Boundary conditions for the wave function of the universe

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We examine possible solutions of the equations of quantum cosmology for the wave function of the universe. One minisuperspace model is employed to analyze the generality of inflationary regimes, and the other is used to analyze the quantum creation of the universe. We show that there exists a large class of wave functions corresponding to classical solutions with prolonged inflation which at the same time can be interpreted as wave functions describing the quantum creation of the universe.

1. INTRODUCTION

The basic requirements for a complete cosmological theory were formulated in Ref. 1. In our opinion, such a theory should describe both the present state of the universe and its quantum creation.

The idea of a quantum origin of the universe seems to be a mandatory feature of one recently proposed cosmological scenario, the scenario based on a model incorporating eternal inflation.[2,3] Without going into detail, we wish to emphasize that although the possibility of the quantum creation of the universe may not in fact be mandatory, it has not been ruled out within the context of this new theory.

In quantum cosmology, the past, present, and future of the universe are described by a wave function satisfying the Wheeler-De Witt (WD) equation. In a quasiclassical region, the quantum treatment provides only minor corrections to the equations of classical cosmology. But the creation of the universe, related as it is to tunneling and decay, can only be described by an essentially quantum cosmology.

One well known feature of the equations of classical cosmology is that they require initial conditions of one kind or another. Rather than eliminating the uncertainties of classical cosmology, the equations of quantum cosmology carry them to a new level—that of choosing boundary conditions for the wave function. A boundary condition that is frequently chosen is the one that leads to the well known Hartle-Hawking (HH) wave function.[4–6] That wave function possesses a number of attractive features, but it is neither unique nor obligatory. Furthermore, it seems to us that such phenomena as quantum tunneling and decay cannot be adequately handled by the HH function. At the same time, it would be desirable to construct just such a wave function, which on the one hand is consistent with present-day cosmological observations (or at least gives a reasonably long period of inflation), and on the other describes the quantum creation of the universe. A comparison of theoretical predictions with the cosmological data for the present epoch (for example, the spectrum of gravitational-wave perturbations) might even enable one (if the inflationary period is not too long) to either confirm or refute the very hypothesis of the quantum creation of the universe experimentally.[7]

In the present paper, we examine the role of boundary conditions for the wave function in two simple minisuperspace models. Both can be treated as limiting cases of one basic model, that of a massive scalar field in a closed ($K = +1$) Friedmann universe. In other words, we consider models with two degrees of freedom: the scale factor $R(t)$ of a uniform, isotropic Friedmann-Robertson-Walker (FRW) metric, and the homogeneous scalar field $\varphi(t)$ with mass M. With the appropriate changes of variable

$$a(t)=\left(\frac{3\pi}{2G}\right)^{1/2} R(t), \qquad \Phi(t)=\left(\frac{4\pi G}{3}\right)^{1/2} \varphi(t),$$
$$m=\left(\frac{3\pi}{2G}\right)^{-1/2} M,$$

the WD equation for the basic model takes the form[5]

$$\left\{\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a}-\frac{1}{a^2}\frac{\partial^2}{\partial\Phi^2}-Ka^2+m^2\Phi^2a^4\right\}\Psi(a,\Phi)=0. \tag{1.1}$$

The preferred choice of operator ordering factor in this equation is $p = 1$.

In the first limiting case, we neglect the term Ka^2 (which accounts for spatial curvature) in (1.1). The resulting equation can be viewed as a flat ($K = 0$) but topologically nontrivial FRW model; an example would be a flat torus with finite volume (see Refs. 5 and 8 for discussions of such a model).

For the second limiting case, we consider a scalar field that varies slowly in time. To this end, we neglect the square of the Φ-field momentum in (1.1), i.e., the term $a^{-2}\partial^2/\partial\Phi^2$, and replace $m^2\Phi^2$ by H^2. The resulting equation can be considered exact for a closed universe with cosmological constant $\Lambda = (9\pi/2G)H^2$ (in this instance, the preferred value for the ordering factor is $p = -1/2$).

Analysis of these two limiting cases enables one to produce a simplified but detailed description of two important stages in the possible evolution of the universe, namely the period of inflation and the quantum creation process. In the first, we will show that a broad class of wave functions (and not just HH functions) corresponds to the classical solution, given a long enough inflationary period. In the second, we will show that in some sense the HF function is close to unique. It gives a coefficient $D > 1$ (to be defined below) for the decay probability, which we believe casts doubt on the interpretation of the HH function as a description of the quantum creation of the universe. But small perturbations of the Hawking boundary condition give a set of wave functions with $D < 1$. It should be emphasized[1)] that the overwhelming majority of wave functions corresponds to $D < 1$, rather than $D > 1$.

Thus, we now raise the issue of the systematic study of all possible wave functions. It is possible that along the way,

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we will find the concept of a "secondary" wave function in the space of all possible wave functions to be a useful one.

