligned}
\tag{4}
$$

the expressions for β_{ij} are derived from (4) by making the substitutions $\Omega_1\leftrightarrow\Omega_2$, $q_1\leftrightarrow q_2$, $\omega_1\leftrightarrow\omega_2$, $k_1\leftrightarrow k_2$. N_0 is the density of the number of particles, and n is the difference between the level populations. We assume that the medium is isotropic, i.e., the scattering tensor $\varkappa_{ij}$ is a scalar: $\varkappa_{ij} = \varkappa\delta_{ij}$. We also assume that $\varkappa$ is a real number. In the case of complex $\varkappa = |\varkappa|e^{i\varphi}$ the phase φ can be eliminated by a simple shift of the field phase by a constant $S_2\to S_2e^{i\varphi/2}$, $P_2\to P_2e^{-i\varphi/2}$; $\Delta = k_2 - q_2 - k_1 + q_1$ is the wave detuning.

For arbitrary values of parameters and field intensities it is impossible to integrate the system (3). In a number of experiments aimed at observing four-wave parametric interaction, the intensity of one of the fields was maintained constant with good accuracy (Ref. 11, Chap. 7). This approximation allows Eq. (3) to be reduced to an equation set integrable for arbitrary $\varkappa(\Omega_i)$ and Δ and under the condition that the phase velocities of the field pairs propagating in one direction be equal.

To save space, only the interaction schemes which lead (in the approximation of constant intensity of one of the fields) to mathematically different sets of equations. As the first step, we rewrite (3) in the following form:

$$
\begin{aligned}
&\partial_{\tilde t}R_+=i(g_1R_+F_3+\delta R_+F_3e^{i\Delta z}),\\
&\partial_XF_+=i(g_2F_+R_3+\varepsilon R_+F_3e^{-i\Delta z}),\\
&\partial_TR_3={}^1/_2i(R_+F_-e^{-i\Delta z}-F_+R_-e^{i\Delta z}),\\
&\partial_XF_3={}^1/_2i(F_+R_-e^{i\Delta z}-R_+F_-e^{-i\Delta z}).
\end{aligned}
\tag{5}
$$

Here

$$
\partial_\eta=\partial_z-V^{-1}\partial_t,\quad \partial_\zeta=\partial_z-U^{-1}\partial_t,\quad a^2=\alpha_{11}\alpha_{22}\beta_{11}^{-1}\beta_{22}^{-1},
$$
$$
F_-=F_+^*,\ R_-=R_+^*,\ \mu=\varkappa(\Omega_1)/\varkappa(\Omega_2).
$$

For the scheme of interaction in which $V_1 = V_2 = V > 0$, $U_1 = U_2 = -U > 0$, we have

$$
\delta=\varepsilon=1,\ g_1=-(\mu^2+a^2)/2\mu a,\ g_2=-(1+\mu^2a^2)/2\mu a,\ s_1=S_1,
$$

$$
\rho_1=P_1,\quad s_2=S_2\left(\frac{\beta_{11}}{\alpha_{11}}\right)^{1/2},\quad \rho_2=P_2\left(\frac{\beta_{22}}{\alpha_{22}}\right)^{1/2},\quad T=\alpha_{11}\int_{-\infty}^{\eta}I_1(\eta)\,d\eta,
$$

$$
X=\alpha_{22}\int_{-\infty}^{\zeta}I_2(\zeta)\,d\zeta,
$$

$$
I_1(\eta)=\frac{a}{\mu}|s_2|^2+\frac{\mu}{a}|s_1|^2,\qquad I_2(\zeta)=\mu a|\rho_2|^2+\frac{1}{\mu a}|\rho_1|^2,
$$

$$
R_3=I_2^{-1}\left(\mu a|\rho_2|^2-\frac{1}{\mu a}|\rho_1|^2\right),
$$

$$
F_3=I_1^{-1}\left(\frac{a}{\mu}|s_2|^2-\frac{\mu}{a}|s_1|^2\right),
$$

$$
\left.\begin{matrix}R_+\\F_+\end{matrix}\right\}=\left.\begin{matrix}\rho_1\rho_2^*I_2^{-1}\\ s_1s_2^*I_1^{-1}\end{matrix}\right\}2\exp\left[i\frac{\mu^2-a^2}{2\mu a}X+i\frac{1-\mu^2a^2}{2\mu a}T\right].
$$

For the phase velocities $V_1 = U_2 = V > 0$, $V_2 = U_1 = -U > 0$ we have

$$
\delta=\varepsilon=-1,\ g_1=-\mu(a^2+1)/2a,\ g_2=-(1+a^2)/2\mu a,
$$

$$
T=\alpha_{11}\int_{-\infty}^{\eta}I_1(\eta)\,d\eta,\qquad X=\alpha_{22}\int_{-\infty}^{\zeta}I_2(\zeta)\,d\zeta,
$$

$$
I_1(\eta)=\frac{1}{a}|\rho_1|^2-a|s_2|^2,\qquad I_2(\zeta)=a|\rho_2|^2-\frac{1}{a}|s_1|^2,
$$

$$
R_3=-\frac{1}{\mu I_2}\left(a|\rho_2|^2+\frac{1}{a}|s_1|^2\right),
$$

$$
F_3=-\mu^3I_1^{-1}\left(a|s_2|^2+\frac{1}{a}|\rho_1|^2\right),
$$

$$
\left.\begin{matrix}R_+\\F_+\end{matrix}\right\}=\left.\begin{matrix}\mu^{-1}I_2^{-1}\rho_1s_2\\ \mu I_1^{-1}\rho_2s_1\end{matrix}\right\}2\exp\left[i\frac{a^2-1}{2a}(T-X)\right].
$$

