Contribution to the theory of polarizability of macroscopic bodies

B. Ya. Balagurov
(Submitted 2 November 1987)
Zh. Eksp. Teor. Fiz. **94**, 95–108 (July 1988)

The properties of the eigenfunctions of local (surface, if retardation is neglected) oscillations are considered for a body of arbitrary shape. The orthogonality conditions are found and a completeness relation is formulated for these functions on the surface of the body. The assumption that the system of surface eigenfunctions is complete makes it possible to corroborate an approach, used earlier by the author [Sov. Phys. JETP, **66**, 182 (1987)], to the polarizability problem. A general expression for the Green's function of the corresponding electrostatic problem is derived by the technique of expansion in a set of surface eigenfunctions.

INTRODUCTION

The problem of polarizability of an arbitrary macroscopic body (inclusion) has been solved by the author earlier[1] by expanding the potential in terms of the eigenfunctions of the local oscillations connected with this body. (For inclusions of simple shape (sphere, cylinder) such oscillations are known as surface modes with retardation neglected—see, e.g., Refs. 2 and 3.) In Ref. 1 are determined the conditions under which these oscillations exist, and light is cast on some of the common properties of their eigenfunctions $\psi_\nu(\mathbf{r})$; in particular, orthogonality relations are established and it is shown that $\psi_\nu(\mathbf{r})$ have a multipole asymptotic behavior. It is noted also in Ref. 1 that the system $\{\psi_\nu(\mathbf{r})\}$ is not complete, so that an arbitrary function of the coordinates cannot be expanded in its terms. According to Ref. 1, however, this circumstance is no obstacle in the polarizability problem, for in this case it suffices to satisfy a much less stringent constraint on the $\{\psi_\nu(\mathbf{r})\}$ system, called in Ref. 1 the expandability condition.

As shown in Ref. 1, satisfaction of the expandability condition permits a consistent and incontrovertible solution of the problem of polarizability of a body of arbitrary shape. At the same time, since the system $\{\psi_\nu(\mathbf{r})\}$ is incomplete, the approach of Ref. 1 cannot be extended to other cases. Thus, for example, an attempt to obtain by a similar method the Green's function of the corresponding electrostatic problem meets with serious difficulties. To overcome them it is necessary to make the method of Ref. 1 regular and consistent, but this cannot be done without constructing a complete system of eigenfunctions of the local (surface) oscillations. There are grounds for assuming that a system of such functions is complete on the surface of an inclusion, so that their properties must be studied on the interface.

The present paper deals with the properties of the eigenfunctions $\psi_\nu(\mathbf{r})$ on the surface of a body of arbitrary shape. (We refer to them as surface eigenfunctions and designate them by Ψ_ν.) We show that surface eigenfunctions are solutions of an integral equation with a nonsymmetric kernel. The equation with the conjugate (transposed) kernel are satisfied by quantities Φ_ν (with the meaning of surface-charge density), so that the functions $\{\Psi_\nu\}$ and $\{\Phi_\nu\}$ constitute mutually orthogonal systems. We show also that it is necessary to add to the system $\{\tilde{\psi}_\nu(\mathbf{r})\}$ an orthogonal finite set of functions $\{\tilde{\psi}_n(\mathbf{r})\}$ that correspond to states having monopole asymptotics and called charge functions. A charge function $\tilde{\psi}_n(\mathbf{r})$ is a potential produced by a charged conducting body and corresponds formally to the value $\varepsilon_\nu = \infty$.

It is assumed here that the set of functions corresponding to multipole and charge states constitutes a complete system on the surface of a body of arbitrary shape (this is true, for example, for a sphere and also in some other exactly solvable cases). This assumption suffices to provide a regular method of solving various electrostatic problem by expansion in a system of surface functions. It is thus possible to show that the completeness relation leads to the expandability condition and by the same token to all the main results of Ref. 1. The method proposed has been used also to find a general expression for the Green's function, making it possible to solve the problem of a body placed in an arbitrary external static field. One can assume that expansion in a set of surface eigenfunctions is useful also in other electrostatic (magnetostatic, etc.) problems. The advantages of such an approach are obvious, for example, when it comes to development of various approximate methods, both analytic and numerical.

2. MULTIPOLE STATES

In Ref. 1 are discussed some general properties of local oscillations connected with macroscopic bodies of finite size. We present now results needed below and pertaining to the eigenfunctions of these oscillations. To be specific and to facilitate the reasoning, we consider the case of a dielectric body (or inclusion). Similar problems involving magnetic permeability, conductivity, heat conduction, diffusion, etc. differ from our case only in notation.

Let a medium with dielectric constant $\varepsilon^{(e)}$ contain an inclusion of arbitrary shape with dielectric constant $\varepsilon^{(i)}$. If such a system is placed in a uniform alternating (quasistationary) electric field of frequency ω, then $\varepsilon^{(e)}$ and $\varepsilon^{(i)}$ depend on ω: $\varepsilon^{(e)} = \varepsilon^{(e)}(\omega)$, $\varepsilon^{(i)} = \varepsilon^{(i)}(\omega)$. We express the coordinate-dependent dielectric constant $\varepsilon(\mathbf{r})$ in the form

$$\varepsilon(\mathbf{r}) = \varepsilon^{(e)}[1-(1-z)\theta(\mathbf{r})], \quad z = z(\omega) = \varepsilon^{(i)}(\omega)/\varepsilon^{(e)}(\omega), \quad (1)$$

where $\theta(\mathbf{r}) = 1$ inside the body and $\theta(\mathbf{r}) = 0$ outside. With (1) taken into account, the equation for the electric potential $\nabla[\varepsilon(\mathbf{r})\nabla\varphi] = 0$ is

$$\nabla^2\varphi(\mathbf{r}) - (1-z)\nabla[\theta(\mathbf{r})\nabla\varphi(\mathbf{r})] = 0. \quad (2)$$

The boundary conditions for $\varphi(\mathbf{r})$ are the usual ones—continuity of the potential itself and of the normal component of the induction $\mathbf{D} = -\varepsilon(\mathbf{r})\nabla\varphi$ on the surface.

 0038-5646/88/071351-07$04.00

According to Ref. 1, for real negative values of $z = z(\omega)$ Eq. (2) has for certain frequencies $\omega = \omega_\nu$ nontrivial nonsingular solutions $\psi_\nu(\mathbf{r})$ even in the absence of an external field. The subscript ν numbers here various solutions corresponding to the quantities (eigenvalues) $\varepsilon_\nu = -z(\omega_\nu)$. The equation for $\psi_\nu(\mathbf{r})$ and ε_ν takes thus the form

$$\nabla^2\psi_\nu(\mathbf{r}) - (1+\varepsilon_\nu)\nabla[\theta(\mathbf{r})\nabla\psi_\nu(\mathbf{r})] = 0. \tag{3}$$

Both $\psi_\nu(\mathbf{r})$ and the normal component of the "induction"

$$\mathbf{n}\nabla\psi_\nu^{(e)} = -\varepsilon_\nu\mathbf{n}\nabla\psi_\nu^{(i)} \tag{4}$$

are continuous on the interface. Here $\mathbf{n}$ is a unit vector in in the direction of the outward normal to the surface, while $\psi_\nu^{(e)}$ and $\psi_\nu^{(i)}$ pertain to the medium and to the inclusion, respectively. We impose on the solutions of Eq. (3) also the condition $\psi_\nu(\mathbf{r}) \to 0$ as $r \to \infty$.

