Selection of the velocity and direction of growth of an isolated dendrite

E. A. Brener, S. É. Esipov, and V. I. Mel'nikov

Institute of Solid-State Physics, Academy of Sciences of the USSR, Chernogolovka, Moscow Province

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR, Chernogolovka, Moscow Province

(Submitted 7 July 1987)
Zh. Eksp. Teor. Fiz. **94**, 236–244 (March 1988)

An analysis is made of the mechanism of selection of the velocity and direction of growth of the two-dimensional dendrite when this mechanism is related to the surface energy anisotropy. The condition of solvability of the initial nonlinear integrodifferential equation describing the shape of the crystallization front is reduced, in the weak anisotropy limit, to the condition that the solution of the linear differential equation is finite near a singularity in a complex plane. The spectrum of the dendrite growth velocities is found. It is shown that a dendrite grows along the direction of the surface energy maximum. A calculation is reported of a correction, linear in the isotropic part of the surface energy, to the parabolic shape of a dendrite.

1. INTRODUCTION

The problem of selection of the growth velocity of a needle-shaped dendrite forming from a supercooled melt is typical of structure formation in nonlinear systems (for a review see, for example, Ref. 1). The steady-state solutions of the Stefan problem for an isolated two-dimensional dendrite represent a family of parabolas $y = -x^2/2\rho$, where the growth velocity is $v \propto 1/\rho$ (Ref. 2). The experimentally observed shape of a dendrite is indeed very close to a parabola,[3,4] but the parabola parameter ρ and the velocity v are governed uniquely by the growth conditions. In the search for a mechanism governing the selection of the growth velocity it is found that an important role is played by a finite surface tension at the phase boundary. The following conclusions are drawn in Refs. 5 and 6 from numerical calculations: 1) there are no steady-state solutions of the type representing a needle-shaped dendrite in the case of a finite isotropic surface tension; 2) when allowance is made for the anisotropy of the surface tension, a discrete spectrum of the growth velocities is obtained (in contrast to a continuous spectrum in the absence of surface tension); 3) only the solution corresponding to the maximum growth velocity exhibits a small-scale stability. In an analytic interpretation of these results it is pointed out that the surface tension plays the role of a singular perturbation and the selection of the growth velocity follows from the condition of solvability of the problem in the presence of this singular perturbation. A qualitative analysis of the growth conditions shows that in the case of low values of the anisotropy α and supercooling Δ parameters the growth velocity obeys $v \propto \Delta^4 \alpha^{7/4}$ (Ref. 7). This solution selection mechanism was first demonstrated in Refs. 8–10 when solving the Saffman-Taylor problem.[8–10]

We shall develop an analytic theory of the selection of the velocity and direction of growth of a needle-shaped dendrite in the limit of a weak anisotropy of the surface tension. We shall find the regular correction to the parabolic shape of a dendrite, calculate the spectrum of growth rates, and find the direction of growth of a dendrite. A brief report of some of the results was given earlier.[11]

2. GROWTH EQUATIONS FOR A TWO-DIMENSIONAL DENDRITE

The distribution of the temperature T in a supercooled melt and a growing crystal is given by the heat conduct equation

$$\partial T/\partial t = D\Delta T. \tag{1}$$

The evolution of heat occurs at the crystallization front $y(x)$ and the boundary condition is

$$c_p D[\mathbf{n}\nabla T_l - \mathbf{n}\nabla T_c] = -Lv_n. \tag{2}$$

Here, c_p and D are the specific heat and the thermal diffusivity, which are identical for both phases; L is the latent heat of fusion; v_n is the normal growth velocity; $\mathbf{n}$ is a unit vector normal to the phase boundary; the indices l and c represent the molten liquid and the crystal, respectively. If we ignore the kinetic effects at the crystallization front, we find that the equilibrium boundary condition becomes

$$T(x,\ y(x)) = T_m + T_m[\gamma_E(\theta)/L]k(x), \tag{3}$$

where T_m is the melting point; $k(x) = y''/(1+y'^2)^{3/2}$ is the curvature of the crystallization front; $\gamma_E(\theta) \equiv \gamma(\theta) + d^2\gamma(\theta)/d\theta^2$; $\gamma(\theta)$ is the anisotropic surface energy: θ is the angle between the normal to the surface and the y axis. It follows from the thermodynamic stability condition that $\gamma_E > 0$. Far from the melt is supercooled and its temperature is $T_0 < T_m$.