Let, $\xi|\rho_1|\gg|\rho_2|$ for any η, i.e., R_3 can be considered constant. This condition allows (5) to be reduced to the shortened system:

$$
\begin{aligned}
&\partial_{\tilde T}R_+=i(g'R_+F_3+\delta F_+),\quad \partial_XF_+=i\nu F_++i\varepsilon F_3R_+,\\
&\partial_XF_3={}^1/_2i(F_+R_--F_-R_+),
\end{aligned}
\tag{6}
$$

where the transformation $F_\pm\to F_\pm e^{\mp i\Delta z}$ was carried out,

$$
g'=g_1R_3^{-1},\ \tilde T=TR_3,\ \nu=\nu_1+g_2R_3,\ \nu_1=\Delta\alpha_{22}^{-1}I_2^{-1}.
$$

Linear dispersion analysis shows that the stakes of the system

$$
R_\pm(X,0)=F_\pm(0,T)=0,\ R_3(X,0)=\pm1,\ F_3(0,T)=\pm1 \tag{7}
$$

with $g'\nu<\delta\varepsilon$ are stable if

$$
F_3(0,T)R_3(X,0)=\delta\varepsilon, \tag{8}
$$

and unstable if

$$
F_3(0,T)R_3(X,0)=-\delta\varepsilon. \tag{9}
$$

In the case $g'\nu = \delta\varepsilon$ the states (7) are in the linear approximation in indifferent equilibrium with respect to the amplitude of the field. In the present paper we study the evolution of the fields between the states (7). The formulations of the Cauchy problem of the evolution of fields with variables X and T are different for the stable and unstable states (7). In the soliton interactions regime we assume that the system is in one of the states (7), $F_+(0,T) = 0$ and the field pulses introduced into the medium are such that the quantity

$$
Q=\int_{-\infty}^{\infty}|R_+(X,0)|\,dX
$$

is sufficiently large for the onset of a discrete spectrum of the corresponding spectral problem (see below). We note that the soliton solutions are able to link both stable and unstable states (7). The instability is eliminated in an order higher than the first in the field amplitude by a nonlinear frequency shift.

The second formulation of the problem involves the non-soliton quasi-self-similar asymptotic state of the system. The system is initially in one of the unstable states (7) or (9), $F_+(0,T) = 0$ is taken out of it by a "pulse" $R_+(X,0)$ of very small area, such that

$$\ln Q^{-1} \gg 1. \tag{10}$$

The system (6) is the principal object of the present paper. All the other physical models discussed below will be represented in the form (6). For these models we will also be utilizing the above formulations of the Cauchy problem.

2. Interaction of two polarized electromagnetic waves in a medium with cubic nonlinearity. In Ref. 9 it was shown that the equations describing the interaction of two wave packets propagating toward each other in a medium with cubic nonlinearity, and of a two level medium with a transition frequency close to the difference of the carrier frequencies of the fields, coincide with the equations of an anisotropic chiral field on group 0(3). In the isotropic case this problem is formally equivalent to the mode considered above under the conditions $V_1 = V_2$, $U_1 = U_2$, $\delta = \varepsilon = 1$, $\Delta = 0$, $g_1 = g_2$. The quantities P_1, P_2 and S_1, S_2-are the polarization components of the first and second fields. Under the condition $|P_1| \gg |P_2|$, the equations of this model reduce to the system (6).

3. Two-photon interaction with a two-level medium. Stimulated Raman scattering and two-photon propagation of USP in a two-level medium was analyzed in a framework of integrable equations in Refs. 6, 7, 13, and 14, while in Refs. 6, 7, and 14 the motion of the level population was considered and the condition of exact resonance was used. The corresponding reduced Maxwell-Bloch equations can be presented in the form (5), where

$$g_1 = g_2 = (b_1 - b_2)/\varkappa_0, \qquad X = \varkappa_0 N_0 z,$$

$$N_0^2 = r_3^2 + r_+ r_-, \qquad T = \varkappa_0 \int_{-\infty}^{\tau} A(\tau')\,d\tau',$$

$$\tau = t - z/c,\ A = |E_1|^2 + \varepsilon |E_2|^2,\ F_3 = (|E_1|^2 - \varepsilon |E_2|^2) A^{-1},$$

$$R_3 = r_3 N_0^{-1}, \qquad R_+ = r_+ \exp\left[i(b_1 + b_2) \int_{-\infty}^{\tau} A(\tau')\,d\tau' \right],$$