The oscillations considered, called local in Ref. 1, can manifest themselves in the form of resonances in the polarizability (see Ref. 1) at $\omega = \omega_\nu$, where ω is determined from the equation $z(\omega_\nu) = -\varepsilon_\nu$. At the same time, the eigenvalues ε_ν (the spectrum) depend only on the shape of the body[1] and are not connected with the specific subject of the problem (the dielectric constant, conductivity, etc.) Moreover, the solutions of Eq. (3) are not connected also with the assumed dependence of z on the frequency ω, since they exist also in the static problem. Local oscillations have thus in electrostatics a formal meaning, so that it is more convenient to speak of certain "states" of the field and of corresponding eigenfunctions $\psi_\nu(\mathbf{r})$ and eigenvalues ε_ν. Note that ε_ν are positive and real.[1] Therefore the functions $\psi_\nu(\mathbf{r})$ can be chosen to be real, as they will hereafter.

We introduce the Green's function of the Laplace operator (see, e.g., Ref. 4):

$$\nabla_{\mathbf{r}}^2 g_0(\mathbf{r}-\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \quad g_0(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{5}$$

Using (5), we can then rewrite (3) in integral form:

$$\psi_\nu(\mathbf{r}) = -(1+\varepsilon_\nu)\int_v d\mathbf{r}'\,\nabla_{\mathbf{r}'}\psi_\nu(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'), \tag{6}$$

with the integration over the volume v of the body. Using as $r \to \infty$ the expansion

$$g_0(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\left\{\frac{1}{r} + \frac{\mathbf{r}\mathbf{r}'}{r^3} + \frac{3(\mathbf{r}\mathbf{r}')^2 - r^2r'^2}{2r^5} + \ldots\right\}, \tag{7}$$

we obtain from (6) an asymptotic expression for $\psi_\nu(\mathbf{r})$ (cf. Ref. 1):

$$r\to\infty:\ \psi_\nu(\mathbf{r}) = \frac{\mathbf{r}\mathbf{d}_\nu}{r^3} + \frac{x_\alpha x_\beta Q_\nu^{\alpha\beta}}{2r^5} + \ldots, \tag{8}$$

$$\mathbf{d}_\nu = \frac{1+\varepsilon_\nu}{4\pi}\mathbf{u}_\nu, \quad \mathbf{u}_\nu = \int_v \nabla\psi_\nu(\mathbf{r})\,d\mathbf{r}, \tag{9}$$

$$Q_\nu^{\alpha\beta} = 3\frac{1+\varepsilon_\nu}{4\pi}\int_v\left\{x_\alpha\frac{\partial\psi_\nu(\mathbf{r})}{\partial x_\beta} + x_\beta\frac{\partial\psi_\nu(\mathbf{r})}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}(\mathbf{r}\nabla\psi_\nu(\mathbf{r}))\right\}d\mathbf{r}. \tag{10}$$

According to (8), if $\mathbf{u}_\nu \neq 0$ ($\mathbf{d}_\nu \neq 0$) the function $\psi_\nu(\mathbf{r})$ has as $r \to \infty$ a dipole character. On the other hand, if $\mathbf{d}_\nu = 0$ in the state ν, the function $\psi_\nu(\mathbf{r})$ has as $r \to \infty$ a quadrupole (octupole, etc.) form. We refer to such functions, which are considered in Ref. 1 and in the present paper, as multipole functions.

As noted in the Introduction, one can expect the eigenfunctions to form a complete system on the surface of the body. It is therefore necessary to study their properties on an interface.

3. SURFACE EIGENFUNCTIONS

We express the function $\psi_\nu(\mathbf{r})$ in terms of its value Ψ_ν on the surface of the inclusion. To this end we transform the integrand of (6) as follows:

$$\nabla_{\mathbf{r}'}\psi_\nu(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}')$$

$$= \nabla_{\mathbf{r}'}\{\psi_\nu(\mathbf{r}')\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}')\} - \psi_\nu(\mathbf{r}')\nabla_{\mathbf{r}'}{}^2 g_0(\mathbf{r}-\mathbf{r}').$$

Taking (5) into account, we obtain then from (6)

$$[1-(1+\varepsilon_\nu)\theta(\mathbf{r})]\psi_\nu(\mathbf{r})$$

$$= -(1+\varepsilon_\nu)\int_s (\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'))\,|_{\mathbf{r}'=\boldsymbol{\rho}'}\Psi_\nu(\boldsymbol{\rho}')\,dS'. \tag{11}$$

Here $\mathbf{n}$ is the same as in Eq. (4), $\boldsymbol{\rho}$ is the value of the radius vector on the inclusion surface, and $\Psi_\nu(\boldsymbol{\rho}) \equiv \psi_\nu(\boldsymbol{\rho})$; the integration in (11) is over the surface area S of the body. According to (11) we have for the function $\psi_\nu(\mathbf{r})$ outside ($\psi_\nu^{(e)}$) and inside ($\psi_\nu^{(i)}$) the inclusion

$$\psi_\nu^{(e)}(\mathbf{r}) = -(1+\varepsilon_\nu)\int_s (\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'))\,|_{\mathbf{r}'=\boldsymbol{\rho}'}\Psi_\nu(\boldsymbol{\rho}')\,dS', \tag{12a}$$

$$\psi_\nu^{(i)}(\mathbf{r}) = \frac{1+\varepsilon_\nu}{\varepsilon_\nu}\int_s (\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'))\,|_{\mathbf{r}'=\boldsymbol{\rho}'}\Psi_\nu(\boldsymbol{\rho}')\,dS'. \tag{12b}$$

From (11) and (12) we can obtain the function $\psi_\nu(\mathbf{r})$ in all of space from its value on the surface of the body.

From (12), letting $\mathbf{r}$ tend to $\boldsymbol{\rho}$ both from outside ($\mathbf{r} \to \boldsymbol{\rho}_e$) and inside ($\mathbf{r} \to \boldsymbol{\rho}_i$) the inclusion, we obtain two integral equations for Ψ_ν:

$$\Psi_\nu(\boldsymbol{\rho}) = -(1+\varepsilon_\nu)\int_s K_1(\boldsymbol{\rho},\boldsymbol{\rho}')\Psi_\nu(\boldsymbol{\rho}')\,dS', \tag{13}$$

$$\Psi_\nu(\boldsymbol{\rho}) = \frac{1+\varepsilon_\nu}{\varepsilon_\nu}\int_s K_2(\boldsymbol{\rho},\boldsymbol{\rho}')\Psi_\nu(\boldsymbol{\rho}')\,dS', \tag{13'}$$

where

$$K_1(\boldsymbol{\rho},\boldsymbol{\rho}') = (\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'))\Big|_{\substack{\mathbf{r}'=\boldsymbol{\rho}'\\ \mathbf{r}=\boldsymbol{\rho}_e}},$$

$$K_2(\boldsymbol{\rho},\boldsymbol{\rho}') = (\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'))\Big|_{\substack{\mathbf{r}'=\boldsymbol{\rho}'\\ \mathbf{r}=\boldsymbol{\rho}_i}}. \tag{14}$$