In the case of steady-state growth of a dendrite at a velocity v along the y axis, we find that using the Green function of Eqs. (1) and (2) and applying the equilibrium condition (3), we obtain the following integrodifferential equation describing the shape of the crystallization front $y(x)$:

 0038-5646/88/030565-05$04.00

$$\Delta+\frac{d_0(\theta)k(x)}{\rho}=\frac{p}{2\pi}\int_{0}^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty}dx'\exp\Big\{-\frac{p}{2t}[(x-x')^2$$

$$+[y(x)-y(x')+t]^2]\Big\}$$

$$=\left(\frac{p}{\pi}\right)\int_{-\infty}^{\infty}dx'\exp\{p[y(x')-y(x)]\}K_0(p\{(x-x')^2$$

$$+[y(x)-y(x')]^2\}^{1/2}),\tag{4}$$

where K_0 is a Macdonald function; all the lengths are measured in terms of ρ; $\Delta=(T_m-T_0)c_pL^{-1}$ is the dimensionless supercooling; $p=v\rho/2D$ is the Peclet number; $d_0(\theta)=\gamma_E(\theta)T_mc_pL^{-2}$ is the capillary length. Following Refs. 5–7, we shall assume that

$$d_0(\theta)=\bar{d}_0(1-\alpha\cos 4\theta),\quad \operatorname{tg}\theta=dy/dx,\tag{5}$$

and that the anisotropy parameter α is small so that $\alpha\ll 1$. In the absence of surface tension ($d_0=0$) the solution of Eq. (4) is a parabola $y=-x^2/2$ (Ref. 2) and the Peclet number is given by the equation

$$\Delta=2p^{1/2}e^p\int_{p^{1/2}}^{\infty}e^{-x^2}dx.\tag{6}$$

This result is obtained from Eq. (4) if $y=-x^2/2$, provided we substitute $t=(x'-x)^2/2\omega$ and integrate initially with respect to x'. This gives an expression independent of x, which reduces to Eq. (6) after integration with respect to ω. It should be pointed out that in the absence of surface tension this solution is valid also for an arbitrary relationship between the thermal diffusivities of the melt and crystal. This is due to the fact that the crystallization front is an isotherm and the temperature throughout the crystal is constant: $T=T_m$. All heat is therefore lost through the melt and it is independent of the thermal characteristics of the crystal, so that the Peclet number is governed by the thermal diffusivity of the melt. The steady-state solution $y=-x^2/2$ corresponding to $d_0=0$ is unstable in the presence of small perturbations of the phase boundary.[12]

Inclusion of a finite surface tension in Eq. (4) distorts the shape of the crystallization front: $y(x)=-x^2/2+\zeta(x)$ A linear equation for $\zeta(x)$ in the limit

$$\sigma=\bar{d}_0/p\rho\ll 1,\quad p\ll 1\tag{7}$$

has the form[7]

$$\sigma\zeta''-\frac{3\sigma x\zeta'}{1+x^2}-\frac{(1+x^2)^{1/2}}{2\pi A(x)}\int_{-\infty}^{\infty}dx'\frac{(x+x')[\zeta(x)-\zeta(x')]}{(x-x')[1+(x+x')^2/4]}=\sigma,$$

$$A(x)=1+8\alpha x^2/(1+x^2)^2.\tag{8}$$

The integral in Eq. (8) is found by the residue method if the function $\zeta(x)$ is divided into terms $\zeta_+(x)$ and $\zeta_-(x)$, which are analytic in the upper and lower half-planes of complex values of x. We then obtain

$$\sigma[\zeta_+''(x)+\zeta_-''(x)]-\frac{3\sigma x}{1+x^2}[\zeta_+'(x)+\zeta_-'(x)]$$

$$-\frac{i(1+x^2)^{1/2}}{A(x)}[(x+i)\zeta_+(x)$$

$$+(x-i)\zeta_-(-x-2i)-(x+i)\zeta_+(-x+2i)-(x-i)\zeta_-(x)]=\sigma\,.\tag{9}$$

A regular correction $\zeta(x)\propto\sigma$ to the shape of the crystallization front is found by solving Eq. (9) ignoring the derivatives. A singular perturbation associated with the derivatives has the effect that Eq. (9) is solvable only for certain values of the parameter σ. In the final analysis, this determines the spectrum of the dendrite growth velocities