$$F_+ = 2\exp\left[i(b_1 + b_2) \int_{-\infty}^{\tau} A(\tau')\,d\tau' \right] \begin{cases} E_1^* E_2, & \varepsilon = 1 \\ E_1^* E_2^*, & \varepsilon = -1 \end{cases}$$

here r_3 is the difference of the level populations of the transition, N_0 is the number of atoms, r_+ is the polarizability of the medium, t is the time and z the space coordinates, E_1 and E_2-are the slow envelopes of the fields with Ω_i, such that $\Omega_1 - \varepsilon\Omega_2 = \omega_0 + \tilde{\nu}_1$, ω_0 is the transition frequency, and $\tilde{\nu}_1$ is the detuning. The values of the constants b_i and $\varkappa_0$ is given in Ref. 7. In many experiments, a constant level-population difference is maintained with good precision[15] (10^{-4}–10^{-7}). Using this fact, it is possible to reduce the Maxwell-Bloch equation to the form (6). In experiments on observation of collective Raman scattering,[16] a different regime of interaction was observed, in which the pumping was practically not exhausted. This regime can be realized by introducing in the medium $N \gg N_0$ pump-field photons. In this case we also arrive at (6) with allowance for the substitutions

$$R_+ \leftrightarrow F_+,\ R_3 \leftrightarrow F_3,\ T \leftrightarrow X,\ \nu_1 = \tilde{\nu}_1/\varkappa_0 N_0.$$

For $\varepsilon = 1$ the stable (unstable) states (7) upon onset of the pumping correspond to the inverted (ground) state of the medium. For $\varepsilon = -1$ the ground (inverted) state of the medium is stable (unstable).

4. Four-wave interaction in a noncentrosymmetric medium with allowance for nonlinear frequency shift. It is known that for three-wave interaction in a medium with quadratic nonlinearity it is necessary, at sufficiently high field intensities, to take into account the nonlinear change of the frequency[17]

$$\Delta\omega = \sum_{i=1}^{3} \alpha_i |E_i|^2,$$

where E_i are the field envelopes and α_i are certain constants. Let the contribution of the field E_3 to this shift be small. This is possible if ω_3 is close to the frequency of the molecular transition or if the field intensities satisfy the condition $|E_{1,2}|^2 \gg |E_3|^2$, where ω_i is the carrier frequency of the field E_i. If the phase velocities of the fields E_1 and E_2 are also equal and the detuning ν is large, $\omega_i \gg \nu = \omega_1 - \varepsilon\omega_2 - \omega_3 \gg \Delta\omega$, the corresponding evolution equations reduce to the system (6). We omit the details to save space, and note only that $|F_+| \propto |E_1 E_2|$ and $|R_+| \propto |E_3|$. This model can be used in plasma theory[17] and to describe stimulated Brillouin scattering. The case $\varepsilon = 1$ corresponds to three-wave interaction, and $\varepsilon = -1$ corresponds to the explosive instability investigated in the ISTM in Ref. 18 for $\alpha_i = 0$.

5. One-photon interaction with allowance for the quadratic Stark effect. The Maxwell-Bloch equations of this model are

$$\begin{gathered} \left(\partial_z + \frac{1}{c}\partial_t \right) E(z,t) = \frac{2\pi\omega_0 N_0}{c}(dR + i\varkappa EN), \\ \partial_t R = -i\tilde{\nu}R + d\hbar^{-1}EN,\ \partial_t N = -d\hbar^{-1}(\bar{E}R + \bar{R}E). \end{gathered} \tag{11}$$

Here E is a slow envelope of a field of frequency $\omega = \omega_0 + \nu_0$, while ω_0 and d are the transition frequency and its dipole moment, $\varkappa = \varkappa_1 - \varkappa_2$, $\varkappa_1$ and $\varkappa_2$ are the polarizabilities of the levels, R is the polarizability of the transition, N is the population difference, and $\tilde{\nu} = \nu_0 + \varkappa\hbar^{-1}|E|^2$. Let $\nu_0 \gg \varkappa\hbar^{-1}|E|^2$, then the system (11) coincides with (6) after making the substitutions

$$g' = \varkappa\hbar\Omega^{1/2}d^{-2},\ \Omega = 2\pi N_0\omega_0 d^2\hbar^{-1},\ T = \Omega^{1/2}c^{-1}z,\ \delta = -1,$$

$$X = (t - z/c)\Omega^{1/2},\ \nu = -\nu_0/\Omega^{1/2},$$

$$R_+ = idE\hbar^{-1}\Omega^{-1}\sqrt{2},\ F_+ = R\sqrt{2},\ \varepsilon = 1.$$

The formulations of the Cauchy problem of USP evolution coincide with the known formulations of problems relating to self-induced transparency[3] and to amplification of a weak light pulse in a laser amplifier.[3,19]

3. CONSTRUCTION OF THE ISTM

The ISTM formalism presented here is a generalization of the corresponding results obtained by Kaup and Newell[20] for a differential nonlinear Schrödinger equation, and agrees with these results for $g'\nu = \delta\varepsilon$.