In view of the singularity at $\boldsymbol{\rho} = \boldsymbol{\rho}'$, the two different methods of taking the limit $\mathbf{r} \to \boldsymbol{\rho}$ in (14) specify two different kernels, so that $K_1 \neq K_2$. For Eqs. (13) and (13') to be compatible, a definite relation must exist between the kernels K_1 and K_2. To find this relation, we multiply (13') by ϵ_ν and add it to (13). The result is the compatibility condition

$$K_1(\boldsymbol{\rho},\boldsymbol{\rho}') - K_2(\boldsymbol{\rho},\boldsymbol{\rho}') = -\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'). \tag{15}$$

In Eq. (15), $\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')$ is the surface delta-functions:

$$\int_s f(\boldsymbol{\rho}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\,dS' = f(\boldsymbol{\rho}), \tag{16}$$

where $f(\boldsymbol{\rho})$ is an arbitrary function specified on the surface S

of the body. It is easily seen that Eq. (15) [with allowance for the definitions in (14)] is a consequence of Eq. (5) for the Green's function of the Laplace operator. By virtue of (15), Eqs. (13) and (13′) are equivalent, so that only one of them need be used, say (13).

Note that the kernels conjugate to $K_1(\rho,\rho')$ (with the permutation $\rho \leftrightarrows \rho'$) are given by

$$K_1(\rho', \rho) = (\mathbf{n}\nabla_{\mathbf{r}} g_0(\mathbf{r}-\mathbf{r}'))\big|_{\substack{\mathbf{r}=\rho \\ \mathbf{r}'=\rho_e'}},$$

$$K_2(\rho', \rho) = (\mathbf{n}\nabla_{\mathbf{r}} g_0(\mathbf{r}-\mathbf{r}'))\big|_{\substack{\mathbf{r}=\rho \\ \mathbf{r}'=\rho_i'}}.$$

We have used here the symmetry of the Green's function: $g_0(\mathbf{r}-\mathbf{r}') = g_0(\mathbf{r}'-\mathbf{r})$. These expressions can also be written in the form

$$K_1(\rho',\rho) = (\mathbf{n}\nabla_{\mathbf{r}} g_0(\mathbf{r}-\rho'))\,|_{\mathbf{r}=\rho_i},$$

$$K_2(\rho',\rho) = (\mathbf{n}\nabla_{\mathbf{r}} g_0(\mathbf{r}-\rho'))\,|_{\mathbf{r}=\rho_e}. \tag{17}$$

Equation (13) for $\Psi_\nu(\rho)$ is a Fredholm integral equation with a nonsymmetric kernel $K_1(\rho,\rho')$. To formulate the orthogonality conditions in this case one must introduce a system conjugate to $\{\Psi_\nu(\rho)\}$ namely $\{\Phi_\nu(\rho)\}$, where $\Phi_\nu(\rho)$ satisfies an integral equation similar to (13) but with conjugate (transposed) kernel $K_1(\rho',\rho)$. To clarify the physical meaning of the functions $\Phi_\nu(\rho)$ we proceed as follows. We transform the integrand of (6) by a second method:

$$\nabla_{\mathbf{r}'}\psi_\nu(\mathbf{r}')\cdot\nabla_{\mathbf{r}'} g_0(\mathbf{r}-\mathbf{r}')$$
$$= \nabla_{\mathbf{r}'}\{g_0(\mathbf{r}-\mathbf{r}')\nabla_{\mathbf{r}'}\psi_\nu(\mathbf{r}')\} - g_0(\mathbf{r}-\mathbf{r}')\nabla_{\mathbf{r}'}{}^2\psi_\nu(\mathbf{r}').$$

Since $\nabla^2\psi_\nu(\mathbf{r}) = 0$ inside the body, it follows from (6) that

$$\psi_\nu(\mathbf{r}) = -(1+\varepsilon_\nu)\int_S g_0(\mathbf{r}-\rho')\,(\mathbf{n}'\nabla_{\mathbf{r}'}\psi_\nu(\mathbf{r}'))\,|_{\mathbf{r}'=\rho_i'}\,dS'. \tag{18}$$

Equation (18) is similar to the standard expression for an electrostatic potential in terms of its normal derivative on the interface.

We introduce the notation

$$\Phi_\nu(\rho) = (\mathbf{n}\nabla\psi_\nu(\mathbf{r}))\,|_{\mathbf{r}=\rho_e}, \quad \Phi_\nu^{(i)}(\rho) = (\mathbf{n}\nabla\psi_\nu(\mathbf{r}))\,|_{\mathbf{r}=\rho_i}. \tag{19}$$

The relation between Φ_ν and $\Phi_\nu^{(i)}$ is, according to (4),

$$\Phi_\nu(\rho) = -\varepsilon_\nu\Phi_\nu^{(i)}(\rho). \tag{20}$$

We apply the operation $\nabla_{\mathbf{r}}$ to both sides of (18), multiply by the unit outward normal $\mathbf{n}$ to the surface S, and take the limit as $\mathbf{r}\to\rho_i$. As a result, with allowance for (17), we get

$$\Phi_\nu(\rho) = -(1+\varepsilon_\nu)\int_S K_1(\rho',\rho)\,\Phi_\nu(\rho')\,dS', \tag{21}$$

where we have used the relation (20). Taking the limit $\mathbf{r}\to\rho_e$ yields for $\Phi_\nu(\rho)$ an equation equivalent to (21), in view of the equality (15).

Comparison of (13) with (21) shows that the quantity conjugate to the potential $\Psi_\nu(\rho)$ is the normal derivative of $\psi_\nu(\mathbf{r})$ on the interface, i.e., apart from a coefficient, the surface density of the polarization charge. From (13) and (21) we find in the usual manner that the functions $\Psi_\nu(\rho)$ and $\Phi_\nu(\rho)$ are orthogonal, if $\varepsilon_\nu \neq \varepsilon_\mu$. Choosing correspondingly the normalization condition, we write the orthonormalization condition in the form

$$\int_S \Psi_\mu(\rho)\,\Phi_\nu(\rho)\,dS = -\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\delta_{\mu\nu}. \tag{22}$$

Thus, the conjugate systems of surface functions $\{\Psi_\nu(\rho)\}$ and $\{\Phi_\nu(\rho)\}$ are mutually orthogonal. It is easy to verify that (22) leads to orthogonality relations for the quantities $\mathbf{e}_\nu(\mathbf{r}) = \nabla\psi_\nu(\mathbf{r})$ determined in Ref. 1.

Note that

$$\int_S (\mathbf{n}'\nabla_{\mathbf{r}'} g_0(\mathbf{r}-\mathbf{r}'))\,|_{\mathbf{r}'=\rho'}\,dS' = \int_v d\mathbf{r}'\,\nabla_{\mathbf{r}'}{}^2 g_0(\mathbf{r}-\mathbf{r}') = \theta(\mathbf{r}), \tag{23}$$

from which it follows, with allowance for the definitions (14), that

$$\int_S K_1(\rho,\rho')\,dS' = 0, \quad \int_S K_2(\rho,\rho')\,dS' = 1. \tag{24}$$

Therefore, integrating (21) over S, we conclude that

$$\int_S \Phi_\nu(\rho)\,dS = 0. \tag{25}$$

Consequently, the total charge on the surface of a body in the multipole state is zero, as is obvious from the asymptotic form of the function $\psi_\nu(\mathbf{r})$, where there is no monopole term—see Eq. (8). If the inclusion consists of individual parts, equations of the type (23)–(25) hold for each of them. The charge of each of the parts is therefore zero.