3. REGULAR PART OF THE PERTURBATION: DEVIATION OF THE DENDRITE SHAPE FROM A PARABOLA

In a calculation of the regular correction $\zeta\propto\sigma$ we shall simplify Eq. (9) by dropping the terms with the derivatives and assume that $A(x)=1$ and also that the anisotropy is weak so that $\alpha\ll 1$. Then, $\zeta(x)$ is described by the following equation:

$$[(x+i)\zeta_+(x)+(x-i)\zeta_-(-x-2i)]$$

$$-[(x+i)\zeta_+(-x+2i)+(x-i)(\zeta_-(x)]=i\sigma(1+x^2)^{-1/2}.\tag{10}$$

The first and second sets of square brackets identify the expressions which are analytic if $\operatorname{Im}x>0$ and $\operatorname{Im}x<0$. Solving this equation by the Wiener-Hopf method, we shall expand the right-hand side using the functions $\Phi_\pm(x)$ which are analytic in the upper and lower half-planes:

$$\Phi_\pm(x)=\mp\frac{\sigma}{2\pi}\int_{-\infty}^{\infty}\frac{dz}{(1+z^2)^{1/2}(x-z\pm i0)}.\tag{11}$$

Equation (10) is then equivalent to a system of two equations:

$$(x+i)\zeta_+(x)+(x-i)\zeta_-(-x-2i)=\Phi_+(x),$$

$$-(x+i)\zeta_+(-x+2i)-(x-i)\zeta_-(x)=\Phi_-(x).$$

Excluding the function $\zeta_+(x)$ from these equations, we obtain an equation which contains only $\zeta_-(x)$. It can be written down conveniently using the substitution

$$z_-(x)=\zeta_-(x)/(x+i).\tag{12}$$

Then, z_- is described by the equation

$$z_-(x)-z_-(x-4i)=\mathcal{F}(x),\tag{13}$$

$$\mathcal{F}(x)=-\frac{1}{x-i}\left[\frac{\Phi_-(x)}{x+i}+\frac{\Phi_+(-x+2i)}{(-x+2i)+i}\right].\tag{14}$$

The solution of the difference equation (13) is

$$z_-(x)=\sum_{k=0}^{\infty}\mathcal{F}(x-4ik).$$

This sum converges because the function $\mathcal{F}(x)$ decreases rapidly in the lower half-plane. Using the explicit form of the function $\mathcal{F}(x)$ and the integral representation of Eq. (11) for $\Phi_\pm(x)$, we can sum over k. Consequently, we find that the function $\zeta_-(x)$ is described by

$$\zeta_-(x)=\frac{\sigma}{2\pi}+\frac{i\sigma(x+i)}{8\pi}\int\limits_{-\infty}^{\infty}\frac{ds}{(s^2+1)^{3/2}}\left[\frac{4i}{x-s-i0}+\psi\left(1+i\frac{x-s}{4}\right)\right.$$
$$\left.-\psi\left(\frac{1}{2}+i\frac{x-s}{4}\right)\right], \qquad (15)$$

where $\psi(x)$ is a logarithmic derivative of the gamma function. In view of the translational invariance of the problem, the function $\zeta(x)$ is defined, apart from an arbitrary constant which we shall select on the basis of the condition $\zeta(\infty)=0$. Therefore, in the approximation which is linear in σ, the shape of the crystallization front is described by

$$y(x)=-x^2/2+\sigma\eta(x), \qquad (16)$$

where

$$\eta(x)=(2/\sigma)\operatorname{Re}\zeta_-(x)|_{\operatorname{Im}x=0}.$$

A graph of the function $\eta(x)$ is shown in Fig. 1. We shall find the asymptote of $\eta(x)$ in the case when $|x|\gg 1$. We shall do this by expanding the expression in square brackets of Eq. (15) up to terms $\sim s^2/x^3$, where $s\ll x$, and we shall truncate the resultant logarithmic integral with respect to s at $s\sim x$. This gives

$$\eta(x)\approx-\frac{1}{\pi}\frac{\ln|x|+1}{x^2}. \qquad (17)$$

This nature of the asymptote of $\eta(x)$ is given in Ref. 9. The correction to the dendrite shape is found by a different method in Ref. 13.