The system (6) can be represented in the form of the

$$\partial_{T'} L - \partial_Z A + [L, A] = 0$$

compatibility condition

$$\partial_Z\Psi = L\Psi = \begin{pmatrix} -i(\eta^2-D), & i\eta q_+\varepsilon \\ i\eta q_-, & i(\eta^2-D) \end{pmatrix}\Psi, \tag{12}$$

$$\partial_{T'}\Psi = A\Psi = \frac{g\eta}{\eta^2-1}\begin{pmatrix} -i(\eta/2)F_3 & iF_+\varepsilon \\ iF_-, & i(\eta/2)F_3 \end{pmatrix}\Psi, \tag{13}$$

where $\Psi=\Psi(Z,\ T',\ \eta)$, $q_+=2gR_+$, $Z=X(2g\varepsilon)^{-1}$, $T'=\delta\tilde{T}$, $D=1-g\nu\varepsilon$, $g=g'\delta$. We define the solutions of the spectral problem (12) (Jost functions) as follows:

$$\Phi\to\begin{pmatrix}1\\0\end{pmatrix}e^{i\xi z},\quad \bar\Phi\to\begin{pmatrix}0\\-1\end{pmatrix}e^{-i\xi z},\quad Z\to-\infty,$$

$$\Psi\to\begin{pmatrix}0\\1\end{pmatrix}e^{-i\xi z},\quad \bar\Psi\to\begin{pmatrix}1\\0\end{pmatrix}e^{i\xi z},\quad Z\to\infty,$$

where $\xi=\eta^2-D$.

The scattering coefficients a and b are given by the relations

$$\Phi=a\bar\Psi+b\Psi,\quad \bar\Phi=-\bar a\Psi+\bar b\bar\Psi, \tag{14}$$

where $a\bar a+b\bar b=1$. The functions Φ and Ψ satisfy the relations

$$\bar\Phi(\eta)=\begin{pmatrix}0&\varepsilon\\1&0\end{pmatrix}\Phi^*(\eta^*),\quad \bar\Psi(\eta)=\begin{pmatrix}0&1\\\varepsilon&0\end{pmatrix}\Psi^*(\eta^*), \tag{15}$$

from which it follows that

$$\bar a(\eta)=a^*(\eta^*),\ \bar b(\eta)=-\varepsilon b^*(\eta^*). \tag{16}$$

The analytic properties of Jost functions are similar to the case investigated by Kaup and Newell,[20] and will not be considered here in detail. We seek a solution in the class of functions that decrease rapidly as $Z\to\pm\infty$. The zeroes of $a(\eta_k)$ correspond to bound states, i.e., to soliton solutions. The discrete eigenvalue lies in quadrant I, and η_k correspondingly in quadrant III of the complex η plane, see Fig. 1. For the zeroes of $a(\eta_k)$ we have from (14)

$$\Phi(\eta_k)=b_k\Psi(\eta_k).$$

Let $q_\pm$ have a compact carrier, leading to the representation

$$\bar\Psi(\eta)\exp[i(\eta^2-D)Z]$$
$$=\begin{pmatrix}1\\0\end{pmatrix}e^{-iM}+\frac{1}{2\pi i}\int_c\frac{d\eta'}{\eta'-\eta}\frac{b(\eta')}{a(\eta')}\Psi(\eta')\exp[(i\eta'^2-iD)Z]. \tag{17}$$

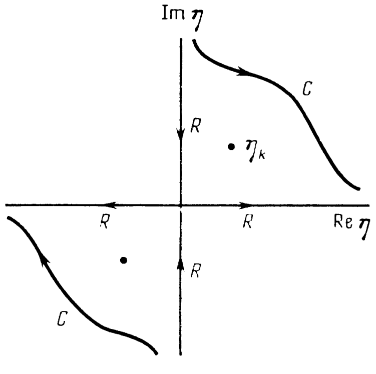

FIG. 1. Integration contours. Contour R was used in the derivation of Eqs. (22) and (23).

The integration contour is shown in the figure, and

$$M(Z)=\frac{\varepsilon}{2}\int_Z^\infty |q_+|^2\,dZ.$$

We represnt $\Psi(\eta)$ in the form

$$\Psi(\eta)=\begin{pmatrix}0\\1\end{pmatrix}\exp[i(\eta^2-D)Z+iM]$$
$$+\int_Z^\infty\exp[i(\eta^2-D)S]\begin{Bmatrix}\eta K_1(Z,S)e^{-iM(Z)}\\K_2(Z,S)e^{iM(Z)}\end{Bmatrix}dS, \tag{18}$$

where $K_{1,2}(Z,S)\to0$, $S\to\infty$. Substituting (18) in (12) we get

$$2K_1(Z,\ Z)=-i\varepsilon q_+(Z)\exp(2iM(Z)), \tag{19}$$

$$(\partial_Z-\partial_S+i\varepsilon|q_+|^2/2)K_1(Z,\ S)=i\varepsilon q_+K_2(Z,\ S)\exp(2iM(Z)), \tag{20}$$

$$(\partial_Z+\partial_S-i\varepsilon|q_+|^2/2)K_2(Z,\ S)=q_-\partial_SK_1(Z,\ S)\exp(-2iM(Z)). \tag{21}$$

It follows from (19)–(21) that there exists a solution $K_{1,2}(Z,S)$ that determines $q_+(Z)$ and vanishes at infinity ($S\to\infty$).