As $r\to\infty$ we obtain from (12a), taking the expansion (7) for $\psi_\nu(\mathbf{r})$ into account, the asymptotic expression (8). Indeed, the quantity $\mathbf{u}_\nu$ in (9) can be represented also in the form

$$\mathbf{u}_\nu = \int_S \mathbf{n}\Psi_\nu(\rho)\,dS. \tag{26}$$

Expression (10) can be transformed similarly. The same asymptotic expansion for $\psi_\nu(\mathbf{r})$ follows also from Eq. (18). The first (monopole) term in expansion (7) to the asymptote $\psi_\nu(\mathbf{r})$ makes no contribution in view of the condition (25). The dipole term of the expansion of $\psi_\nu(\mathbf{r})$ coincides with (8) and (9) since, on the other hand,

$$\mathbf{u}_\nu \equiv \int_v \frac{\partial}{\partial x_\alpha}\left\{\mathbf{r}\frac{\partial\psi_\nu(\mathbf{r})}{\partial x_\alpha}\right\}d\mathbf{r} = \int_S \rho\Phi_\nu^{(i)}(\rho)\,dS, \tag{26′}$$

where account is taken of the fact that $\nabla^2\psi_\nu(\mathbf{r}) = 0$ inside the inclusion. We ascertain in similar fashion that the quadrupole terms of the expansion are also equal. If the quantity

$$\sigma_\nu(\rho) = -\frac{1+\varepsilon_\nu}{4\pi\varepsilon_\nu}\Phi_\nu(\rho) = \frac{1+\varepsilon_\nu}{4\pi}\Phi_\nu^{(i)}(\rho)$$

is introduced, the expressions for the dipole moment $\mathbf{d}_\nu$ and for the quadrupole moment $\hat{\mathbf{Q}}_\nu$ take the standard form

$$\mathbf{d}_\nu = \int_S \rho\sigma_\nu(\rho)\,dS, \quad Q_\nu^{\alpha\beta} = \int_S (3\rho_\alpha\rho_\beta - \rho^2\delta_{\alpha\beta})\,\sigma_\nu(\rho)\,dS. \tag{27}$$

Consequently, the quantity $\sigma_\nu(\rho)$ has the meaning of the surface density of the polarization charges in the state ν.

4. CHARGE (MONOPOLE) STATES

It follows from (25) that the function $\tilde{\Psi} = \text{const}$ is orthogonal to all Φ_ν. At the same time, by virtue of the condi-

tion (24), $\tilde{\Psi} = \text{const}$ is not a solution of Eqs. (13) and (13') for finite ε_ν, and consequently is not contained in the system $\{\Psi_\nu\}$. This means that the system of multipole surface functions is not complete and it is necessary to add to it $\tilde{\Psi} = \text{const}$, and also the conjugate function $\tilde{\Phi}(\rho)$. (Thus, for the spherical inclusion considered in Sec. 7, none of the functions $\{\Psi_\nu\}$ is proportional to the zeroth spherical harmonic, i.e., $\tilde{\Psi} = \text{const}$.) From the physical point of view the situation with a potential constant on the surface of a body is that of a metallic ($\varepsilon^{(i)} \to \infty$) inclusion corresponding to $z = \infty$ (or $\varepsilon_\nu = \infty$). The foregoing arguments show thus that the case $\varepsilon_\nu = \infty$ must be considered separately.

We shall tag the Ψ and Φ pertaining to $\varepsilon_\nu = \infty$ by a tilde. Going in (13) to the limit as $\varepsilon_\nu \to \infty$, we get

$$\int_S K_1(\rho, \rho')\,\tilde{\Psi}_n(\rho')\,dS' = 0. \tag{28}$$

The subscript n numbers here various solutions of Eq. (28). By the same method we obtain from (21) an equation for the conjugate functions

$$\int_S K_1(\rho', \rho)\,\tilde{\Phi}_n(\rho')\,dS' = 0. \tag{29}$$

As follows from (20), $\tilde{\Phi}_n^{(i)}(\rho)$ is equal to zero.

From (12) and (29), and also from (21) and (28), follow orthogonality relations

$$\int_S \Psi_\nu(\rho)\,\tilde{\Phi}_n(\rho)\,dS = 0, \quad \int_S \tilde{\Psi}_n(\rho)\,\Phi_\nu(\rho)\,dS = 0, \tag{30}$$

which are valid for those states ν to which finite ε_ν correspond. We assume also that the function $\tilde{\Psi}_\nu$ and $\tilde{\Phi}_\nu$ are orthogonalized, with

$$\int_S \tilde{\Psi}_n(\rho)\,\tilde{\Phi}_m(\rho)\,dS = -\delta_{nm}. \tag{31}$$

The set of functions $\{\tilde{\Psi}_n, \tilde{\Phi}_n\}$ is thus orthogonal to $\{\Psi_\nu, \Phi_\nu\}$ and must be added to the system of multipole surface functions.

Consider a non-sectionalized body consisting of one connective part. As already noted, the value $\varepsilon_\nu = \infty$ ($z = \infty$) corresponds to a metallic inclusion on the surface of which the potential is constant ($\tilde{\Psi} = \text{const}$). In the absence of an external electric field, nontrivial (different from a constant in all of space) solutions for the potential exist if the conductor has a charge $q \neq 0$, where

$$q = \int_S \tilde{\sigma}(\rho)\,dS, \quad \tilde{\sigma}(\rho) = -\frac{1}{4\pi}\tilde{\Phi}(\rho). \tag{32}$$

We call such a state (with $q \neq 0$) a charge state. The values of and q are connected by the relation $q = C\tilde{\Psi}$, where C is the capacitance of the conductor.[5] In our case of a non-sectionalized inclusion there is only one independent solution $\tilde{\Psi}$ corresponding to a potential $\tilde{\psi}^{(e)}(\mathbf{r}) \to 0$ as $r \to \infty$, and from the normalization condition (31) it follows that $\Psi = (4\pi C)^{-1/2}$ and $q = (C/4\pi)^{1/2}$.

If the inclusion consists of N individual parts (bodies), the potential $\tilde{\Psi}(\rho)$ takes on constant (in general, different) values $\tilde{\Psi}^{(a)}$ ($a = 1, \ldots, N$) on the surfaces of these bodies. Accordingly, each of the parts carries a charge $q^{(a)}$ given by (32), where the integration is carried out in this case over the surface S_a of the ath body. Assume that we known N different solutions of the conductor electrostatics equations, corresponding to situtions in which one of the bodies has a unit charge and the rest are uncharged. In accordance with the superposition principle, a linear combination of such solutions yields a potential also for the case when each of the bodies has a charge $q^{(a)}$. Consequently, Eqs. (28) and (29) should have N linearly independent solutions, so that we have N eigenfunctions $\tilde{\Psi}_n$ and accordingly N functions $\tilde{\Phi}_n$, where $n = 1, \ldots, N$.