Recall that in the classical regime, the model in question is homogeneous—the scale factor a and the scalar field Φ are functions of time only. But it is well known that inhomogeneities play a very important role. Zero-point quantum fluctuations are amplified during the inflationary stage, and they lead to various kinds of inhomogeneities. Perturbations with wavelengths less than the current Hubble radius r_H may be responsible for producing the observed structure of the universe. Perturbations with wavelengths longer than r_H can make the universe significantly irregular on scales much greater than the Hubble radius r_H. One pertinent question has to do with the range of applicability of the simplest minisuperspace models.

Let us consider the "dangerous" long-wave perturbations, $\lambda > r_H$. We now show that the minisuperspace approach employed in the present paper is very widely applicable, since long-wavelength perturbations are not immediately enhanced. The mean squared amplitude of metric fluctuations produced during the inflationary stage is given by the integral of a "flat" spectrum:

$$\langle h^2 \rangle \approx H^2 \int_{\nu_m}^{\nu_H} \frac{d\nu}{\nu}, \tag{1.2}$$

where $\nu_H = c/r_H$, and H is the Hubble parameter (in Planck units) during the inflationary stage. The most important contribution to the integral (1.2) comes from low frequencies, $\nu_m \ll \nu_H$. The quantity $\langle h^2 \rangle$ attains values of order unity if the integration is carried out down to low enough frequencies ν_m, corresponding to wavelengths $\lambda_m \approx r_H \exp(1/H^2)$. (For $H \approx 10^{-5}$, we get $\lambda_m \approx r_H \exp(10^{10})$.) Starting with (1.2), we can also estimate the duration of the inflationary period $\Delta t = t_2 - t_1$ needed to obtain such a broad spectrum. Since

$$\langle h^2 \rangle \approx H^2 \ln \frac{\nu_H}{\nu_m} \approx H^2 \ln \frac{\exp(Ht_2)}{\exp(Ht_1)} = H^2 (H\Delta t),$$

the requirement that $\langle h^2 \rangle \approx 1$ leads to the condition $H\Delta t \approx H^{-2}$. We thereby obtain for Δt a much longer period of inflation, $H\Delta t \approx 70$, than the minimum necessary. This implies that minisuperspace models are widely applicable.

The possible inhomogeneity of the universe on scales greater than r_H, and the constraints imposed on this possible inhomogeneity by the observed isotropy of the microwave background radiation, $\Delta T/T < 10^{-4}$, were investigated in Ref. 12 (see also Refs. 13 and 14) before the idea of inflation came upon the scene. It was shown there that larger perturbations, of order unity, were consistent with the $\Delta T/T$ data only if the typical wavelengths of these perturbations were large enough; specifically, they should satisfy $\lambda \gtrsim 10^2 r_H$.

The model of a highly inhomogeneous universe consisting of a collection of almost noninteracting parts has become a popular topic of recent research.[2,3] The assertion is that individual regions in which inflation has come to an end are essentially isolated from a huge inflationary volume filled with a fluctuating scalar field whose amplitude takes on very high values. The proponents of this view stress that within the scope of such a hypothesis, one can manage without any assumptions about the quantum origin of the universe. In our opinion, this alternative approach more likely indicates that a minisuperspace model can be used only as a so-called intermediate asymptote, while in general a full superspace treatment is required. As a first step, we can examine a minisuperspace in which only one or a few degrees of freedom characterizing the inhomogeneities are taken into account. We hope to return to this problem in the future.

With these prefatory remarks, let us proceed to a detailed study of these two minisuperspace models.

2. FLAT MODEL ($K=0$) WITH A MASSIVE SCALAR FIELD

Neglecting the Ka^2 term in Eq. (1.1), we obtain the WD equation

$$\left\{ \frac{1}{a}\frac{\partial}{\partial a} a \frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial \Phi^2} + m^2\Phi^2 a^4 \right\} \Psi(a, \Phi) = 0, \quad p = +1. \tag{2.1}$$

In a $K = 0$ model there is no potential barrier separating classically distinct regions. We will therefore investigate Eq. (2.1) with the aim of studying inflationary and noninflationary solutions, and not in order to analyze the problem of creation. The problem of quantum creation (by tunneling or decay) will be examined in the next section. Reference 15 contains a detailed analysis of all classical solutions of the present model (see also Refs. 16–18).

In the quasiclassical approximation, the wave function $\Psi(a, \Phi)$ is of the form

$$\Psi(a, \Phi) = \exp\,[iS(a, \Phi) + i\sigma(a, \Phi) + \ldots], \tag{2.2}$$

where S and σ satisfy the equations

$$-\left(\frac{\partial S}{\partial a}\right)^2 + \frac{1}{a^2}\left(\frac{\partial S}{\partial \Phi}\right)^2 + m^2\Phi^2 a^4 = 0, \tag{2.3}$$

$$i\frac{\partial^2 S}{\partial a^2} - 2\frac{\partial S}{\partial a}\frac{\partial \sigma}{\partial a} + \frac{i}{a}\frac{\partial S}{\partial a} - \frac{i}{a^2}\frac{\partial^2 S}{\partial \Phi^2} + \frac{2}{a^2}\frac{\partial S}{\partial \Phi}\frac{\partial \sigma}{\partial \Phi} = 0. \tag{2.4}$$