Substituting (18) in (17) and using the symmetry properties (15), we obtain the Marchenko-Ge'fand-Levitan equations ($y>Z$)

$$K_2^*(Z,y)+\int_Z^\infty\left(D+i\frac{\partial}{\partial y}\right)F(y+S)K_1(Z,S)\,dS=0, \tag{22}$$

$$-\varepsilon K_1(Z,y)+F^*(Z+y)+\int_Z^\infty K_2^*(S,Z)F^*(S+y)\,dS=0. \tag{23}$$

The function $F(Z)$ is of the form ($\eta^2=D+\xi$)

$$F(Z)=\frac{1}{2\pi}\int_c\frac{b(\eta)}{a(\eta)}e^{+i\xi z}\,d\eta. \tag{24}$$

4. SOLITON AND NON-SOLITON QUASI-SELF-SIMILAR ASYMPTOTIC SOLUTIONS

We substitute $\eta\to-\eta$ in (12) and (13) and obtain the simplest soliton solution corresponding to the eigenvalue $\eta=(D+\xi)^{1/2}$ of the spectral problem (12). We put

$$c_1=-b(\xi)/[2a'(\xi)(D+\xi)^{1/2}],\ a'(\xi)=(d/d\xi)a(\xi). \tag{25}$$

The function $F(Z)$ takes then the form

$$F(Z)=c_1\exp[-i\xi Z]. \tag{26}$$

The solution of the system of integral equations (22) and (23) with kernel (26) is of the form

$$K_1(Z,y)=c_1^*\exp[i\xi^*(Z+y)]$$
$$\left\{\varepsilon-\frac{D+\xi}{(\xi^*-\xi)^2}|c_1|^2\exp[2i(\xi^*-\xi)Z]\right\}^{-1}. \tag{27}$$

Using (19), we obtain the single-soliton solution

$$q_+(Z)=2i\varepsilon K_1(Z,Z)\exp\left\{2i\varepsilon\int_Z^\infty|K_1(Z',Z')|^2\,dZ'\right\}. \tag{28}$$

This solution agrees with the corresponding stationary solution of the nonlinear Schrödinger equation[2] at $g=0$ and the differential nonlinear Schrödinger equation[20] at $D=0$ ($\xi=2g\omega\varepsilon$). The dependence of q_+ on T is determined by the c_1 evolution which we obtain from the set of equations (13). For $F_\pm(0,T')=0$ we obtain

$$c_1(T)=c_1(0)\exp\left[i\frac{D+\xi}{1-D+\xi}g\int_{-\infty}^{T'}F_3(0,T')\,dT'\right]. \tag{29}$$

We rewrite the solution in a form more amenable to analysis:

$$q_+(Z,T)=4i\xi_0\varepsilon\exp\left[-i\left(2\xi_1 Z+\varphi+\operatorname{Re}B+\varepsilon\int_Z^\infty|q_+(Z_1'T)|^2dZ'-\theta_0(Z,T)\right)\right]\times[(e^{-2\psi}+We^{2\psi})^2+\xi_0^2e^{4\psi}]^{-1/2}, \tag{30}$$

where

$$\xi_1=\operatorname{Re}\xi,\quad \xi_0=\operatorname{Im}\xi,\quad Z_0=-\ln\left(\frac{|c_1|}{2\xi_0}\right)\frac{1}{2\xi_0},$$

$$c_1=|c_1|e^{i\varphi},\quad W=\varepsilon D+\varepsilon\xi_1,$$

$$\psi=\xi_0\left[(Z-Z_0)-\frac{1}{2}\frac{\Upsilon g}{\xi_0^2+(g\nu\varepsilon-\xi_1)^2}\right],$$

$$\Upsilon=\delta R_3\int_{-\infty}^{T}F_3(0,T)\,dT,$$

$$\theta_0=\arcsin\{\xi_0e^{2\psi}[(e^{-2\psi}+We^{2\psi})^2+\xi_0^2e^{4\psi}]^{-1/2}\},$$

$$B=\frac{D+\xi^*}{1-D-\xi^*}g\Upsilon.$$

Here $-\xi_0\partial\psi/\partial T$ is the soliton velocity, which can be assumed to be independent of T if $\partial_TF_3(0,T)\ll F_3(0,T)/T$.

The maximum field intensity is

$$I_{\max}=\frac{8\xi_0^2}{W+(W^2+\xi_0^2)^{1/2}} \tag{31}$$

and is reached at

$$Z=g\Upsilon\{2[\xi_0^2+(g\nu\varepsilon-\xi_1)^2]\}^{-1}+Z_0+{}^1/_8\ln|W^2+\xi_0^2|.$$

It is seen from (31) that a nonlinear frequency shift makes the soliton amplitude dependent on the linear detuning. As $\xi_0\to0$ two limiting values of $I_{\max}$ are possible, depending on the sign of W:

$$\lim_{\xi_0\to0}I_{\max}=8g^{-1}(|W|-W). \tag{32}$$

The soliton degenerates in this case into a plane wave with an amplitude that is constant of $W<0$.

