Evolution of light pulses in a nonlinear amplifying medium

V. S. Grigor'yan, A. I. Maĭmistov, and Yu. M. Sklyarov

Scientific-Research Institute of the Physics of Condensed Media at the State University, Erevan

(Submitted 17 June 1987)
Zh. Eksp. Teor. Fiz. **94**, 174–182 (March 1988)

Numerical experiments on a nonlinear amplifying dispersive medium (in the case when the gain exhibits saturation and there are linear losses in the medium) revealed a new effect which is the formation of a solitary wave (soliton) with all its parameters independent of the amplified initial pulse and governed only by the parameters of the amplifying medium such as the gain and its saturation parameter. These numerical experiments were carried out for initial pulses with a regular amplitude and phase modulation as well as for stochastic pulses with different rms deviations of random parameters. The envelope of the resultant solitary wave was described by a smooth bell-shaped curve. The dependences of the main parameters of this wave (maximum amplitude, pulse duration, propagation velocity, etc.) and the parameters of the medium were found. An increase in the gain increased the amplitude of the wave and reduced its duration.

1. INTRODUCTION

The nonlinear Schrödinger equation (NSE), which has been thoroughly studied, is the basic equation in investigations of weakly nonlinear waves in weakly dispersive media.[1] Various examples of the use of this equation can be found in nonlinear optics, plasma physics, theory of superconductivity, physics of low temperatures, and gravity waves on water. However, the number of nonlinear dispersive media and of the effects observed in them is considerably greater. More complex systems with important practical applications include nonlinear amplifying dispersive media, in which in addition to dispersion and nonlinearity of the refractive index saturable gain and losses are also important. However, it should be pointed out that the problem of evolution of pulses in such media has been largely ignored. Since in such cases, as in the majority of nonlinear dispersive media, the evolution equations are nonintegrable, the most effective methods for their investigation are numerical.

A model for the amplification of a short light pulse in a dispersive Kerr medium, where the gain may reach saturation in the field of the pulse itself but there are linear losses, was proposed recently.[2] It was shown that in this situation we can expect the formation of a light pulse with parameters governed by those of the medium and independent of the parameters of the initial pulse being amplified. The conclusion of the formation of a steady-state solitary wave is valid in the case of a low gain (much less than the reciprocal of the dispersion), which makes it possible to utilize a theory of perturbations of solitons based on the NSE. In real amplifying media the gain may be greater than or of the order of the reciprocal of the dispersion length and the method employed in Ref. 2 is inappropriate. Although a solitary wave found in Ref. 2 represents the exact solution of the evolution equation (for arbitrary values of the gain), the problem of its stability when the gain in the sense mentioned above is not small has yet to be considered. Moreover, the general evolution of pulses in such media subjected to given initial pulses is not yet clear.

We therefore carried out numerical experiments on the propagation of pulses in a nonlinear dispersive inverted medium exhibiting both gain saturation and nonlinear losses.

2. BASIC EQUATION

It was shown in Ref. 2 that evolution of a light pulse in a weakly dispersive inverted medium with a Kerr nonlinearity and linear losses is described by the following equation under conditions of weak saturation of the gain:

$$i\frac{\partial q}{\partial z}-\frac{1}{2}\frac{\partial^2 q}{\partial t^2}-q|q|^2=-i\alpha q\int_{-\infty}^{t}|q|^2\,dt'+i\beta q, \tag{1}$$

where $q=t_p(n_2k/2n_0^2|k''|)^{1/2}E$, t_p is the characteristic duration of a pulse; E is the slowly varying envelope of the optical field pulse; N_2 represents a nonlinear correction to the refractive index $n=n_0+n_2|E|^2$; k is the wave vector; $k''=\partial^2k/\partial\omega^2$; the variables z and t are normalized and related to the longitudinal coordinate ξ and time τ by

$$z=|k''|\xi/t_p^2,\quad t=(\tau-k'\xi)/t_p,$$

the coefficients β and α (α is proportional to the density of amplifying centers) represent respectively the linear gain (normalized to the reciprocal dispersion length $t_p^2/|k''|$) and the gain saturation "rate." Equation (1) readily yields the following expression for the pulse energy $u=\int_{-\infty}^{\infty}|q|^2dt$:

$$\partial U/\partial z=2U(\beta-{}^1/_2\alpha U),$$

from which it follows that

$$U=\frac{2\beta}{\alpha}U_0e^{2\beta z}\Big/\Big[U_0\Big(e^{2\beta z}-1\Big)+\frac{2\beta}{\alpha}\Big],$$

where $U_0=U(z=0)$. It is clear from Eq. (2) that the energy of a pulse reaches the value $2\beta/\alpha$ irrespective of the initial energy provided βz is sufficiently large. One of the possible ways in which the field can then behave is the formation of a solitary wave (soliton) of energy $2\beta/\alpha$. For example, if the right-hand part of Eq. (1) is small and can be regarded as a perturbation of the NSE, it is then found from analytic calculations[2] that establishment of the energy of a pulse (irrespective of the initial energy) results in the creation of a solitary wave in the form of a pulse of secant shape irrespective of the initial shape of the pulse envelope. If the right-

 0038-5646/88/030530-05$04.00

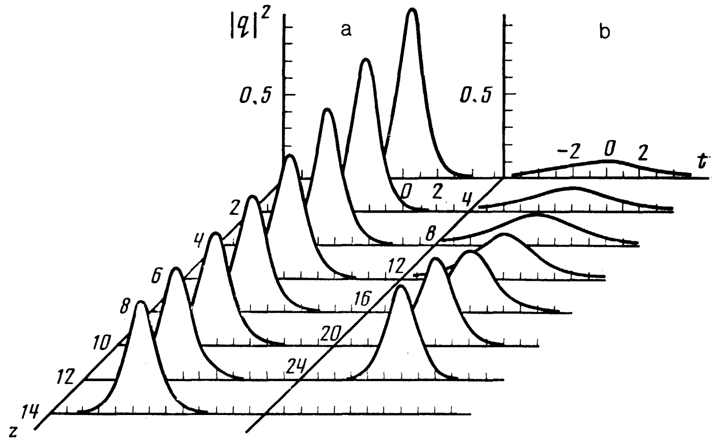

FIG. 1. Evolution of a secant pulse $q(0,t) = \eta\,\mathrm{sech}(\eta t)$ in a medium characterized by $\alpha = 0.1$ and $\beta = 0.08$ when $\eta = 1$ (a) and $\eta = 0.3$ (b).

hand part of Eq. (1) is not small, the method of perturbation of solitons based on the NSE is inapplicable and the most effective method for the solution of Eq. (1) is clearly by numerical calculations.

In the present study we carried out numerical modeling of the propagation of a pulse employing a modification of an implicit Crank-Nicholson scheme.[3]

3. CASE OF SMALL VALUES OF α AND β

We shall assume that the initial pulse satisfies the conditions necessary for the formation of a one-soliton pulse based on the NSE.[4] If β is sufficiently small, we can assume that the gain has little effect on the length of formation of a one soliton pulse. We can follow the evolution of a pulse by assuming that the initial pulse is a single soliton based on the NSE. Consequently, we shall assume that this initial pulse is in the form of $q(0,t) = \eta\,\mathrm{sech}(\eta t)$. Numerical calculations demonstrate (Fig. 1) that, irrespective of the initial amplitude, a secant pulse is formed in the medium. The dependence of the field on z and t in such a pulse is described by

$$q=\frac{\beta}{\alpha}\exp\left\{i\left[\xi_0 t+\frac{1}{2}\left(\xi_0^2-\frac{\beta^2}{\alpha^2}\right)z\right]\right\}\mathrm{sech}\left[\frac{\beta}{\alpha}(t+\xi_0 z+\alpha z)\right]. \quad (3)$$

The maximum intensity $|q|^2_{\max}\beta^2/\alpha^2 = 0.64$ corresponds to $\alpha = 0.1$ and $\beta = 0.08$. Similar results are obtained for other values of α and β. In other words, numerical calculations confirm the results of perturbation theory based on the NSE. The formation of a solitary wave of Eq. (3) occurs if the initial pulse satisfies the condition of formation of a one-soliton pulse based on the NSE [when $\alpha = \beta = 0$ in Eq. (1)] characterized by arbitrary parameters. However, further calculations revealed that if $\alpha, \beta \gtrsim 1$, the solution in the form of a secant pulse of Eq. (3) is unstable in the medium under discussion.