4. SINGULAR PERTURBATION IN THE CASE OF ISOTROPIC SURFACE TENSION. ABSENCE OF STEADY-STATE SOLUTIONS

In the preceding section we ignored the derivatives in Eq. (9) and found the regular correction to the shape of the crystallization front $\zeta(x)\propto\sigma$. Inclusion of the derivatives gives rise to corrections in powers of σ, which are small almost throughout the complex x plane. However, this is not true in small regions around singularities $x=\pm i$. By way of example, we shall consider the vicinity of point $x=i$. We can see from Eq. (10) that at $x\simeq i$ all the functions on the left-hand side are regular with the exception of $\zeta_-(x)$, so that

$$\zeta_-(x)\propto(x-i)^{-3/2}, \quad |x-i|\ll 1.$$

Therefore, the neglect of the derivatives is unjustified near $x=i$ and an analysis of the complete Eq. (9) is required. Since the mean contribution to the terms deduced from the integral term is made by $\zeta_-(x)$, we have to investigate the differential equation containing just this function.

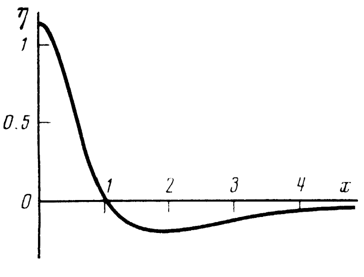

FIG. 1. Regular correction $\eta(x)$ to the parabolic shape of a dendrite.

We shall assume that $|x-i|\ll 1$ and make the substitutions

$$x=i(1-\sigma^{2/7}t), \quad \psi=\sigma^{-4/7}t^{-3/4}\zeta_-(x(t)).$$

Then, ψ is described by the equation

$$\psi''-(2^{1/2}t^{3/2}+21/16t^2)\psi=-t^{-3/4}. \qquad (18)$$

If $1\ll|t|\ll\sigma^{-2/7}$, the general solution of this equation contains terms of the type

$$\psi_{1,2}\propto t^{-3/8}\exp[\pm{}^4/_7\cdot 2^{1/4}t^{7/4}],$$

which rise most rapidly along the rays $\arg t=0, \pm 4\pi/7$. Matching to the solution of the Wiener-Hopf Eq. (15) requires suppression of the exponentially growing terms along these three rays. We have only two integration constants which is insufficient to satisfy these requirements. In fact, the general solution of Eq. (18) is

$$\psi=C_1t^{1/2}J_{5/7}\left(\frac{4}{7}\cdot 2^{1/4}it^{7/4}\right)+C_2t^{1/2}J_{-5/7}\left(\frac{4}{7}\cdot 2^{1/4}it^{7/4}\right)$$
$$+\Gamma\left(\frac{6}{7}\right)\Gamma\left(\frac{11}{7}\right)t^{7/4}\sum_{n=0}^{\infty}\left(\frac{2^{5/2}}{49}\right)^n\frac{t^{7n/2}}{\Gamma(n+{}^6/_7)\Gamma(n+{}^{11}/_7)}.$$

Asymptotes of the Bessel functions J_ν are known and the asymptote of a series can be calculated using the relationship

$$\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+\alpha)\Gamma(n+\beta)}=\frac{1}{2\pi^{1/2}}x^{3/4-(\alpha+\beta)/2}\exp(2x^{1/2}), \quad x\gg 1. \qquad (19)$$

Selecting the constants C_1 and C_2 in such a way as to suppress the exponential rise along the rays $\arg t=\pm 4\pi/7$, we find for real values $t\gg 1$ that

$$\psi=\frac{1}{\pi^{1/2}}\left(\frac{49}{2^{5/2}}\right)^{13/8}\Gamma\left(\frac{6}{7}\right)\Gamma\left(\frac{11}{7}\right)\left[\sin\frac{2\pi}{7}\operatorname{ctg}\frac{3\pi}{7}+\frac{1}{2}\right]$$
$$\times t^{-3/8}\exp\left(\frac{4}{7}\cdot 2^{1/4}t^{7/4}\right).$$

Therefore, direct calculations demonstrate that it is impossible to match the solution of the differential Eq. (18) with the solution of the Wiener-Hopf equation and, consequently, the initial Eq. (8) cannot be solved for an isotropic surface tension. This analytic result is in agreement with numerical calculations reported in Ref. 5 and with the WKB approximation discussed in Refs. 7 and 14.