It is known that for a stationary soliton solution it is necessary to use a "pulse" $R_+(X,0)$ of sufficiently high area $Q\gtrsim1$. Besides the soliton solutions, the system (6) contains non-soliton solutions that describe the decay of the unstable state (7) or (9). We assume that this decay is initiated by a weak "jolt" $Q\ll1$, after which the system evolves to a stable state. During the linear stage of the interaction, an instability develops, viz., F_+ increases exponentially. Let us estimate the characteristic scales of the quantities at which $|F_+|$ is small: $|F_+|\ll1$. Solving (6), we get

$$F_+(X,T)=\frac{F_0}{(2\pi\lambda_0)^{1/2}}\exp(\lambda_0+i\nu X+ig_1TF_3^0), \tag{33}$$

where

$$\lambda_0=2(-XT\delta\varepsilon R_3^0F_3^0)^{1/2},\quad F_3^0=F_3(0,0),\quad R_3^0=R_3(0,0).$$

The function F_0 can be defined in terms of $R_+(X,0)$, $F_+(0,T)$. For the function $R_+(X,0)$, which meets the condition (10)

$$\ln\left[\int_{-\infty}^{\infty}|F_+(0,T)|\,dT\right]^{-1}\gg1,$$

the solution (33) is valid in the self-similar-variable range $\lambda_0\gtrsim\lambda_0^{(0)}\gg1$. We use this property to obtain an asymptotic solution that describes the shape of the first field spike $R_+(X,T)$. The field $R_+(X,T)$ consists (at the exit from the system) of an infinite sequence of spikes having an amplitude that decreases as $\eta_0\to\infty$. Under real conditions, however, relaxation processes cause the contribution of the succeeding spikes to the field energy to become small (model 5 of Sec. 2). In any case, by analyzing the form of the first spike one can obtain important information on the character of the interaction and on the properties of the medium. Note that at sufficiently large T the linear solution can be "joined" to the asymptotic solution of the system (6) (cf. Ref. 19), but to explain the singularities of the behavior of the solution it is expedient to obtain the form of the first field spike $R_+(X,T)$ explicitly.

We seek the solution of Eqs. (22) and (23) with that part of the kernel which is determined by the continuous spectrum of the scattering problem

$$F(Z)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\xi)\exp(-i\xi Z)\,d\xi;$$

here $\rho(\xi)=b(\xi)/a(\xi)\eta$, $\eta=(D+\xi)^{1/2}$.

Let $q_\pm(X,0)$ be small: $|q_\pm(X,0)|\ll1$; then, iterating (12), we obtain accurate to $O(|q_\pm(X,0)|^2)$

$$b(\xi)=i\eta\int_{-\infty}^{\infty}q_-(Z)\exp(2i\xi Z)\,dZ,\qquad a(\xi)\approx1.$$

Taking into account the evolution of b with respect to T, which takes a form similar to (29), we get

$$F(Z,Y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\xi)\exp\left[-i\xi Z+i\frac{D+\xi}{1-D-\xi}g\Upsilon\right], \tag{34}$$

where

$$\rho(\xi)=-i\int_{-\infty}^{\infty}q_-(Z)\exp(2i\xi Z)\,dZ,\qquad \Upsilon=\delta R_3\int_{-\infty}^{T}F_3(0,T)\,dT.$$

We assume that $\rho(\xi)$ changes slowly enough (not faster than a power law) and Z is larger than the characteristic dimensions of the "perturbation" $q_-(Z)$. Under these assumptions the dynamics of the interaction process is determined mainly by a self-similar asymptotic relation. We obtain K_2 from (22) and, substituting it and the kernel of (34) in (23), we arrive at the following integral equation ($\xi=2g\omega\varepsilon$, $Z=\varepsilon X(2g)^{-1}$):

$$\varepsilon K_1(X,X')-F^*(X+X')$$
$$+\int_X^{\infty}\int dS\,dS'\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int d\omega\,d\omega'\,|\rho(2g\omega+g\nu\varepsilon)|^2$$
$$\times K_1(X,S')(D+2g\omega+g\nu\varepsilon)$$
$$\times\exp\left\{i\left[-\omega(S+S')-\frac{Y}{2\omega}-\frac{\nu}{2}(S+S')\right]\right.$$
$$\left.+i\left[\omega'^*(S+X')+\frac{Y}{2\omega'^*}+\frac{\nu}{2}(S+X')\right]\right\}=0,\qquad Y=\tilde{Y}\varepsilon. \tag{35}$$

We multiply (35) by $\exp(-i\omega X')$ and integrate with respect to X'. We obtain the function

$$K(X)=\int_X^{\infty}K_1(X,X')e^{i\omega' X'}\,dX'.$$

Next, substituting $K(X)$ in (35), we determine $K_1(X,X')$. We calculate the integrals with respect to ω and ω' and by the saddle-point method[21] for large λ_0. This solution is valid accurate to $O(\lambda_0^{-3/2})$ for $\lambda_0^{(0)}\gg 1$.