4. CASE WHEN $\alpha, \beta \gtrsim 1$

The relationships describing the propagation of pulses in media characterized by $\alpha, \beta \gtrsim 1$ in the case when α and β are fixed were identified by varying the range of types of initial pulses differing in respect of their profile as well as energy and duration; similar calculations were then carried out for other pairs of α and β. The numerical experiments were carried out both for initial pulses with a regular amplitude and phase modulation and for stochastic pulses.

a) Pulses with a regular amplitude-phase modulation

We selected the initial pulses so that they differed significantly from one another in respect of the law governing the time dependences of the amplitude at the leading and trailing edges. This enabled us to cover the widest possible range of initial pulses. A characteristic pattern of the evolution of a pulse with amplitude modulation is shown in Fig. 2. It is clear from Fig. 2 that even in a distance approximately equal to three dispersion lengths ($z = 3.05$) a pulse transforms into one with a symmetric bell-shaped profile and remains practically constant. In the course of its propagation the pulse shifts forward, in other words, it acquires a velocity higher than $\partial\omega/\partial k$. The velocity of propagation of a pulse becomes constant for $z > 2.85$ (the numbers under the maxima in Fig. 2 represent the positions of the maximum of a pulse along the t axis). We can also see that fairly rapid smoothing and "cutoff" of the internal substructure of a pulse takes place. At the leading edge of a pulse it is possible to identify a smooth symmetric subpulse, and gradually damped oscillations remain at the trailing edge of a pulse. In addition to the case shown in Fig. 2, we investigated also the propagation of a number of initial pulses of different types with both symmetric and asymmetric profiles: Lorentzian, Gaussian, as well as pulses described by more complex curves. In all the investigated cases irrespective of the initial pulse we found that the same solitary wave is obtained for given values of α and β. On the whole, the propagation pattern is similar to that shown in Fig. 2 and there is only a change in the characteristic length of formation of a solitary wave. It should be mentioned that for pulses characterized by an internal substructure this length is greater than for smooth pulses. For example, in the case of an initial pulse of the shape $\pi t^2\exp(-t)$, where $t > 0$, this length is equal to two dispersion lengths. As a rule, it was found that the characteristic length of formation of a solitary wave does not exceed 4–6 dispersion lengths.

We also investigated the propagation of phase-modulated pulses. In this case we considered the most important, from the practical point of view, type of phase modulation, which is quadratic modulation. Numerical calculations were carried out for initial pulses with the profile

$$q(0,t) = \exp(-t^2 \pm i\delta t^2)$$

characterized by $\delta = 1$, 5, 10, and 20. In the presence of phase modulation we found that at low values of z a pulse

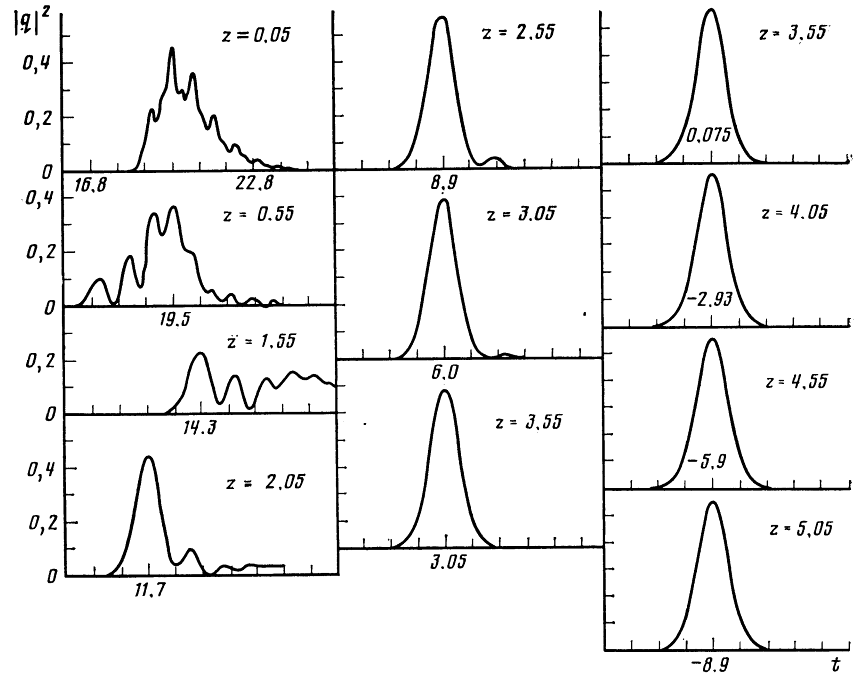

FIG. 2. Typical example of the evolution of a pulse with a regular amplitude modulation in a medium characterized by $\alpha = 5$ and $\beta = 2$, calculated for different values of z:

$$q(0,t) = \begin{cases} 0, & t<0 \\ t^2(2+\sin 8t)e^{-t}, & t\geqslant 0 \end{cases}$$

acquires a velocity which increases on increase in the value of δ. However, the pulse slows down quite rapidly (already for $z\approx 0.5$) the pulse slows down and a smooth bell-shaped solitary wave is formed. We found no significant increase in the length of formation of a solitary wave on increase in δ. For all values of δ it was approximately 4 dispersion lengths. It should be pointed out that in this case, as before, all the parameters of a solitary wave were found to be independent of the parameters of the initial pulse, but were determined solely by the coefficients α and β.

We may thus conclude that a solitary wave independent of the initial pulse forms in a nonlinear amplifying dispersive medium of the type discussed here. Naturally, one should then ask the question of the degree of such independence. One of the ways of investigating this aspect is to consider the worst (from the point of formation of a coherent solitary wave) cases when noise pulses are applied to the input of the medium. This problem is important also from the practical point of view, because pulses generated by electromagnetic radiation sources have some noise component.

b) Stochastic pulses

Our stochastic initial pulses were of the type

$$q(0,t)=\xi(t)\exp(-t^2),$$

where $\xi(t) = a(t) + ib(t)$ represents a complex random Gaussian process with average values $\bar{a}$ and $\bar{b}$, rms deviations σ_a and σ_b of its real and imaginary parts, respectively, and a correlation coefficient R between these parts. Figures 3 and 4 demonstrate the evolution of pulses characterized by $\bar{a} = \bar{b} = 1$ and $R = 0$ in the case of different variances σ_a and σ_b. The graphs plotted in Figs. 3 and 4 provide an opportunity for identifying the general relationships governing the propagation of stochastic pulses in the medium under discussion. We found that, firstly, in all the cases when the variance σ ranges from 0.1 to 0.8 (on the assumption that $\sigma_a = \sigma_b = \sigma$), an identical (for given values of α and β) solitary wave is formed and it is the same as in the case of regular amplitude-phase modulation. Secondly, an increase in variance of the random quantity increases the characteristic length of formation of a solitary wave. If $\sigma = 0.1$, this characteristic length is approximately equal to 4 dispersion lengths, whereas for $\sigma = 0.5$ it is equal to 8 dispersion lengths, and for $\sigma = 0.8$ it increases to 12 dispersion lengths.

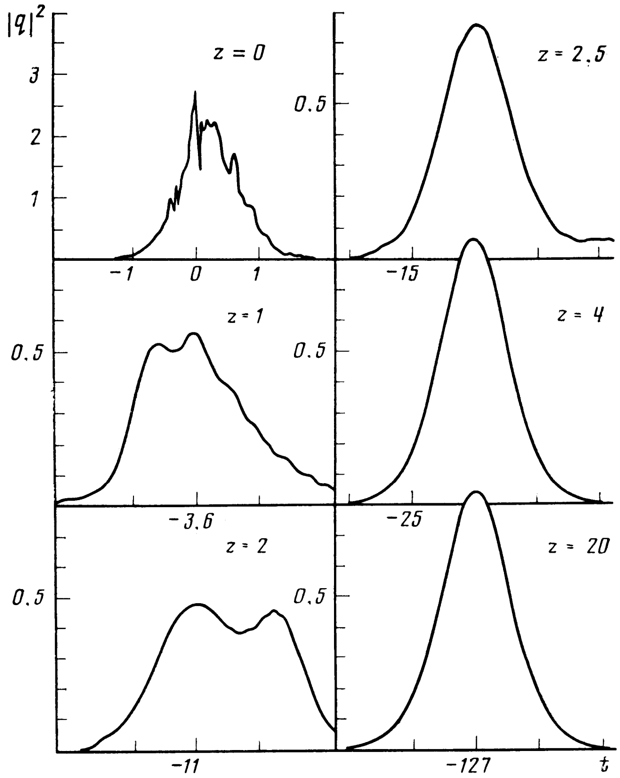

FIG. 3. Evolution of stochastic pulses in a medium characterized by $\alpha = 5$ and $\beta = 3$ shown for different values of z when $\sigma = 0.1$.