Equation (2.3) is the Hamiltonian-Jacobi equation for the action S. A real solution of (2.3), which describes the classical dynamics of the model, can be represented in the form

$$S(a, \Phi) = -a^3 f(\Phi). \tag{2.5}$$

An unimportant additive constant has been omitted here. The function $f(\Phi)$ satisfies the ordinary differential equation

$$9f^2 - (df/d\Phi)^2 = m^2\Phi^2. \tag{2.6}$$

The classical equations of motion are obtained from (2.5) and the system Lagrangian

$$L = \frac{1}{2}(-a\dot{a}^2 + a^3\dot{\Phi}^2 - a^3 m^2 \Phi^2)$$

in the usual way:

$$\frac{\partial L}{\partial \dot{a}} = -a\dot{a} = \frac{\partial S}{\partial a}, \qquad \frac{\partial L}{\partial \dot{\Phi}} = a^3\dot{\Phi} = \frac{\partial S}{\partial \Phi}.$$

This then gives the relations

$$\dot{a}/a = 3f, \quad \dot{\Phi} = -f'. \tag{2.7}$$

The prime here signifies differentiation with respect to Φ, and the dot denotes differentiation with respect to time. We have used the dimensionless time t (i.e., the time expressed in Planck units), which is related to the physical time T by

$$dt/dT=(3\pi/2G)^{1/2}.$$

We make use of the definitions (2.7), and assume that we have expansion; i.e., we have $f>0$, $S<0$ corresponding to $\dot{a}>0$. But the functions $f<0$, $S>0$ can also describe expansion if we make the change of variable $t\to -t$ in Eq. (2.7). Differentiating (2.7) with respect to t and using (2.6), we can obtain the equations of motion in the usual form:

$$\ddot{\Phi}+3(\dot{a}/a)\dot{\Phi}+m^2\Phi=0,\quad (\dot{a}/a)^2=\dot{\Phi}^2+m^2\Phi^2,$$

$$(\dot{a}/a)^{\cdot}+(\dot{a}/a)^2=-2\dot{\Phi}^2+m^2\Phi^2. \tag{2.8}$$

These equations of motion are invariant under the transformation $t\to -t$. The three equations of motion can be combined to give one, in which the time parameter t does not appear:

$$\Phi\frac{d^2\Phi}{d\alpha^2}+\left(3\Phi\frac{d\Phi}{d\alpha}+1\right)\left[1-\left(\frac{d\Phi}{d\alpha}\right)^2\right]=0.$$

(For the sake of convenience, we use the variable $\alpha=\ln a$ from here on.) This equation completely describes the classical trajectories. The direction of motion is determined by the choice of direction of time.

All trajectories of the model (2.8) in the $(\Phi, \dot{\Phi})$ phase plane have previously been found,[15] and it has been shown that in the case of expansion (i.e., $\dot{a}>0$), the trajectories all start out from two ejecting nodes. Apart from these trajectories, there are also two attracting separatrices that originate at two saddle points. The solutions of Eq. (2.6) have the following asymptotic behavior for trajectories that start out from the nodes:

$$f\approx Ce^{\pm 3\Phi},\quad C^2e^{\pm 6\Phi}\gg m^2\Phi^2,\quad C=\text{const},$$

and for the separatrices

$$f\approx \pm{}^1/_3 m\Phi,\quad 9\Phi^2\gg 1.$$

Different values of C select different trajectories leaving the nodes. Consequently, the choice of a definite solution of Eq. (2.6) gives a definite function S and a definite classical trajectory.

We will distinguish different solutions of Eq. (2.6) by the subscript n, which varies continuously and takes on two values, corresponding to the separatrices. By virtue of the linearity of the WD equation, we can symbolically write a more general solution of Eq. (2.1) to lowest order in the form

$$\Psi=\sum_n \exp(iA_n+iS_n),\qquad S_n=-\exp(3\alpha)f_n,\qquad A_n=\text{const}.$$

One important property of the classical equations for the case in which $K=0$ is their scale-invariance; that is, the function $a(t)$ itself does not appear in Eq. (2.8). This property has its counterpart in the lowest-order approximation to the quantum version of the theory. In fact, every quasiclassical wavefunction $\Psi_n=\exp(iS_n)$ can be put into correspondence with a family of normals to the surfaces $S_n=\text{const}$. These surfaces are constructed in the minisuperspace (α, Φ), with metric tensor

$$G^{\mu\nu}=e^{-3\alpha}\,\mathrm{diag}(-1,+1),\quad \mu,\nu=1,2,\quad x^1=\alpha,\quad x^2=\Phi. \tag{2.9}$$