To calculate the integrals, we change to new variables:

$$\omega\to\Lambda=2\omega(-X/Y)^{1/2},\quad \omega'\to\Lambda'=2\omega'(-X/Y)^{1/2}.$$

We deform the integration contours in such a way that they pass through the points $\Lambda=i$ and $\Lambda'=-i$. We obtain ultimately

$$K_1(X,X')=\frac{1}{2}\left(\frac{-Y}{X}\right)^{1/2}\tilde{F}^*(X+X')$$
$$\times\left\{\varepsilon+\frac{1+2g\omega_+\varepsilon}{[2-i\nu(-X/Y)^{1/2}]^2}\tilde{F}^*(2X)\tilde{F}(2X)\right\}, \tag{36}$$

where

$$\tilde{F}(X+X')=\frac{1}{(2\pi\lambda')^{1/2}}\rho_0^*(2g\omega_-'+g\nu\varepsilon)$$
$$\times\exp\left[\lambda'+iY+i\nu\frac{(X+X')}{2}\right],$$

$$\rho_0(\vartheta)=-i\varepsilon\int_{-\infty}^{\infty}R_-(X,0)e^{-2i\vartheta\varepsilon X}\,dX,\qquad \lambda'=(-2(X+X')Y)^{1/2},$$
$$\omega_\pm'=\pm i\left(\frac{-Y}{2(X+X')}\right)^{1/2},$$
$$\omega_\pm=\pm i(-Y/4X)^{1/2}.$$

The solution for $q_+(X,T)$ takes the form (28) with the value $K_1(X,X')$ (36) on the characteristic $X=X'$. In the calculation of the integrals with respect to ω and ω' we assume that $\rho_0(\vartheta)$ is regular at zero. In this case, if $\rho_0(\vartheta)=r(\vartheta)\vartheta^{-2|a|-1}$ and $r(\vartheta)$ is a function regular at zero, Eq. (36) acquires a factor $\exp[-i\pi|a|]$ (Ref. 21). Thus, we have obtained an asymptotic solution (which is correct for $\lambda\gg 1$), modulated by a slowly varying function that is determined by the initial condition ($R_-(X,0)$), and having a power-law dependence on the variable $\sigma=(-Y/X)^{1/2}$. The maximum value of the modulus of the beam amplitude for variable λ and fixed σ is

$$\max_\lambda|q_+|=\frac{\sigma(4+\theta^2)}{\sqrt{2}\{[[4(1-\tilde{g}\nu)+\theta^2]^2+[\theta(4-\tilde{g}\nu)+4g\sigma]^2]^{1/2}+[4(1-\tilde{g}\nu)-\theta^2]\varepsilon\}^{1/2}} \tag{37}$$

where

$$\theta=\nu/\sigma,\quad \tilde{\lambda}=\lambda-\tilde{\lambda}_0-\ln(2\pi\lambda)^{1/2},\quad \lambda=2(-YX)^{1/2},$$
$$\tilde{\lambda}_0=-\ln|\rho_0^*(2g\omega_-+g\nu\varepsilon)|,\quad \tilde{g}=g\varepsilon=g'\delta\varepsilon,$$
$$|q_+|=\sigma(4+\theta^2)^2/\{[\varepsilon e^{-\tilde{\lambda}}(4+\theta^2)^2+(4-\theta^2-4\tilde{g}\nu)e^{\tilde{\lambda}}]^2$$
$$+[\theta(4-\tilde{g}\nu)+4\tilde{g}\sigma]^2e^{2\tilde{\lambda}}\}^{1/2}. \tag{38}$$

For $\theta\ll 1$ we obtain from (37)

$$\max_\lambda|q_+|=\frac{4}{\sqrt{2}}\frac{\sigma}{[(\tilde{W}^2+g^2\sigma^2)^{1/2}+\tilde{W}]^{1/2}}[1+O(\theta^2)], \tag{39}$$

where $\tilde{W}=(1-\tilde{g}\nu)\varepsilon$. Note the analogy with the soliton case (31). For $|g\sigma|\gg|\tilde{W}|$ we obtain from (39)

$$\max_\lambda|q_+|=\frac{4}{\sqrt{2}}\left(\frac{\sigma}{|g|}\right)^{1/2}\left[1+O\left(\frac{|\tilde{W}|}{|g\sigma|}\right)\right] \tag{40}$$

and for $|g\sigma|\ll|\tilde{W}|$,

$$\max_\lambda|q_+|=\left[1+O\left(\left|\frac{g\sigma}{|\tilde{W}|}\right|\right)\right]\begin{cases}2\sigma|\tilde{W}|^{-1/2}, & \tilde{W}>0\\ 4|\tilde{W}|^{1/2}|g|^{-1/2}, & \tilde{W}<0\end{cases}. \tag{41}$$

