Chromoelectric string as a result of phase transition in nonabelian gauge field theory

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The mechanism for the appearance of a relativistic string binding quarks in a hadron is analogous to a second-order phase transition in statistical physics. The number of colors N plays the role of the inverse temperature. The critical value is $N_c = 1$.

Nonabelian $SU(N)$-gauge theory simplifies[1] in the limit of a large number of colors $N \gg 1$. In this limit the basic properties of the theory are preserved and the $O(1/N)$ corrections are small. The central unsolved problem consists in evaluating the term of zeroth (leading) order in $1/N$. When the usual fields of quarks and gluons are used as dynamical variables, there arise in leading order an infinite number of planar diagrams with all possible gluon exchanges, whose total contribution is unknown. On the other hand, it was shown in Ref. 2 that the $1/N$ expansion represents the quasiclassical limit in the parameter $1/N$. That means that the functional integral (statistical sum) over the color variables should be performed by the saddle-point method (in Euclidean space R^4). In doing this it is necessary to find the stable configurations of fields that extremizes the action S. To this end it is convenient to pass to new, more adequate variables, in terms of which the fluctuations are small, of order $\sim 1/N$. An analogous approach was used by Landau[3] in the theory of phase transitions. Moreover, if the resultant extremum S^{eff} is connected with a (spontaneous) reduction of the original gauge symmetry to a smaller subgroup, there arises a phenomenon similar to a second-order phase transition in condensed media. We shall show below that it is precisely such a mechanism that results in the appearance of a chromoelectric string with quarks at its ends in leading order in $1/N$.

We consider the Euclidean correlator $K(1..., n)$, containing n field operators of constituent mesons $i\psi_c^+(x)\psi_c(x)$, where ψ_c stands for the field of the quark $q(c = 1, ..., N)$, $(\psi^+\psi)^* = \psi^T\psi^* = -\psi^+\psi$:

$$K(1,\dots,n)=B^{-1}\int d\mu(A)D\psi D\psi^+ \cdot [i\psi^+(x_1)\psi(x_1)\dots i\psi^+(x_n)\psi(x_n)]\exp(-S_{Y-M}), \quad (1)$$

$$S_{Y-M}=\int d^4x\left[\frac{1}{4}G_{\mu\nu}{}^aG_{\mu\nu}{}^a+i\psi^+(\hat{D}+m_0)\psi\right],$$

$$G_{\mu\nu}{}^a=\partial_\mu A_\nu{}^a-\partial_\nu A_\mu{}^a-ef^{abc}A_\mu{}^bA_\nu{}^c,$$

$$\gamma_0=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\quad \gamma_m=\begin{pmatrix}0 & -\sigma_m\\ \sigma_m & 0\end{pmatrix},\quad \{\gamma_\mu,\gamma_\nu\}=-2\delta_{\mu\nu}, \quad (2)$$

$D_\mu = \partial_\mu + ie(\lambda^a/2)A_\mu^a \equiv \partial_\mu + iA_\mu$, B is a normalization constant, $e^2 = e_0^2/N$. Integrating over ψ and throwing away internal $O(1/N)$ quark loops, we write the connected part of (1) in the form[4,5] (for simplicity we use scalar quarks)

$$K(1,\dots,n)=\int Dx_\mu(\gamma)D\lambda(\gamma) \cdot \left\{\langle C(\Gamma)\rangle_A\exp\left[-\frac{1}{2}\oint_\Gamma d\gamma(\dot{x}^2/\lambda+\lambda m_0^2)\right]\right\}. \quad (3)$$

We made use here of the representation of the quark propagator in the field A by an integral over paths $x_\mu(\gamma)$ and metrics $\lambda(\gamma)$ on the contour Γ.[6,7] The variable γ parametrizes the closed contour Γ, which arises by confluence of the q and $\bar{q}$ trajectories at the points $x_1, ..., x_n$, with $x_\mu(0) = x_\mu(1)$, $\dot{x}_\mu = dx_\mu/d\gamma$. Further, we used in (3) the notation

$$\langle C(\Gamma)\rangle_A=B^{-1}\int d\mu(A)C(\Gamma)\exp(-S_{Y-M}[A]),$$
$$C(\Gamma)=\mathrm{Tr}\left[P\exp\left(-ie\oint_\Gamma dx_\mu A_\mu\right)\right]. \quad (4)$$