The vector normal N_μ to $S_n=\text{const}$ can be obtained by acting on $\Psi_n=\exp(iS_n)$ with the momentum operators $\hat{\pi}_\alpha$ and $\hat{\pi}_\Phi$:

$$\hat{\pi}_\alpha\Psi_n=\frac{1}{i}\frac{\partial}{\partial\alpha}\Psi_n=\frac{\partial S_n}{\partial\alpha}\Psi_n=N_\alpha\Psi_n,$$

$$\hat{\pi}_\Phi\Psi_n=\frac{1}{i}\frac{\partial}{\partial\Phi}\Psi_n=\frac{\partial S_n}{\partial\Phi}\Psi_n=N_\Phi\Psi_n.$$

In the case at hand we have $K=0$, and the family of normals and associated tangent vectors $N^\alpha=3f$, $N^\Phi=f'$ are independent of α, and transform back into themselves under $\alpha\to\alpha+\text{const}$.

Since the vector (N^α, N^Φ) points in the same direction as the momentum vector (π^α, π^Φ), the normals trace out classical trajectories in (α, Φ) space. Therefore, invariance of the family of normals under the displacement $\alpha\to\alpha+\text{const}$ signifies that for a given S_n, the curves traced out by the normals are all copies of the same classical solution in the $(\Phi, \dot{\Phi})$ plane. This is a manifestation of the fact that the classical solutions are independent of $a(t)$.

Integrating the relation $d\alpha/d\Phi=N^\alpha/N^\Phi=-3f/f'$ along every classical path in the (α, Φ) plane, we obtain $z(\alpha,\Phi)=\text{const}$, where $z\equiv\alpha+3\int(f/f')d\Phi$.

It will also prove useful to introduce the expression for the square of the normal vector,

$$N^2=N^\alpha N^\alpha G_{\alpha\alpha}+N^\Phi N^\Phi G_{\Phi\Phi}=-m^2\Phi^2e^{3\alpha}.$$

For the solutions $f\approx Ce^{\pm 3\Phi}$, the normal vector becomes isotropic, with $N^2\approx 0$.

Our most immediate problem is to construct a quantity that is conserved along every classical trajectory in the $(\Phi, \dot{\Phi})$ phase space. We might be able to integrate the quantity thus constructed as a relative weight along a given classical trajectory, thereby comparing the extensibility of the inflationary and non-inflationary solutions. With this in mind, let us return to Eq. (2.4). The general solution for σ_n can be expressed in terms of the function $f_n(\Phi)$:

$$\sigma_n(\alpha,\Phi)=(i/2)(3\alpha+\ln f_n')+\tilde{B}_n(z),$$

where $\tilde{B}_n$ is an arbitrary function of its argument z, satisfying the condition that Ψ_n be quasiclassical. Taking advantage of this arbitrariness, $\tilde{B}_n(z)$ can incorporate the constant A_n from the solution of (2.5), giving $B_n(z)=\tilde{B}_n(z)+A_n$. In what follows we shall assume that this has been done.

In the present approximation, the general solution of the WD equation can be written in the form

$$\Psi=\sum_n\Psi_n=\sum_n\exp[i(S_n+\sigma_n)].$$

For each Ψ_n, we define a current:

$$j_\mu^{(n)}=\frac{i}{2}\left(\Psi_n^*\frac{\partial\Psi_n}{\partial x^\mu}-\frac{\partial\Psi_n^*}{\partial x^\mu}\Psi_n\right)$$

$$\approx-\exp[i(\sigma_n-\sigma_n^*)]\frac{\partial S_n}{\partial x^\mu},\quad \mu=1,2.$$

For every n, the current $j_\mu^{(n)}$ (the superscript n will be omitted where this will not be confusing) possesses two important properties. First, it satisfies the continuity equation

$$G^{\mu\nu} j_{\mu;\,\nu}=0, \tag{2.10}$$

which is fully covariant in (α, Φ) space; using (2.9), this equation reduces to

$$e^{-3\alpha}(-j_{\alpha,\,\alpha}+j_{\Phi,\,\Phi})=0\ . \tag{2.11}$$

Second, in (α, Φ) space, curves of current j^μ are identical to the normal curves N^μ to the surface S_n = const. The components of the current j_μ take the values

$$j_\alpha=3\frac{f}{|f'|}\chi^2(z), \quad j_\Phi=\frac{f'}{|f'|}\chi^2(z),$$

where $\chi(z) = \exp(-\operatorname{Im} B(z))$.