5. ANISOTROPIC SURFACE TENSION

The problem changes qualitatively if we allow for a weak anisotropy of the surface tension. For an anisotropy of the simplest kind described by Eq. (5) near a point $x=i$, we have

$$A(x)\approx 1+2\alpha/(x-i)^2,$$

which shows that the values $|x-i|\propto\alpha^{1/2}$ are important. Making the substitutions

$$x=i(1-\alpha^{1/2}t), \quad \zeta_-(x(t))=\alpha t^{3/4}\psi(t),$$

we obtain the following equation for $\psi(t)$:

$$d^2\psi/dt^2+P^2(t)\psi=-t^{-3/4}, \qquad (20)$$

$$P^2(t)=-[21/16t^2+2^{1/2}\lambda t^{7/2}/(t^2-2)]. \qquad (21)$$

The small parameters σ and α are excluded by introducing

$$\lambda=\alpha^{7/4}/\sigma. \tag{22}$$

The parameter σ depends on the radius of curvature ρ, so that determination of the values of λ is equivalent to selection of the solution with very specific parameters ρ and v.

As demonstrated in the preceding section, matching of the solution of Eq. (20) to the solution of the Wiener-Hopf equation for $1 \ll |t| \ll \alpha^{-1/2}$ requires suppression of the exponentially growing solutions of Eq. (20) along the rays $\arg t = 0, \pm 4\pi/7$. Introduction of the anisotropy, in addition to the two integration constants, provides us also with the parameter λ. For specific values of this parameter we can satisfy the boundary conditions formulated above. The spectrum of λ found in this way determines the spectrum of the growth velocities of the dendrite under investigation.

We shall now summarize the results obtained. We began with the integrodifferential Eq. (8) containing small parameters α and σ. We found the regular correction to the shape of the crystallization front $\zeta(x) \propto \sigma$ independent of the weak anisotropy α. The terms with the derivatives in Eq. (18) behave as a singular perturbation concentrated near the points $x = \pm i$. If $\alpha = 0$, the presence of this perturbation makes the problem unsolvable. In the case of finite but small values of α an investigation of the singular perturbation reduces to solution of the differential Eq. (20) describing the spectrum of the parameter λ. Near a singularity the function $\zeta \propto \alpha$ is much greater than the regular correction $\zeta \propto \sigma$ because $\sigma \propto \alpha^{7/4} \ll \alpha$.

Equation (20) is the inhomogeneous Schrödinger equation defined in the complex plane t with a cut along the semiaxis $(-\infty, \sqrt{2})$. Its solution and determination of the spectrum of λ can be performed either numerically or using the WKB approximation in which it is assumed formally that $\lambda \gg 1$. Numerical integration gives the following spectrum of λ:

$$\lambda_0=0.48, \quad \lambda_1=5.8, \quad \lambda_2=17.5, \quad \lambda_3=34.4.$$

These results apply to a situation when the function $d_0(\theta)$ [see Eq. (5)] is minimal along the growth direction. If the angle between these directions is φ, then

$$d_0(\theta)=\bar{d}_0\{1-\alpha\cos[4(\theta-\varphi)]\}.$$

The numerical calculations were not carried out for the case when $\varphi \neq 0$. The general structure of the problem does not exclude, at least in principle, the existence of a discrete spectrum of the angles φ. We shall show in the next section that in the semiclassical limit $\lambda \gg 1$ within the range $0 \leqslant \varphi \leqslant \pi/4$ the solutions exist only for $\varphi = 0$. Therefore, there is a unique direction of growth which corresponds to the minimum value of d_0 [i.e.. to the minimum of the function $\gamma(\theta) + \gamma''(\theta)$] and to the maximum of the surface tension $\gamma(\theta)$.

We shall conclude this section with the expression for the growth velocity of an isolated dendrite. If $\Delta \ll 1$, it follows from Eq. (6) that $\Delta \approx (\pi p)^{1/2}$. Combining this relationship with Eqs. (7) and (22), we obtain

$$v_n=2D\Delta^4\alpha^{7/4}/\pi^2\lambda_n\bar{d}_0. \tag{23}$$

We must point out the following important circumstance. Equation (8) is derived on the assumption that $p, \Delta \ll 1$ (Ref. 7). An analysis of Eq. (3) shows that, in view of the small size of singular regions near $x = \pm i$, the linear Eq. (20) for the singular part of the function $\zeta(x)$ applies as long as $p \ll \alpha^{-1/2}$, i.e., also when $p \sim 1$. In this case the solution for the regular correction to the crystallization front shape cannot be obtained, but the dependence of the growth velocity on the parameters of the problem can be determined using only the exact relationship between p and Δ given by Eq. (6).