For the purpose of going over to S^{eff} it is convenient to rewrite $C(\Gamma)$ as an integral over Grassman fields $\xi_c(\gamma)$, describing the color spin of the quarks[4,8]:

$$C(\Gamma)=\int iD\xi D\xi^{\cdot} i\xi_c(1)\xi_c(0)\exp\{-i\xi_d{}^{\cdot}(0)\xi_d(0)-S[\xi]\},$$
$$S[\xi]=\oint_\Gamma d\gamma\, \xi_c{}^{\cdot}(\gamma)\,[d/d\gamma+ie(\lambda^a/2)A_\mu{}^a\dot{x}_\mu]_{cd}\xi_d(\gamma),$$
$$(\xi_1\xi_2)^{\cdot}=\xi_1{}^{\cdot}\xi_2{}^{\cdot}. \quad (5)$$

Following these manipulations Eq. (3) takes on a form convenient for the application of the saddle-point method:

$$K(1,\dots,n)=B^{-1}\int d\mu(A)Dx(\gamma)D\lambda(\gamma)[iD\xi(\gamma)D\xi^{\cdot}(\gamma)]i\xi_c(1)\xi_c{}^{\cdot}(0) \cdot \exp\{-S[A,\xi,x]-i\xi_d{}^{\cdot}(0)\xi_d(0)\}, \quad (6)$$

$$S[A,\xi,x]=\oint_\Gamma d\gamma[{}^1/{}_2(\dot{x}^2/\lambda+\lambda m_0{}^2)+\xi^{\cdot}(\gamma)D_\gamma\xi(\gamma)] +{}^1/{}_4\int d^4x\, G_{\mu\nu}{}^aG_{\mu\nu}{}^a. \quad (7)$$

The choice of more suitable variables is based on the follow-

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ing arguments. It follows from $\delta S/\delta\xi^* = 0$ that

$$\xi_c^{cl}(\gamma)=\left\{P\exp\left[-ie\frac{\lambda^a}{2}\int_0^\gamma d\gamma\ \frac{dx_\mu}{d\gamma'}A_\mu^a(x(\gamma'))\right]\right\}\ \xi_d(0), \quad (8)$$

where $\xi(0)$ is arbitrary.

The field (8) maps the contour Γ into the group $SU(N)$, which is trivial since $\pi_1[SU(N)] = 0$ for $N\geqslant 2$. An exception is provided by the case of spontaneous symmetry breaking down to the local subgroup $U(1)$, since $\pi_1[U(1)] = Z$. Hence quantization arises[9] of the chromoelectric current of the gauge field on an arbitrary surface Σ with boundary $\partial\Sigma = \Gamma$. In the quasiclassical approach one must have a stable extremum of the action S. It is known that stable field configurations consist of topologically nontrivial solutions of the classical equations of motion.

This quantization of the current can only stabilize the field configuration if it is the full current of the field that is being quantized, and not some arbitrary part of it. Since the contour Γ represents a one-dimensional boundardy $\partial\Sigma$, such a requirement will be satisfied by quasi-two-dimensional fields that "live" on surfaces Σ with injection $x_\mu = x_\mu(\eta^i)$ and $\partial\Sigma = \Gamma$,

$$[A_\mu^a(x(\eta))]_\Sigma=\left[\frac{\partial x_\mu}{\partial\eta^i}A^{a,i}(\eta)\right]_\Sigma\ (i=1,2,\ \mu=1,2,3,4). \quad (9)$$

We therefore make a change of variable, introducing under the integral sign in (6) the unit functional

$$1=\int\prod_\eta D[A(x(\eta))]_\Sigma\delta\{[A_\mu^a(x(\eta))]_\Sigma-A_\mu^a(x)\}$$

$$=\int\prod Dx_\Sigma(\eta)\int d\mu[A(\eta)]_\Sigma J_\Sigma[A(\eta),x(\eta)]\delta$$

$$\cdot\left\{\left[\frac{\partial x_\mu}{\partial\eta^i}A^{a,i}(\eta)\right]-A_\mu^a(x)\right\}, \quad (10)$$

with the Jacobian of the transformation given by

$$J[A,x]=\det\left|\frac{\partial x_\mu}{\partial\eta^i}\right|\det\left|\frac{\delta A_\mu^a(x(\eta))}{\delta x_\nu(\eta)}\right|=g^{1/2}\det\left|\frac{\delta A_\mu^a}{\delta x_\nu}\right|.$$