In the other limit, $\theta\gg 1$, it follows from (37) ($\tilde{U}=24-16g\nu+g^2\nu^2$) that

$$\max_\lambda|q_+|=\nu^{3/2}[\nu(1-\varepsilon)+{}^1/_2\theta^{-2}\tilde{U}]^{-1/2}[1+O(\vartheta^{-2})], \tag{42}$$

whence we have for $\varepsilon=-1$, $\nu\neq 0$

$$\max|q_+|=2^{-1/2}\nu[1+O(\theta^{-2})] \tag{43}$$

and for $\varepsilon=1$

$$\max|q_+|=\frac{\sqrt{2}\,\nu^{5/2}}{\sigma\tilde{U}^{1/2}}[1+O(\theta^{-2})]. \tag{44}$$

If $\nu=0$, we obtain in place of (42)–(44) ($g\sigma\ll 1$)

$$\max|q_+|=2\sigma(1+\varepsilon+g^2\sigma^2/2)^{-1}[1+O(|g\sigma|^2)]. \tag{45}$$

For $\nu\neq 0$, $\tilde{U}=0$, $\theta\gg 1$ we find in place of (44)

$$\max_\lambda|q_+|=\frac{\nu}{2(|1-\tilde{g}\nu|)^{1/2}}\left[1+O\left\{\max\left[\left(\frac{g\sigma}{\theta^2}\right)^2,\theta^{-2}\right]\right\}\right].$$

Let us obtain the maximum value of $|q_+(X,T)|$ attainable when σ is varied, and for a fixed λ. For $\theta\ll 1$ the maximum of $|q_+|$ is reached at infinity and is equal to

$$\max_\sigma|q_+|=(4/|g|)e^{-\tilde{\lambda}}[1+O(\theta)]; \tag{46}$$

if $\theta \gg 1$, then

$$\max_{\sigma} |q_+| = \frac{\nu}{|4-\tilde{g}\nu|} e^{\tilde{\lambda}} \left[1 + O\left(\frac{|g|\sigma}{\theta}\right)\right]. \tag{47}$$

5. DISCUSSION OF SOLUTIONS AND COMPARISON WITH EXPERIMENTAL RESULTS

The behavior of the soliton and non-soliton quasi-self-similar solution obtained in the preceding section is determined mainly by the two parameters g' and $g'\nu - \varepsilon\delta$. As seen from the asymptotic relations above, the limiting values of the amplitude can change qualitatively when the signs of $\widetilde{W}$ and W are reversed. The presence of a nonlinear frequency shift leads to stabilization of the "explosive instability," which takes place at $\varepsilon = -1, g = 0$. This effect is known in plasma theory.[17] What is new is that a stationary soliton can be formed within the framework of the considered model. The very same mechanism determines the feasibility of self-induced transparency for two-photon absorption in a two-level medium. For the quasi-self-similar solution describing the decay of an unstable initial state at $g = 0$, $\sigma \to \infty$, the amplitude of the spike (within the scope of the given mathematical model) tends to infinity in proportion to σ. This was demonstrated in Ref. 19 for a two-level laser amplifier. For $g \neq 0$, the character of the asymptotic relations changes qualitatively [see (39)–(45)]. Let us estimate the field amplitudes at which the nonlinear frequency shifts makes a contribution on the order of unity. For ruby, the dipole moment is $d \approx 6 \cdot 10^{-21}$ cgs esu and $\varkappa \sim 10^{-23}$ cm^3; we obtain the a field amplitude $|E_0| \approx 10^6$ V/cm, lower by two orders than the breakdown field. If linear detuning is present together with the nonlinear shift, the conditions for observing the effect become substantially more favorable. At $|g\nu| \sim 1$ one can observe an appreciable (on the order of unity) change of the soliton form even in weak fields $|E| \ll |E|_0$. Let us estimate the parameters of the medium: the condition $g\nu \approx 1$ is met for $d \approx 10^{-18}$–10^{-21} cgs esu, $\varkappa \approx 10^{-22}$–$10^{-25}$ cm^3, the detuning lies in the interval 10^7–10^{14} s$^{-1} \ll \omega_0 = 10^{15}$ s^{-1}. It can be seen that the effect can be discerned under the usual conditions for observation of self-induced transparency. For two-photon interaction we have $|g| \sim 1$ (e.g., for various transitions in cesium vapor $g_1 = -1.43$, $g_1 = -0.51$, Ref. 7). In the case of a four-photon interaction $|g|$ are functions of the carrier frequencies and can take on values $\gg 1$. Values $g\nu = \pm 1$ are easily reached in these models.

The dependence of the USP on the "number of particles" can exert a substantial influence on the pulse evolution in those problems where an important role is played by interference of fields. For example, in the case of a transition degenerate in the angular-momentum projections, such an interference can be the mechanism that "blurs" the pulse shapes and which is observed in experiments on collective Raman scattering in ortho-hydrogen.[22] One should expect here a more appreciable shortening of the pulse "tail." Interference can take place when the fields are not uniformly distributed relative to the transverse coordinates, i.e., it must be taken into account in the nondegenerate case.

We note in conclusion that the results of the present paper can be used also outside the scope of nonlinear optics. In particular, the set of equations (6) can be obtained by simplifying the equations used to describe wave interaction in a plasma,[8] an anisotropic chiral field,[9] and also for the description of stimulated Brillouin scattering.

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Translated by D. Medeiros