6. SEMICLASSICAL SPECTRUM OF VELOCITIES AND SELECTION OF THE GROWTH DIRECTIONS

Equation (20) can be regarded as an inhomogeneous Schrödinger equation with zero energy and the potential $-P^2(t)$. The behavior of this potential at $t>0$ is shown schematically in Fig. 2. At $t=0$ there is a second-order pole and a branching point. At $t_1 \approx 2^{7/11}(21/\lambda)^{2/11} \ll 1$ there is a turning point and a pole at $t=\sqrt{2}$. The interval $t_1 < t < \sqrt{2}$ is the range of allowed motion. We can find the spectrum of λ if, firstly, we solve Eq. (20) in the range $|t| \ll 1$ including the singularity $t=0$ and the turning point $t=t_1$ and, secondly, if we solve Eq. (20) for $|t-\sqrt{2}| \ll 1$ and, finally, match the semiclassical asymptotes of these solutions in the range of allowed motion.

We shall consider the region $\operatorname{Im} t \geqslant 0$ and assume that the solution of Eq. (20) obtained for real values of t is real. Then, in view of the symmetry the finite nature of the solution along the ray $\arg t = -4\pi/7$ follows asymptotically from the solution that it is finite along the ray $\arg t = 4\pi/7$.

If $|t| \ll 1$, Eq. (20) becomes

$$t^2\psi''+(\lambda t^{11/2}/2^{3/2}-21/16)\psi=-t^{5/4}/4. \tag{24}$$

The general solution of this equation is

$$\psi=C_1t^{1/2}J_{5/11}\left(\frac{4\lambda^{1/2}t^{11/4}}{11\cdot 2^{3/4}}\right)+C_2t^{1/2}J_{-5/11}\left(\frac{4\lambda^{1/2}t^{11/4}}{11\cdot 2^{3/4}}\right)$$

$$+\Gamma\left(\frac{10}{11}\right)\Gamma\left(\frac{15}{11}\right)t^{5/4}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{3n/2}\lambda^n t^{11n/2}}{11^n\,\Gamma(n+{}^{10}/_{11})\,\Gamma(n+{}^{15}/_{11})}. \tag{25}$$

This expression rises most rapidly for anti-Stokes lines $\arg t = 2\pi/11, 6\pi/11$. When we go over from $|t| \ll 1$ to $|t| \gg 1$, the lines follow the rays $\arg t = 0, 4\pi/7$. The constants C_1 and C_2 are calculated on the assumption that the solution is finite for the anti-Stokes lines. Using Eq. (19) and the known asymptotes of the Bessel functions J_ν, we find that the function ψ in the region of allowed motion is described by the following quasiclassical asymptote:

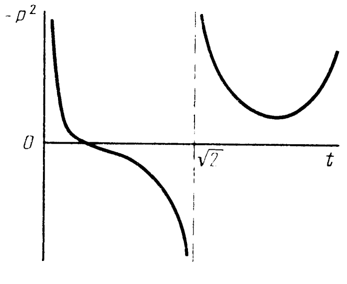

FIG. 2. Qualitative behavior of the potential $-P^2(t)$.

$$\psi \propto P^{-1/2}(t)\exp\left\{i\left[\int_{t_1}^{t} P(t')\,dt' - 5\pi/44\right]\right\}, \tag{26}$$

where the real factor is omitted.

We shall now consider Eq. (20) near the point $t=\sqrt{2}$. The substitutions $t=\sqrt{2}+s$, $|s|\ll 1$ yield the equation

$$\psi'' - 2^{3/4}\lambda\psi/s = -2^{-3/8}, \tag{27}$$

whose general solution that decreases in the limit $s\to\infty$ is

$$\psi = Cs^{1/2}K_1(2^{11/8}\lambda^{1/2}s^{1/2}) + s/2^{9/8}\lambda. \tag{28}$$

In the region of allowed motion we find from Eq. (28) the following semiclassical asymptote:

$$\psi \propto P^{-1/2}\exp\left\{-i\left[\int_{t}^{\sqrt{2}} P(t')\,dt' + 3\pi/4\right]\right\}. \tag{29}$$