Thereafter the correlator (6) will contain integration over two-dimensional fields $A_i^a(\eta)$ for fixed surface Σ followed by summation over all surfaces (i.e., over $x_\mu(\eta)\equiv x_\Sigma$):

$$K(1,\dots,n)=B^{-1}\int Dx(\gamma)D\lambda(\gamma)Dx_\Sigma(\eta)d\mu$$

$$\cdot[A(\eta)]_\Sigma J_\Sigma(A,x)iD\xi D\xi^* i\xi_c(1)$$

$$\cdot\xi_c^*(0)\exp\{-i\xi_d(0)\xi_d^*(0)-S_\Sigma^{eff}\}, \quad (11)$$

where

$$S_\Sigma^{eff}[A,\xi,x]=\frac{1}{4}\int_\Sigma d^2\eta\, g^{1/2}g^{il}g^{kn}G_{ik}^aG_{ln}^a$$

$$+\oint_{\partial\Sigma}d\gamma\left[\frac{1}{2}\left(\frac{\dot{x}^2}{\lambda}+\lambda m_0^2\right)\right.$$

$$\left.+\xi_c^*\left(\frac{d}{d\gamma}+ie\frac{\lambda^a}{2}A_i^a\dot{\eta}^i\right)_{cd}\xi_d\right],$$

$$G_{ik}^a=\partial_iA_k^a/\partial\eta^i-\partial A_i^a/\partial\eta^k-\varepsilon f^{abc}A_i^bA_k^c, \quad (12)$$

$g_{ik} = (\partial x_\mu/\partial\eta^i)(\partial x_\mu/\partial\eta^k)$ is the metric induced by the insertion $x_\mu = x_\mu(\eta)$ into R^4. For fixed surface Σ the δ-function in (10) picks out fields defined only on that surface. The volume element d^4x is replaced by $d^2\eta\, g^{1/2}\delta^2$, where δ is the size of the cells that R^4 is divided into when evaluating the functional integral in (1). Correspondingly we redefine the field and charge in (12): $A_i^a\to A_i^a\delta$, $e\to e/\delta\equiv\varepsilon$, as is dictated by the requirement that the action $S^{\rm eff}$ be dimensionless (it is understood that no physical quantities can depend on δ [9]). Here and above it is understood that

$$e^2=e^2(\delta)\approx\frac{16\pi^2}{{}^{11}/_3N\ln(1/\delta^2\Lambda^2)}\equiv\frac{e_0^2}{N}.$$

The variation $\delta S^{\rm eff}/\delta A_i^a = 0$ gives rise to the equations of motion for $A^{cl}(\eta)$ on Σ:

$$\partial_i(g^{1/2}G^{a,ik})+\varepsilon f^{abc}A_i^cg^{1/2}G^{b,ik}=0, \quad (13)$$

and the boundary conditon on $\partial\Sigma = \Gamma$:

$$[g^{1/2}G^{a,ik}(\eta(\gamma))e_{is}+\varepsilon T^a(\gamma)\delta_s^k]\dot{\eta}^s=0, \quad (14)$$

where $T^a(\gamma)$ is the color spin of the quark and e_{is} is the antisymmetric unit tensor. Equation (13) has a solution, which spontaneously breaks the original gauge symmetry down to the local $SU(N-1)$ subgroup:

$$G^{a,ik}(\eta)=\varepsilon e^{ik}g^{-1/2}(\eta)I^a(\eta)\ \text{ for } D_i^{ab}I^b(\eta)=0. \quad (15)$$

It follows from Eq. (14) that the square of the covariantly-constant vector $I^a(\eta)$ is equal to the square of the color spin of the quark:

$$I^2\equiv I^aI^a=T^aT^a=(N^2-1)/2N. \quad (16)$$

As a result of condition (14) the potential A^{cl}, corresponding to (15), reduces on $\partial\Sigma = \Gamma$ in essence to an abelian $U(1)$ field, which ensures topological quantization of the phase $\xi^{cl}(\gamma)$ in Eq. (8) (see Ref. 9, Sec. 4). At the extremum $S^{\rm eff}$ reduces to the well-known action for string and quarks.