Let us now write down the integral form of Eq. (2.11):

$$\int_W e^{-3\alpha}(-j_{\alpha,\alpha}+j_{\Phi,\Phi})\,e^{+3\alpha}\,d\alpha\,d\Phi=\int_W(-j_{\alpha,\alpha}+j_{\Phi,\Phi})\,d\alpha\,d\Phi=0. \tag{2.12}$$

The region of integration W in (α, Φ) space is chosen as follows. Its boundary ∂W consists of the two current curves z_1 = const, z_2 = const and the two straight lines $\Phi = \Phi_1$ and $\Phi = \Phi_2$ (see Fig. 1). The lower current curve corresponds to $z = z_1$, and the upper to $z = z_2 = z_1 + \delta$. Proceeding then to the limit $\delta \to 0$, we find from (2.12) that we can obtain a Q characterizing any given solution Ψ_n and any given current curve z = const:

$$Q=\lim_{\delta\to 0}\frac{1}{\delta}\int_{z_1}^{z_2} j_\Phi(z, \Phi_1)\,dz$$

$$=\lim_{\delta\to 0}\frac{1}{\delta}\int_{z_1}^{z_2} j_\Phi(z, \Phi_2)\,dz=j_\Phi(z_1, \Phi_1)$$

$$=j_\Phi(z_1, \Phi_2)=\chi^2(z)\frac{f'}{|f'|}.$$

Restoring the subscript n, we may write more accurately

$$Q_n=\chi_n{}^2(z)\frac{f_n{}'}{|f_n{}'|}.$$

The number Q_n is conserved along any current curve $z(n)$ = const. The particular value of Q_n is determined by the chosen boundary conditions for the wave function, and specifically by the function $\chi_n(z)$.

Recall that to lowest order, i.e., when $\Psi_n = \exp(iS_n)$, the curves $z(n)$ = const are all copies of a single phase trajectory in (Φ, Φ) phase space. Since Q_n depends in general on z, the curves $z(n)$ = const now become different, carrying as they do different values of Q_n. This is consistent with the fact that even to only the next-highest order, i.e., when $\Psi_n = \exp[i(S_n + \sigma_n)]$, the scale factor a acquires a sizable absolute value. Compactification of a flat three-dimensional space, as by reduction to a flat torus, introduces length scales that are related to a. These quantities do not change the local classical evolution of the system [Eq. (2.8) remains the same], but they do change the absolute value of the action, and they have an effect on quantum corrections.

Over a limited class of boundary conditions for the wave function Ψ_n, namely when one assumes that χ_n is independent of z (but not of n!), all trajectories z = const for given n take on the same value Q_n. In this special case, which

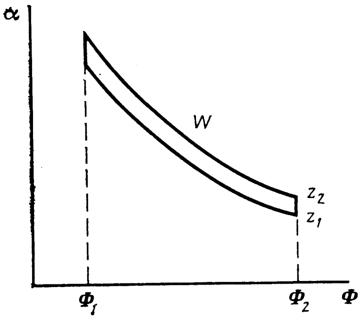

FIG. 1. Region of integration W in (α, Φ) space.

we will consider below, a certain value of Q_n is ascribed to a given phase trajectory in the (Φ, Φ) plane, and is conserved along that trajectory. The number Q_n can therefore be specified at any point on the trajectory, in particular where it crosses a quantum boundary given by the condition $\Phi^2 + m^2\Phi^2 \approx 1$.[15] It was precisely on the quantum boundary that the inflationary and noninflationary solutions were equated in Ref. 15, where it was assumed that all points on the quantum boundary were equally likely.

The assertion that Q_n is conserved along an entire classical trajectory in the (Φ, Φ) plane must be refined if either of the functions $f_n(\Phi)$ is multiple-valued. Each branch describes its own section of a classical trajectory between two adjacent turning points ($f_n' = 0$), or in other words between neighboring points at which $\Phi = 0$. We assume that Ψ_n incorporates all branches of the function f_n needed to specify all oscillatory modes of the field Φ and any trajectory.

Up to this point, we have only considered individual functions Ψ_n. We now embark on a discussion of an arbitrary wave function

$$\Psi=\sum_n \Psi_n.$$

This can be represented in the form

$$\Psi=\sum_{n'}\Psi_{n'}+\sum_{n''}\Psi_{n''}, \tag{2.13}$$

where the functions $\Psi_{n'}$ specify phase trajectories with a fairly long inflationary period (as in Ref. 15, we call such trajectories *favorable*), and the functions $\Psi_{n''}$ specify all the rest (unfavorable). There are infinitely many phase trajectories overall, but we will start out by discussing a finite number N (although N can be as large as we please). In Eq. (2.13), there will then be N' terms, $0 \leqslant N' \leqslant N$, describing favorable trajectories, and $N - N'$ describing unfavorable ones.

We can justify the consideration of a finite number of classical trajectories in the following manner. Assume that for some reason (related, perhaps, to the fairly low accuracy of astrophysical observations) the difference between nearby trajectories is not too important for our purposes. Then an entire bundle of our neighboring trajectories (where the thickness of the bundle depends on the accuracy of the theoretical predictions or experimental results) can be replaced by a single trajectory from that bundle—the mean, for example. Since each bundle has a finite thickness, we ultimately obtain a finite number of trajectories characterizing the problem at hand.