Matching the solutions (26) and (29), we find the condition for the semiclassical spectrum of λ:

$$\int_{t_1}^{\sqrt{2}} P(t)\,dt = \pi(n+4/11), \quad n=0,1,\ldots, \tag{30}$$

where t_1 is the root of the equation $P(t)=0$. We shall now give the first few values of λ deduced from Eq. (30):

$$\lambda_0=0.79, \ \lambda_1=7.3, \ \lambda_2=19.7, \ \lambda_3=38.5.$$

If $n\gg 1$, it follows from Eq. (30) that

$$\lambda_n = \pi\left[\frac{7\Gamma(7/8)}{3\cdot 2^{7/8}\Gamma(5/8)}\right]^2 n^2 \approx 3n^2.$$

A spectrum of similar type is obtained in the WKB approximation if we analyze the Saffman-Taylor problem.[15]

If the direction of growth makes an angle of φ with the direction of the minimum of $d_0(\theta)$ (see preceding section), then in the potential of Eq. (21) we have to replace t^2-2 with $t^2-2\exp(-4i\varphi)$ in the denominator of the second fraction. This replacement alters radically the qualitative behavior of the function $\psi(t)$. This is because the solutions of Eq. (20) obtained for $\varphi=0$ oscillate in the cut between the singularities t_1 and $\sqrt{2}$, so that the quantization condition of Eq. (30) is analogous to the condition that the function $\psi(t)$ is single-valued along a contour which begins from the point t_1 and returns to the same point around $t=\sqrt{2}$. If $\varphi\neq 0$, the distribution of the Stokes lines in the plane t changes and in the cut $[t_1, \sqrt{2}\exp(-2i\varphi)]$ the oscillations of $\psi(t)$ are supplemented by, for example, a monotonic rise, so that when we follow this contour, we find that there is a change not only in the phase but also in the amplitude of the function $\psi(t)$. Consequently, it is not possible to obtain a function $\psi(t)$ which is single-valued going round the point $t=\sqrt{2}\exp(-2i\varphi)$ if we select one parameter λ. Nevertheless, we can attempt to do this by simultaneous selection of λ and φ, which would correspond to selection of several directions of the dendrite growth, but in the limit $\lambda\gg 1$ the semiclassical integral of the type given by Eq. (30) is proportional to $\lambda^{1/2}\exp(-7i\varphi/2)$ and for φ in the interval $(0, \pi/4)$ it assumes real values only at $\varphi=0$. Therefore, the difference between φ and zero results in disappearance of the spectrum of λ. It therefore means that $\varphi=0$, i.e., the direction of the maximum of the surface tension is the only possible direction of dendrite growth.

[1]J. S. Langer, Rev. Mod. Phys. **52**, 1 (1980).
[2]G. P. Ivantsov, Dokl. Akad. Nauk SSSR **58**, 567 (1947).
[3]M. E. Glicksman, Mater. Sci. Eng. **65**, 45 (1984)
[4]H. Honjo, S. Ohta, and Y. Sawada, Phys. Rev. Lett. **55**, 841 (1985).
[5]D. A. Kessler and H. Levin, Phys. Rev. B **33**, 7867 (1986).
[6]D. A. Kessler and H. Levine Phys. Rev. Lett. **57**, 3069 (1986).
[7]A. Barbieri, D. C. Hong, and J. S. Langer, Phys. Rev. A **35**, 1802 (1987).
[8]J. W. McLean and P. G. Saffman, J. Fluid Mech. **102**, 455 (1981).
[9]J. M. Vanden-Broeck, Phys. Fluids **26**, 2033 (1983).
[10]D. A. Kessler and H. Levine, Phys. Rev. A **32**, 1930 (1985); **33**, 2621, 2634 (1986).
[11]E. A. Brener, S. E. Ésipov, and V. I. Mel'nikov, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 595 (1987) JETP Lett. **45**, 759 (1987)].
[12]J. S. Langer and H. Muller-Krumbhaar, Acta Metall. **26**, 1689 (1978).
[13]M. Ben Amar and B. Moussallam, Physica D (Utrecht) **25**, 155 (1987).
[14]M. Ben Amar and Y. Pomeau, Europhys. Lett. **2**, 307 (1986).
[15]A. T. Dorsey and O. Martin, Phys. Rev. A **35**, 3989 (1987).

Translated bv A. Tybulewicz