$$S^{eff}[A^{cl},\xi^{cl},x]=k_0\int_\Sigma d^2\eta\, g^{1/2}+{}^1/_2\oint_{\partial\Sigma}d\gamma\,(\dot{x}^2/\lambda+\lambda m_0^2). \quad (17)$$

Here $k_0 = \varepsilon^2I^2/2$ is the bare string tension coefficient. As a result of current quantization the action $S_{Y\text{-}M}[A]$ in (12) and (17) is also quantized[9]:

$$k_0\int_\Sigma d^2\eta\, g^{1/2}=\hbar\pi|Q|,\quad Q=0,\pm1,\pm2\dots. \quad (18)$$

The index Q [the number of "windings" of the phase $\xi^{cl}(\eta)$] has a gauge-invariant representation in terms of the first Chern class

$$Q=\frac{\varepsilon}{4\pi}\int_\Sigma d^2\eta\, e^{ik}I^a(\eta)G_{ik}^a(\eta) \quad (19)$$

and defines the different topological sectors.

As a result the string field $A^{cl}(\eta)$ is stable against small fluctuations δA.[10] [In the gauge in which $I^a = \text{const}$, the field A^{cl} has the form[9]

$$A_i^a(\eta)=(I^ae_{ki}/2\varepsilon)g_R^{1/2}g_R^{kl}\partial_l(\ln(g_R^{1/2})), \quad (20)$$

where g_R is the metric of constant curvature $R = -2\varepsilon^2$ on Σ.]

The calculation of the contribution of Gaussian fluctuations δA to $S^{\rm eff}$ for $N\gg 1$ was performed by the author and will be described in a separate paper. Here we just give the answer (for conformal coordinates $\eta_\alpha = u, v$):

$$\Delta S^{eff} = \frac{N^2|R|}{2^4} \int_\Sigma d^2\eta \, g_R^{1/2} \partial_\alpha \ln g_R^{1/2} \partial^\alpha \ln g_R^{1/2} \langle \eta | \Box^{-2} | \eta \rangle,$$

$$\Box = \nabla_\alpha \nabla^\alpha, \qquad g_R^{1/2}(u, v) = 4\left[1 - \frac{|R|}{2}(u^2+v^2)\right]^{-2}. \tag{21}$$

Since $|R| \sim e^2 \sim 1/N$, $[\partial_\alpha \ln(g_R^{1/2})]^2 \sim 1/N^2$, the contribution of the fluctuations is $\Delta S^{\text{eff}} \sim 1/N$, i.e., small for $N \gg 1$, as it should be in line with the logic of the quasiclassical expansion in the $1/N$ parameter.

The expressions (15)–(20) describe for $Q \neq 0$ the nonperturbative phase (or the confinement phase), the sector with $Q = 0$ corresponds to the perturbative phase.

The inverse of the number of colors $1/N$ plays in our approach the same role as the constant $\hbar$ in the usual quasiclassical approach or the temperature T in statistical physics (this last can be seen from the analogy between the functional integral and the statistical sum).

The appearance of the nonperturbative phase is connected with the lowering of the original symmetry, so that we have a phenomenon analogous to second order phase transition in statistical physics. The role of the order parameters is here played by the quantity $\rho = (I^a I^a)^{1/2}$. In the standard theory of phase transitions[3] the temperature dependence of the order parameter near the transition point T_c to the nonsymmetric phase has the form $\rho \sim (T_c - T)^{1/2}$. In our case the same behavior occurs near T_c [see (16)], with $T_c \equiv 1/N_c = 1$. In this manner, in passing to the abelian theory ($N = 1$) the confinement phase automatically disappears.

The appearance of several characteristic features [quantization of the action (18), constancy of the curvature R of the surface Σ, the additional Jacobian in (11)] permit one to hope that the present approach will be free of the well-known difficulties that beset quantum string theories (unphysical dimension of the space R^d, appearance of tachyons).

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