Since both favorable and unfavorable trajectories contribute to Eq. (2.13), every wave function can be characterized by a number that expresses the "degree of inflation" of that function, and that number should satisfy the following requirements:

a) $0 \leqslant P \leqslant 1$, with $P = 1$ if Ψ contains only favorable solutions, and $P = 0$ if it contains only unfavorable ones;

b) P is the same for wave functions differing only by an overall normalization factor. A reasonable definition of P is

$$P = \sum_{n'=1}^{N'} |Q_{n'}| \Big/ \sum_{n=1}^{N} |Q_n|. \tag{2.14}$$

It can readily be shown that this expression satisfies both of the foregoing requirements.

So far we have been discussing the degree of inflation of some single wave function, but of course we do not know which of the functions (2.13) will actually be realized. We may therefore ask what the mean value of P is when the wave function is chosen randomly. Answering this question would enable us to assess the extent to which inflation is a typical property of wave functions of the universe.

In order to do so, we introduce the solution space of the WD equation. Recall that we are considering those wave functions Ψ that describe an ensemble of data from N classical trajectories. Every point in this space is characterized by coordinates $\{\chi_n\}$, where by virtue of requirement (b) above, the value of P is independent of the radius of the sphere

$$\sum_{n=1}^{N} \chi_n{}^2 = \text{const} .$$

If we take a unit sphere for the sake of definiteness and assume that there are no preferred points on the sphere, we obtain

$$\bar{P} = \int_S P(\chi_1, \ldots, \chi_N)\, d\omega = \int_S \sum_{n'=1}^{N'} \chi_{n'}{}^2\, d\omega = \frac{N'}{N}, \tag{2.15}$$

for the mean value of P, where dw is an element of area on the sphere S defined by

$$\left(\sum_{n=1}^{N} \chi_n{}^2 = 1\right).$$

Thus, if P is chosen in the form (2.14) and we assume equal likelihood for all points on S, the mean degree of inflation is simply the ratio of the number of favorable trajectories to the total number of trajectories. Trajectories can be chosen in the following way: partition the quantum boundary in $(\Phi, \dot{\Phi})$ phase space into N individual sections, and replace each bundle of trajectories traversing a given section by a single one. Then for sufficiently large N,

$$\bar{P} = 1 - b\,(m/m_p),$$

where $b =$ const is of order unity; in other words, we revert to the result obtained in Ref. 15.

As a second example, let us now consider a wave function analogous to the Hartle-Hawking function. The HH function itself was constructed for the case $K = +1$. Neglecting terms with spatial curvature, we obtain

$$\Psi \sim \frac{\exp(-3d/2)}{m^{1/2}} \left[\exp\left(-i\frac{m}{3}\Phi e^{3\alpha}\right) + \exp\left(+i\frac{m}{3}\Phi e^{3\beta}\right)\right]. \tag{2.16}$$

for $K = 0$. The function (2.16) differs somewhat from the corresponding expression in Ref. 6, since for the latter it was assumed that $\Phi =$ const. Both terms in (2.16) describes separatrices. In the present instance, we immediately obtain $P = 1$, since we have chosen a single unique (favorable) trajectory. However, we know that many other trajectories exist that are not separatrices but that are perfectly suitable from the standpoint of the duration of the inflationary period. Thus, in that regard, the HH function is no better than many others.

We can generalize the preceding equations to the case of arbitrary P and to a treatment of the continuum of classical trajectories as follows. We replace the sum over n and n' by integration over the continuous versions of these parameters. The solution space then becomes infinite-dimensional, and the sets $\{n\}$ and $\{n'\}$ take on the cardinality of the continuum. A value of P is specified at every point in this space, and requirement (b) separates out the set of surfaces on which the numbers P are distributed in the same way. For definiteness, we choose one of these surfaces and specify a metric thereon. The expression for P on this surface is a continuous integral, instead of the integral (2.15). The specification of P in one form or another reflects the extent of our information about the correspondence between the theoretical predictions for some choice of Ψ and the observational data for the actual universe.

In the present section, we have examined the case $K = 0$. However, difficulties will ensue if one attempts to carry out this procedure for computing Q_n when $K \neq 0$, due to a lack of shift invariance in the α-coordinate. Rather than the function $f(\Phi)$, one then has a function of two variables $\tilde{f}(\alpha, \Phi)$. But these may well be merely technical difficulties. For $K \neq 0$, the modulus of the current $|j|$ takes on the role of the charge of the classical trajectory, and is a function of the boundary conditions.

3. CLOSED MODEL WITH A Λ TERM

Neglecting the momentum of the Φ field, we may write Eq. (1.1) in the simplified form

$$\left\{\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da} - a^2 + H^2a^4\right\}\Psi(a) = 0, \tag{3.1}$$

where

$$H^2 = \frac{2G}{9\pi}\Lambda = \text{const}$$

and the value of p has yet to be fixed. As will be explained below, this model is a suitable one for discussing the quantum tunneling process.