The relations between $\tilde{\Psi}_n^{(a)}$ and $q_n^{(a)}$ are[5]

$$q_n^{(a)} = \sum_b C_{ab}\tilde{\Psi}_n^{(b)}.$$

Here $\hat{C}$ is a matrix made up of coefficients of the capacitance and the electrostatic-induction.[5] Equation (31) takes in this case the form

$$4\pi\sum_a \tilde{\Psi}_n^{(a)}\, q_m^{(a)} = 4\pi \sum_a \sum_b C_{ab}\tilde{\Psi}_n^{(a)}\,\tilde{\Psi}_m^{(b)} = \delta_{nm}.$$

Since the matrix $\hat{C}$ is symmetric and positive-definite,[5] it can be reduced to the diagonal unit matrix $\hat{U}^T\hat{C}\hat{U} = \hat{1}$, where $\hat{U}$ is the diagonalizing matrix. The orthonormalization relation (31) is therefore satisfied if the potentials are chosen to be $\tilde{\Psi}_n^{(a)} = (4\pi)^{-1/2}U_{an}$.

For the potential $\tilde{\psi}_n(\mathbf{r})$ outside the surface of the body we get from (18) (with (20) taken into account) by taking the limit

$$\tilde{\psi}_n(\mathbf{r}) = \int_S g_0(\mathbf{r}-\rho')\,\tilde{\Phi}_n(\rho')\,dS'. \tag{33}$$

From (33) and (7) it follows that as $r \to \infty$

$$\tilde{\psi}_n(\mathbf{r}) = \frac{q_n}{r} + \frac{\mathbf{r}\tilde{\mathbf{d}}_n}{r^3} + \frac{x_\alpha x_\beta \tilde{Q}_n^{\alpha\beta}}{2r^5} + \ldots, \tag{34}$$

where q_n is the total charge of the inclusion and is determined by Eqs. (32) with $\tilde{\sigma}(\rho)$ replaced by $\tilde{\sigma}_n(\rho)$, i.e., $\tilde{\Phi}$ by $\tilde{\Phi}_n$. (Since (34) contains the term q_n/r, the charge states can also be named monopole states.) The dipole and quadrupole moments $\tilde{\mathbf{d}}_n$ and $\hat{Q}_n$ are defined in analogy with (27) in which $\sigma_\nu(\rho)$ is replaced by $\tilde{\sigma}_n(\rho)$. For the potential inside the inclusion we get from (12b) in the limit as $\varepsilon_n \to \infty$

$$\tilde{\psi}_n^{(i)}(\mathbf{r}) = \int_S (\mathbf{n}'\nabla_{\mathbf{r}'} g_0(\mathbf{r}-\mathbf{r}'))\,|_{\mathbf{r}'=\rho'}\,\tilde{\Psi}_n(\rho')\,dS'. \tag{35}$$

It follows hence, with (23) taken into account, that $\tilde{\psi}_n^{(i)}(\mathbf{r}) = \tilde{\Psi}_n^{(a)} = \text{const}$ inside the ath body, as it should for a conductor. Therefore the value of $\mathbf{u}_n$ determined in analogy with (9) and (26) is zero for charge states.

The multipole and charge (monopole) states considered above seem to account for all the solutions of Eq. (2) in the absence of an electric field. It can therefore be assumed that the eigenfunctions of these states on the surface of the body constitute a complete system.

5. COMPLETENESS RELATION

We make now an assumption fundamental for the described method, that the aggregate of the surface eigenfunctions corresponding to multipole and charge states form a complete system. This means that any function $f(\rho)$ specified on the surface of a body can be expanded in a series either in Ψ_ν and $\tilde{\Psi}_n$:

$$f(\rho)=\sum_{\nu}A_{\nu}\Psi_{\nu}(\rho)+\sum_{n}\tilde{A}_{n}\tilde{\Psi}_{n}(\rho), \tag{36}$$

or in an analogous series in Φ_ν and $\tilde{\Phi}_n$. The expansion coefficients in (36) are determined with the aid of the orthogonality conditions (22), (30), and (31). In order for the series in the right-hand side of (36) to converge to the function $f(\rho)$, it is necessary to satisfy the equation

$$\sum_{\nu}\frac{1+\varepsilon_{\nu}}{\varepsilon_{\nu}}\Psi_{\nu}(\rho)\Phi_{\nu}(\rho')+\sum_{n}\tilde{\Psi}_{n}(\rho)\tilde{\Phi}_{n}(\rho')=-\delta(\rho-\rho'), \tag{37}$$

which is the completeness condition for the system of surface eigenfunctions. It is thus assumed that (37) is met on the surface of a body of arbitrary shape.

It is possible to obtain as a consequence of (37) various useful expansions. We make in (37) the substitution $\rho\to\rho''$, multiply by $(n''\nabla_{r''}g_0(\mathbf{r}_i-\mathbf{r}''))dS''$, where $\mathbf{r}''=\rho''$, and integrate over S. Next, using (12b) and (35), we obtain

$$(\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}_i-\mathbf{r}'))\,|_{\mathbf{r}'=\rho'}$$
$$=-\sum_{\nu}\psi_{\nu}^{(i)}(\mathbf{r})\Phi_{\nu}(\rho')-\sum_{n}\tilde{\psi}_{n}^{(i)}(\mathbf{r})\tilde{\Phi}_{n}(\rho'). \tag{38a}$$

Here $\mathbf{r}=\mathbf{r}_i$ denotes that r belongs to an inclusion. For $\mathbf{r}$ not belonging to an inclusion ($\mathbf{r}=\mathbf{r}_e$) we obtain similarly

$$(\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}_e-\mathbf{r}'))\,|_{\mathbf{r}'=\rho'}=\sum_{\nu}\frac{1}{\varepsilon_{\nu}}\psi_{\nu}^{(e)}(\mathbf{r})\Phi_{\nu}(\rho'). \tag{38b}$$

We now make in (37) the substitutions $\rho\to\rho'$ and $\rho'\to\rho''$, multiply by $g_0(\mathbf{r}-\rho'')dS''$, and integrate over S. Using (18) and (33), we obtain then

$$g_0(\mathbf{r}-\rho')=-\sum_{\nu}\psi_{\nu}(\mathbf{r})\Psi_{\nu}(\rho')-\sum_{n}\tilde{\psi}_{n}(\mathbf{r})\tilde{\Psi}_{n}(\rho'), \tag{39}$$

where ρ' belongs to the interface and $\mathbf{r}$ is arbitrary.