It can be shown that if $\Psi(a)$ is a solution of Eq. (3.1) for some p, then $\tilde{\Psi}(a) = a^{1-p}\Psi(a)$ is a solution of the same equation for $\tilde{p} = 2 - p$. The present authors have previously found exact solutions for $p = 3$ and $p = -1$ (the latter was first discussed in Ref. 20). For $p = -1$, the general solution of (3.1) takes the form

$$\Psi(a) = u^{1/2}\left[A_1 H_{1/3}^{(1)}\left(\frac{u^{3/2}}{3H^2}\right) + A_2 H_{1/3}^{(2)}\left(\frac{u^{3/2}}{3H^2}\right)\right], \quad H^2a^2 \geqslant 1, \tag{3.2}$$

$$\Psi(a) = (-u)^{1/2}\left[\tilde{A}_1 I_{1/3}\left(\frac{(-u)^{3/2}}{3H^2}\right) + \tilde{A}_2 K_{1/3}\left(\frac{(-u)^{3/2}}{3H^2}\right)\right], \quad H^2a^2 \leqslant 1, \tag{3.3}$$

where $u \equiv H^2a^2 - 1$, $I_{1/3}$, $K_{1/3}$, and $H^{(1),(2)}_{1/3}$ are respectively modified Bessel functions and Hankel functions, and A_1, A_2, and $\tilde{A}_1$, $\tilde{A}_2$ are the two pairs of arbitrary complex coefficients of the two linearly independent solutions. The continuity of $\Psi(a)$ and $d\Psi(a)/da$ at $a = 1/H$, gives the relations between these coefficients:

$$\tilde{A}_1=-A_1(1+i\cdot 3^{1/2})-A_2(1-i\cdot 3^{1/2}),\quad \tilde{A}_2=i(2/\pi)(A_2-A_1). \tag{3.4}$$

From here on we will deal mainly with the exact solution (3.2), (3.3) for $p = -1$, but the principal results remain valid for other p as well.

In order to make the subsequent treatment more accessible, let us briefly discuss an auxiliary model that includes, along with the Λ term, a certain amount of the conformally invariant massless scalar field φ. The total wave function will then depend on both a and φ, but it is factorizable: $\overline{\Psi}(a, \varphi) = \chi(\varphi)\Psi(a)$. The part that depends solely on a then satisfies the equation

$$\left\{\frac{1}{a^p}\frac{d}{da}a^p\frac{d}{da}-a^2+H^2a^4+\varepsilon\right\}\Psi(a)=0, \tag{3.5}$$

where ε is a constant that represents the addition made by the massless scalar field. Equation (3.5) can be looked upon as the Schrödinger equation for a particle of energy ε, moving in an external potential $V(a) = a^2 - H^2a^4$. The potential barrier for $0 < \varepsilon < 1/(4H^2)$ separates two classically distinct regions with turning points a_1 and a_2 (see Fig. 2). In the quantum version of the theory, there is a possibility of tunneling between these two regions.

Returning to our case with $\varepsilon = 0$, we should probably speak of quantum decay rather than quantum tunneling, since for $\varepsilon = 0$ there is only one classically allowed region, $a > 1/H$. (This situation could be called quantum creation "out of nothing.")

In conventional quantum mechanics, the tunneling probability is given in the quasiclassical approximation by $|\Psi(a_2)|^2/|\Psi(a_1)|^2$. In this expression, one uses a wave function appropriate to the specific problem at hand.[19]

We may define the probability of decay as

$$D=|\Psi(1/H)|^2/|\Psi(0)|^2,\quad p<1.$$

For the given wave function to be interpretable as a descriptor of quantum decay, it must at least give $D < 1$, rather than $D > 1$.

An exact expression for D is readily obtained for the case $p = -1$, using Eq. (3.3). We introduce the notation

$$\tilde{A}_1=|\tilde{A}_1|e^{i\beta_1},\quad \tilde{A}_2=|\tilde{A}_2|e^{i\beta_2},\quad |\tilde{A}_1|/|\tilde{A}_2|=x,\quad \beta_1-\beta_2=\beta.$$

The expression for D simplifies when $H \ll 1$. To leading order in H, then, we obtain

$$D\approx\gamma H^{-2/3}(x^2e^{2/3H^2}+2\pi x\cos\beta+\pi^2e^{-2/3H^2})^{-1}, \tag{3.6}$$

where γ is a constant of order unity. Since $-1 \leqslant \cos\beta \leqslant 1$, D essentially depends only on x.