Note that the following equation holds by virtue of (5):

$$\int_{v}\{g_0(\mathbf{r}'-\mathbf{r}'')\nabla^2_{\mathbf{r}''}g_0(\mathbf{r}-\mathbf{r}'')-g_0(\mathbf{r}-\mathbf{r}'')\nabla^2_{\mathbf{r}''}g_0(\mathbf{r}'-\mathbf{r}'')\}d\mathbf{r}''$$
$$=[\theta(\mathbf{r})-\theta(\mathbf{r}')]g_0(\mathbf{r}-\mathbf{r}'). \tag{40}$$

From this we have for $\mathbf{r}=\mathbf{r}_e$ and $\mathbf{r}'=\mathbf{r}_i'$

$$g_0(\mathbf{r}_e-\mathbf{r}_i')=\int_{S}\{g_0(\mathbf{r}_e-\rho'')(\mathbf{n}''\nabla_{\mathbf{r}''}g_0(\mathbf{r}_i'-\mathbf{r}''))\,|_{\mathbf{r}''=\rho''}$$
$$-g_0(\mathbf{r}_i'-\rho'')(\mathbf{n}''\nabla_{\mathbf{r}''}g_0(\mathbf{r}_e-\mathbf{r}''))\,|_{\mathbf{r}''=\rho''}\}\,dS''. \tag{41}$$

In the derivation of (41) it was taken into account that the integrand in (40) can be expressed in the form of a total derivative, so that the integral over the inclusion volume v is transformed into a surface integral. Substitution of (38) and (39) in (41) leads, with allowance for the orthogonality relations (22), (30), and (31), to the expansion

$$g_0(\mathbf{r}_e-\mathbf{r}_i')=-\sum_{\nu}\psi_{\nu}^{(e)}(\mathbf{r})\psi_{\nu}^{(i)}(\mathbf{r}')-\sum_{n}\tilde{\psi}_{n}^{(e)}(\mathbf{r})\tilde{\psi}_{n}^{(i)}(\mathbf{r}'), \tag{42}$$

which is valid for the $\mathbf{r}$ and $\mathbf{r}'$ indicated in (42). Similarly, for $\mathbf{r}=\mathbf{r}_i$ and $\mathbf{r}'=\mathbf{r}_e'$ we get for $g_0(\mathbf{r}_i-\mathbf{r}_e')$ an expression that differs from (42) by permutation of the subscripts $i\rightleftarrows e$. The use of (42) leads to an expansion for the kernel $K_1(\rho,\rho')$:

$$K_1(\rho,\rho')=\sum_{\nu}\frac{1}{\varepsilon_{\nu}}\Psi_{\nu}(\rho)\Phi_{\nu}(\rho').$$

An expansion for $K_2(\rho,\rho')$ can be found similarly.

Consider the identity

$$\int_{v}\mathbf{r}'\nabla_{\mathbf{r}'}{}^2g_0(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'=\mathbf{r}\theta(\mathbf{r}).$$

Using the relation

$$x_{\alpha}'\frac{\partial^2 g_0(\mathbf{r}-\mathbf{r}')}{\partial x_{\beta}'^2}=\frac{\partial}{\partial x_{\beta}'}\left\{x_{\alpha}'\frac{\partial g_0(\mathbf{r}-\mathbf{r}')}{\partial x_{\beta}'}\right\}-\frac{\partial g_0(\mathbf{r}-\mathbf{r}')}{\partial x_{\alpha}'},$$

we transform the integral into a surface one, so that

$$\mathbf{r}\theta(\mathbf{r})=\int_{S}\rho'(\mathbf{n}'\nabla_{\mathbf{r}'}g_0(\mathbf{r}-\mathbf{r}'))\,|_{\mathbf{r}'=\rho'}\,dS'-\int_{S}g_0(\mathbf{r}-\rho')\,d\mathbf{S}'.$$

Substitution of (38) and (39) leads, with allowance for (26), (26'), and the definition of $\tilde{\mathbf{d}}_n$ from (34), to the relation

$$\mathbf{r}\theta(\mathbf{r})=4\pi\left\{\sum_{\nu}\psi_{\nu}^{(i)}(\mathbf{r})\mathbf{d}_{\nu}+\sum_{n}\tilde{\psi}_{n}^{(i)}(\mathbf{r})\tilde{\mathbf{d}}_{n}\right\}\theta(\mathbf{r}). \tag{43}$$

Equation (43) is an expansion in the eigenfunctions of the vector $\mathbf{r}$ inside the inclusion.

Differentiation of (43) with respect to coordinate leads, with allowance for $\nabla\tilde{\psi}_n^{(i)}=0$, to the expandability condition - see Eq. (18) of Ref. 1. Thus, use of the method proposed in the present paper corroborates the approach of Ref. 1 to the determination of the polarization of macroscopic bodies. Accordingly, all the results of Ref. 1 can be obtained from the completeness condition (37).

Expansion in terms of the complete set of surface eigenfunctions provides a systematic method of solving various electrostatic problems involving a dielectric body of given shape. The general procedure for solving such problems is the following. The sought function $f(\mathbf{r})$ must be expressed in terms of its value $F(\rho)$ on the surface S of the body. Next, by taking the limit $\mathbf{r}\to\rho$, an equation is obtained for $F(\rho)$. This equation is solved by expansion in a set of surface eigenfunctions. Finally, the function $f(\mathbf{r})$ is determined from the obtained expression for $F(\rho)$. This method can be used to solve also other more complicated problems, such as the determination of the Green's function of Eq. (12).

6. THE GREEN'S FUNCTION

The Green's function $g(\mathbf{r},\mathbf{r}')$ satisfies the equation

$$\nabla_{\mathbf{r}}^2 g(\mathbf{r},\mathbf{r}')-(1-z)\nabla_{\mathbf{r}}[\theta(\mathbf{r})\nabla_{\mathbf{r}}g(\mathbf{r},\mathbf{r}')]=\delta(\mathbf{r}-\mathbf{r}'). \tag{44}$$

Equation (44) can be expressed with the aid of (5) in integral form

$$g(\mathbf{r},\mathbf{r}')=g_0(\mathbf{r}-\mathbf{r}')-(1-z)\int_{v}d\mathbf{r}''\,\nabla_{\mathbf{r}''}g_0(\mathbf{r}-\mathbf{r}'')\cdot\nabla_{\mathbf{r}''}g(\mathbf{r}'',\mathbf{r}'). \tag{45}$$

The integrand can be represented as

$$\nabla_{\mathbf{r}''}\{\nabla_{\mathbf{r}''}g_0(\mathbf{r}-\mathbf{r}'')\cdot g(\mathbf{r}'',\mathbf{r}')\}-\nabla_{\mathbf{r}''}{}^2g_0(\mathbf{r}-\mathbf{r}'')\cdot g(\mathbf{r}'',\mathbf{r}'),$$

so that we obtain from (45)

$$[1-(1-z)\theta(\mathbf{r})]g(\mathbf{r},\mathbf{r}')$$
$$=g_0(\mathbf{r}-\mathbf{r}')-(1-z)\int_{S}(\mathbf{n}''\nabla_{\mathbf{r}''}g_0(\mathbf{r}-\mathbf{r}''))\,|_{\mathbf{r}''=\rho''}g(\rho'',\mathbf{r}')dS''. \tag{46}$$