Now consider the plane with coordinates $y = \tan^{-1}x$ and β (Fig. 3). The straight line $y_0 \approx \exp(-1/3H^2)$ separates the regions $D > 1$ and $D < 1$. Every specific choice of coefficients $\tilde{A}_1$, $\tilde{A}_2$ determines a specific wave function and an associated value D. For example, the Hartle-Hawking function entails the choice $\tilde{A}_1 = 0$, $y = 0$, giving $D = \exp(2/3H^2) \gg 1$, and that is why the HH function can scarcely be interpreted as the describing process for tunneling or decay. But Eq. (3.6) and Fig. 3 make it clear that a minor change in the Hawking boundary condition $\tilde{A}_1 = 0$ is all that is needed to obtain $y \gtrsim \exp(-1/3H^2)$, with a corresponding wave function having $D < 1$. It would be reasonable to presume that there are no preferred values of y or β.[2)] We can therefore introduce the idea of a probability for the realization of wave functions with $D < 1$ and $D > 1$. Taking that point of view, the probability P of finding a wave function with $D > 1$ is very low:

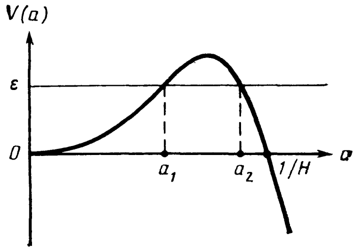

FIG. 2. The form of the potential $V(a)$.

$$P(D>1)\approx\exp(-1/3H^2), \tag{3.7}$$

inasmuch as it is determined by the very narrow band $\Delta y \approx \exp(-1/3H^2)$ in the (y, β) coefficient space. Figure 3 shows directly that the vast majority of wave functions have $D < 1$, rather than $D > 1$. (There is one wave function among those with $D < 1$ that corresponds to the boundary condition described by Vilenkin[20]: $A_1 = 0$, $D \approx \exp(-2/3H^2)$.)

The result we have obtained is valid for other values of p as well. This becomes obvious if in the definition of D, we replace (3.3) by the quasiclassical approximation for the wave function. In this approximation, the wave function is well known to be independent, in general, of the ordering factor p:

$$\Psi(a)\approx\tilde{A}_1\exp\left[\frac{1}{3H^2}(1-H^2a^2)^{3/2}\right]+\tilde{A}_2\exp\left[-\frac{1}{3H^2}(1-H^2a^2)^{3/2}\right].$$

We assumed above that all values of y and β were equally likely. We now give up this assumption and consider a more general case. Let us introduce a measure $\rho(y, \beta)$ in

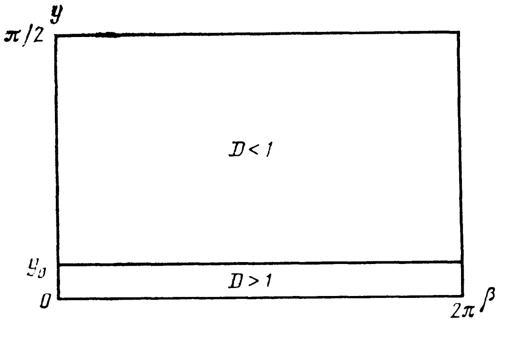

FIG. 3. The regions $D < 1$ and $D > 1$ in the (y, β)-plane.

(y, β)-space; this is essentially a measure in the solution space of (3.2) and (3.3). Let the function $\rho(y, \beta)$ be normalized by

$$\int_0^{\pi/2} dy \int_0^{2\pi} d\beta\, \rho(y, \beta) = 1. \tag{3.8}$$

Then in the general case, the probability of realization of a solution with $D < 1$ is given by

$$P(D<1) = \int_0^{\pi/2} dy \int_0^{2\pi} d\beta\, \rho(y, \beta)\, \theta\left(1 - \frac{|\Psi(1/H; y, \beta)|^2}{|\Psi(0; y, \beta)|^2}\right), \tag{3.9}$$

where

$$\theta(v) = \begin{cases} 1, & v>0 \\ 0, & v<0 \end{cases}.$$

The case of uniformly distributed y and β considered above corresponds to choosing a measure of the form $\rho(y, \beta) = 1/\pi^2$. The introduction of a measure in wave-function space is a manifestation of our ignorance of the actual state of the universe. One reasonable approach to reducing the extent of this ignorance, and thereby bringing ρ to some definite form, would be to examine the evolution of fluctuations given different choices of boundary conditions.

4. CONCLUSION

In Sec. 2, we considered a model with $K = 0$ as an approximation to the realistic model $K = +1$ in a region where we could neglect the curvature of space. Introducing the idea of the "degree of inflation" of a given wave function, we determined the probability of occurrence of an inflationary stage, given a random choice of solution for the WD equation. We then found that a fairly extended period of inflation is typical of a broad class of wave functions.

In Sec. 3, we examined the quantum creation process, employing for this purpose an approximate model with a Λ term. We introduced the concept of a realization probability for a wave function having some value of D, and showed that under the simplest assumptions about $\rho(y, \beta)$, the overwhelming majority of solutions corresponds to $D < 1$.

We may conclude, then, that at least within the framework of the minisuperspace models that have been considered, the wave function describing both the inflationary stage and quantum creation is in no way overly exotic.

We feel duty-bound to note the support and interest in this work shown by our teacher Ya. B. Zel'dovich to the very end of his life. We also thank L. V. Rozhanskii for critical remarks and advice.

[1)] Note that the HH function has previously been criticized by a number of authors.[9–11]

[2)] For an improvement on the choice of measure in (y, β)-space, see G. W. Gibbons and L. P. Grishchuk, Nucl. Phys. B (in press).

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