By taking the limit $\mathbf{r}\to\rho_e$ we get from (46) an equation for $g(\rho,\mathbf{r}')$:

$$g(\rho,\mathbf{r}')=g_0(\rho-\mathbf{r}')-(1-z)\int_S K_1(\rho,\rho'')\,g(\rho'',\mathbf{r}')\,dS'', \quad (47)$$

where K_1 is the same as in (13) and (14). We solve (47) by expanding $g(\rho,\mathbf{r}')$ in series in $\{\Psi_\nu(\rho),\tilde{\Psi}_n(\rho)\}$. As a result, using (13), (28), and (39) we get

$$g(\rho,\mathbf{r}')=-\sum_\nu\frac{1+\varepsilon_\nu}{z+\varepsilon_\nu}\Psi_\nu(\rho)\psi_\nu(\mathbf{r}')-\sum_n\tilde{\Psi}_n(\rho)\tilde{\psi}_n(\mathbf{r}'). \quad (48)$$

Finally, substituting (48) in (46) we obtain, with allowance for (38), an expression for the Green's function:

$$g(\mathbf{r}_e,\mathbf{r}')=g_0(\mathbf{r}-\mathbf{r}')-(1-z)\sum_\nu\frac{\psi_\nu^{(e)}(\mathbf{r})\psi_\nu(\mathbf{r}')}{z+\varepsilon_\nu}, \quad (49a)$$

$$g(\mathbf{r}_i,\mathbf{r}')=\frac{1}{z}g_0(\mathbf{r}-\mathbf{r}')+\frac{1-z}{z}\sum_\nu\varepsilon_\nu\frac{\psi_\nu^{(i)}(\mathbf{r})\psi_\nu(\mathbf{r}')}{z+\varepsilon_\nu}$$

$$+\frac{1-z}{z}\sum_n\tilde{\psi}_n^{(i)}(\mathbf{r})\tilde{\psi}_n(\mathbf{r}'). \quad (49b)$$

Here $\mathbf{r}=\mathbf{r}_e$ is located outside the body, $\mathbf{r}=\mathbf{r}_i$ belongs to the inclusion, and $\mathbf{r}'$ is arbitrary. If $\mathbf{r}'=\mathbf{r}_i'$, expression (49a) can be transformed with the aid of the expansion (42) into

$$g(\mathbf{r}_e,\mathbf{r}_i')=-\sum_\nu\frac{1+\varepsilon_\nu}{z+\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\psi_\nu^{(i)}(\mathbf{r}')-\sum_n\tilde{\psi}_n^{(e)}(\mathbf{r})\tilde{\psi}_n^{(i)}(\mathbf{r}'). \quad (49c)$$

If $\mathbf{r}=\mathbf{r}_i$ and $\mathbf{r}'=\mathbf{r}_e'$ we obtain for $g(\mathbf{r}_i,\mathbf{r}_e')$ an expression that differs from (49c) by the subscript permutation $i\rightleftarrows e$.

Knowledge of the Green's function yields a solution for a body placed in an arbitrary external field $\mathbf{E}_0(\mathbf{r})$. Let $\mathbf{E}_0(\mathbf{r})=-\nabla\varphi_0(\mathbf{r})$, where the potential $\varphi_0(r)$ obeys the equation $\nabla^2\varphi_0(\mathbf{r})=0$. Putting $\varphi(\mathbf{r})=\varphi_0(\mathbf{r})+\psi(\mathbf{r})$ we obtain then from (2) an equation for $\psi(\mathbf{r})$:

$$\nabla^2\psi(\mathbf{r})-(1-z)\nabla[\theta(\mathbf{r})\nabla\psi(\mathbf{r})]=(1-z)\nabla[\theta(\mathbf{r})\nabla\varphi_0(\mathbf{r})].$$

Solving this equation with the aid of the Green's function, we get

$$\psi(\mathbf{r})=-(1-z)\int_v d\mathbf{r}'\,\nabla_{\mathbf{r}'}g(\mathbf{r},\mathbf{r}')\cdot\nabla_{\mathbf{r}'}\varphi_0(\mathbf{r}').$$

Since $\nabla^2\varphi_0(\mathbf{r})=0$, the expression for $\psi(\mathbf{r})$ can be transformed into

$$\psi(\mathbf{r})=-(1-z)\int_S g(\mathbf{r},\rho')\,(\mathbf{n}'\nabla_{\mathbf{r}'}\varphi_0(\mathbf{r}'))\,|_{\mathbf{r}'=\rho'}\,dS'. \quad (50)$$

For a uniform field we have $\varphi_0(\mathbf{r})=-\mathbf{E}_0\cdot\mathbf{r}$, so that in this case we get from (5) and (48) the result of Ref. 1 for $\psi(\mathbf{r})$. For the polarizability tensor $\hat{\Lambda}$ we obtain correspondingly the spectral expansion

$$\Lambda_{\alpha\beta}(z)=-4\pi(1-z)\sum_\nu\frac{d_{\nu\alpha}d_{\nu\beta}}{z+\varepsilon_\nu} \quad (51)$$

obtained in Ref. 1.

An expansion in a system of eigenfunctions can be used also to solve the so-called Dirichlet and Neumann boundary-value problems. Thus, for example, if a normal derivative of the potential $(\partial\varphi/\partial n)|_S=\Phi(\rho)$, is specified on the surface of an inclusion, the potential outside the body (the external Neumann problem) can be represented in the form

$$\varphi(\mathbf{r})=\int_S G_N^{(e)}(\mathbf{r},\rho')\Phi(\rho')\,dS',$$

$$G_N^{(e)}(\mathbf{r},\rho')=-\sum_\nu\frac{1+\varepsilon_\nu}{\varepsilon_\nu}\psi_\nu^{(e)}(\mathbf{r})\Psi_\nu(\rho')$$

$$-\sum_n\tilde{\psi}_n^{(e)}(\mathbf{r})\tilde{\Psi}_n(\rho'). \quad (52)$$

A similar treatment is possible for the internal Neumann problem and also for the external and internal Dirichlet problems.

The method of expansion in a set of surface eigenfunctions makes thus possible an analysis of a rather extensive group of electrostatic (magnetostatic, etc.) problems. The need for solving the Laplace equation arises also in other branches of physics, particularly in hydrodynamics. By way of example, we consider translational motion of a solid with velocity $\mathbf{V}$ in an incompressible ideal liquid (see Ref. 6). For the velocity potential φ we have in this case the external Neumann problem $\Phi(\rho)=\mathbf{n}\cdot\mathbf{v}$. From (52) we get for $\varphi(\mathbf{r})$

$$\varphi(\mathbf{r})=-4\pi\sum_\nu\frac{1}{\varepsilon_\nu}(\mathbf{V}\mathbf{d}_\nu)\psi_\nu^{(e)}(\mathbf{r}).$$

We have used here relation (26), the definition (9), and the fact that $\tilde{\mathbf{u}}_n$ is zero. In the dipole approximation, as $\mathbf{r}\to\infty$ and with allowance for (8), this yields $\varphi(\mathbf{r})\approx-(\mathbf{A}\mathbf{r})/r^3$, $\mathbf{A}=-\hat{\Lambda}(0)\mathbf{V}$, where $\hat{\Lambda}(0)$ is the polarizability tensor (51) for $z=0$. According to Ref. 6, one can express in terms of the vector $\mathbf{A}$ the total momentum $\mathbf{P}$ of the liquid, viz. $\mathbf{P}=\rho\{4\pi\mathbf{A}-v\mathbf{V}\}$, where v is the volume of the body. On the other hand, the momentum P can be expressed in terms of the joined-masses tensor $\hat{m}$: $\mathbf{P}=\hat{m}\mathbf{V}$ (Ref. 6). As a result, introducing the dimensionless polarizability tensor $\alpha=v^{-1}\hat{\Lambda}$ we obtain

$$\hat{m}=-M\{\hat{1}+4\pi\hat{\alpha}(0)\}. \quad (53)$$

Here $M=\rho v$ is the mass of the liquid displaced by the body. In the case of an ellipsoid we have for the principal values of the polarizability tensor[5]

$$4\pi\alpha^{(\gamma)}(z)=-(1-z)/[1-(1-z)n^{(\gamma)}],$$

where $\gamma=x,y,z$ and $n^{(\gamma)}$ are the depolarization coefficients. For the principal values of the joined-mass tensor of an ellipsoidal body it follows then from (53) that $m^{(\gamma)}/M=n^{(\gamma)}/(1-n^{(\gamma)})$, which agrees with Ref. 7. For a sphere ($n^{(\gamma)}=1/3$), in particular, we obtain $m=M/2$ (see Refs. 6 and 7).

In the two-dimensional ($D=2$) case the method proposed here calls for a somewhat different formulation. The charge states and the Green's functions were considered here under the conditions $\tilde{\psi}_n(\mathbf{r})\to0$, $g_0(\mathbf{r}-\mathbf{r}')\to0$, and $g(\mathbf{r},\mathbf{r}')\to0$ as $r\to\infty$, which cannot be met for $D=2$. In this case the problem can be formulated in the following standard manner. We require that the eigenfunctions as well as the Green's function vanish on the surface of a cylinder of radius R, with R large enough. This condition sets the values of $\tilde{\Psi}_n$: the zeroth Green's function takes the form

$$\tilde{g}_0(\mathbf{r},\mathbf{r}')=\frac{1}{2\pi}\left\{\ln|\mathbf{r}-\mathbf{r}'|-\ln\left|\frac{R}{r}\mathbf{r}-\frac{r}{R}\mathbf{r}'\right|\right\}$$

and remains symmetric: $\bar{g}_0(\mathbf{r},\mathbf{r}') = \bar{g}_0(\mathbf{r}',\mathbf{r})$. In this formulation, replacement of $g_0(\mathbf{r}-\mathbf{r}')$ by $\bar{g}_0(\mathbf{r},\mathbf{r}')$ makes all the main results of the present paper applicable also in the two-dimensional case (for the region $r \leqslant R$). To obtain a solution valid in all of space, it is necessary to let R tend to infinity in the final equations.

7. SPHERICAL INCLUSION

The eigenfunctions $\psi_\nu(\mathbf{r})$ and $\tilde{\psi}_n(\mathbf{r})$ and the eigenvalues can be obtained in explicit form for certain bodies with simple shapes. We consider by way of example a spherical inclusion of radius R. The multipole eigenfunctions were obtained in Ref. 1:

$$\begin{aligned} \psi^{(i)}_{\lambda lm}(\mathbf{r}) &= [(2l+1)R]^{-1/2}(r/R)^l Y^{(\lambda)}_{lm}(\theta,\varphi), \quad r<R, \\ \psi^{(e)}_{\lambda lm}(\mathbf{r}) &= [(2l+1)R]^{-1/2}(R/r)^{l+1} Y^{(\lambda)}_{lm}(\theta,\varphi), \quad r>R. \end{aligned} \tag{54}$$

Here $0 \leqslant m \leqslant l$ for $\lambda = 1$ and $1 \leqslant m \leqslant l$ for $\lambda = 2$ and $l \geqslant 1$. Corresponding to the functions (54) are eigenvalues $\varepsilon_{\lambda lm} = \varepsilon_l = (l+1)/l, l = 1,2,\ldots$ which are degenerate in λ and m. In Eq. (54), $Y^{(\lambda)}_{lm}$ are real spherical functions (see, e.g., Ref. 1), for which the following relation holds:

$$\sum_{\lambda m} Y^{(\lambda)}_{lm}(\mathbf{n}) Y^{(\lambda)}_{lm}(\mathbf{n}') = \frac{2l+1}{4\pi} P_l(\mathbf{nn}'), \tag{55}$$

where $\mathbf{n} = \mathbf{r}/r$.

In this case there is one charge state corresponding to the function

$$\begin{aligned} \tilde{\psi}^{(i)}(\mathbf{r}) &= (4\pi R)^{-1/2} = R^{-1/2} Y_{00}(\mathbf{n}), \quad r<R, \\ \tilde{\psi}^{(e)}(\mathbf{r}) &= \frac{1}{r}\left(\frac{R}{4\pi}\right)^{1/2} = \frac{R^{1/2}}{r} Y_{00}(\mathbf{n}), \quad r>R, \end{aligned}$$

where $Y_{00}(\mathbf{n}) = (4\pi)^{-1/2}$ is a zero-order spherical harmonic. This state corresponds to a charge $q = (R/4\pi)^{1/2}$, and in the chosen coordinate frame we have $\tilde{\mathbf{d}} = 0$ and $\hat{Q} = 0$.

The completeness relation (37) is met in this case, since it takes, with allowance for (55), the form of the known identity

$$\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\mathbf{nn}') = \delta(\mathbf{n}-\mathbf{n}').$$

Also valid are Eqs. (38), (39), and (42). Thus, for example, Eq. (42) (for $r > R > r'$) reduces to the known expansion

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\mathbf{nn}'), \quad r>r'.$$

Finally, substitution of (54) and (56) in Eqs. (49) yields for the Green's function $g(\mathbf{r},\mathbf{r}')$ expressions that coincide with the corresponding results of Ref. 8.

One can consider similarly also certain other cases, when the variables in the Laplace equations are separable. Particular interest attaches to the classical problem of the potential of a point charge (i.e., of the Green's function) in the presence of a dielectric wedge, which is solved in Ref. 9 by rather complicated and unwieldy integral-equations method. Yet the use of expansion in a system of surface eigenfunctions solves this rather difficult problem by a relatively elementary method (and with much fewer calculations). The approach proposed here, in constrast to that of Ref. 9, encounters no fundamental difficulties when it comes to even more complicated problems of this type.

[1] B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **93**, 316 (1987) [Sov. Phys. JETP **66**, 182 (1987)].
[2] V. V. Bryksin, D. N. Mirlin, and Yu. A. Firsov, Usp. Fiz. **113**, 29 (1974) [Sov. Phys. Usp. **17**, 305 (1974)].
[3] V. M. Agranovich, *ibid.* **115**, 199 (1975) [**18**, 99 (1975)].
[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. 2, McGraw, 1953.
[5] L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, 1984.
[6] L. D. Landau and E. M. Lifshitz, Hydrodynamics [in Russian], Nauka, 1986, §11.
[7] H. Lamb, *Hydrodynamics*, 6th ed. Cambridge Univ. Press, 1932, §114.
[8] V. V. Batygin and I. N. Toptygin, Collected Problems in Electrodynamics [in Russian], Nauka, 1970, Problems 157 and 158.
[9] G. A. Grinberg, Selected Topics in the Mathematical Theory of Electric and Magnetic Phenomena [in Russian], AN SSSR, 1948.

Translated by J. G. Adashko