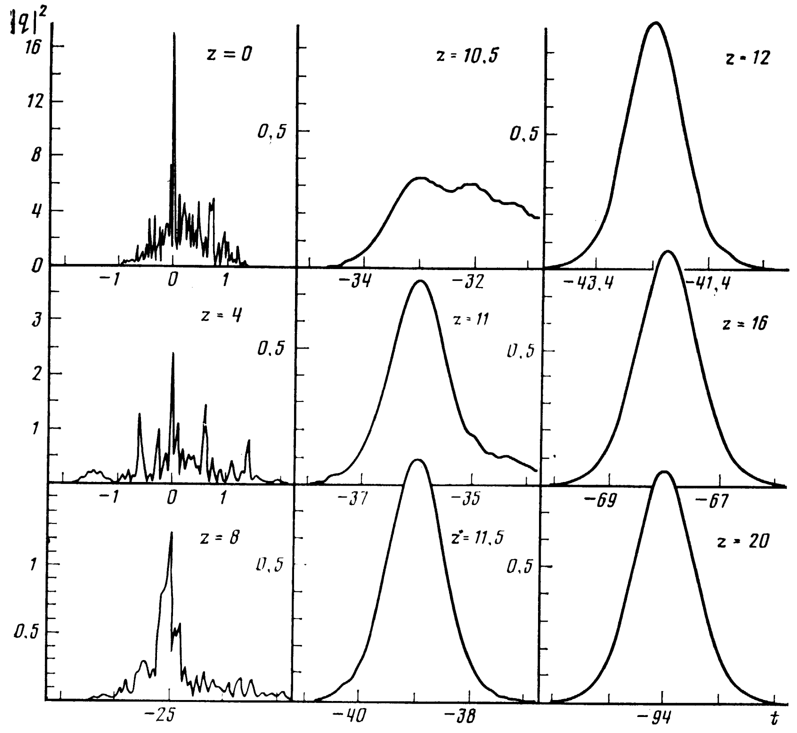

FIG. 4. Same as in Fig. 3, but for $\sigma = 0.8$.

A further increase in σ increases rapidly the errors of the numerical solution of Eq. (1), so that we were unable to investigate the propagation of stochastic pulses characterized by $\sigma > 0.8$. The kinetics of formation of a solitary wave from a noise pulse is illustrated clearly in Figs. 3 and 4. We first observe an increase in the duration of characteristic peaks of a pulse, which is followed by separation of one subpulse at the leading edge, whereas chaotic pulsations remain at the trailing edge of the pulse. In the course of subsequent propagation the chaotic pulsations at the trailing edge of the pulse are damped out, whereas the separated subpulse transforms into a smooth bell-shaped solitary wave.

5. CHARACTERISTICS OF THE RESULTANT SOLITARY WAVE

The results of the reported numerical experiments demonstrate that a solitary wave with a profile independent of that of the initial pulse forms in a nonlinear amplifying dispersive medium described by Eq. (1). All the parameters of this solitary wave are governed solely by the characteristic of the medium, which are the coefficients α and β. The dependence of the maximum square of the field amplitude $|q|^2_{\max}$ α and β is described, subject to an error of 2–3%, by the following formula valid in the range $0 < \alpha$ and $\beta < 5$:

$$|q|^2_{max}=\beta^{4/3}/\alpha. \tag{4}$$

Since the energy of a solitary wave is $U = 2\beta/\alpha$, it follows that its characteristic duration amounts to

$$\tau_b=U/|q|^2_{max}=2\beta^{-1/3}. \tag{5}$$

In contrast to a solitary wave in the form of a secant pulse of Eq. (3), the wave discussed here exhibits modulation of the phase φ. Like the variation of $|q|^2$, the change in the frequency correction $\varphi = \partial\varphi/\partial t$ with t is governed solely by the coefficients α and β. Figures 5b–5d show typical time depen-

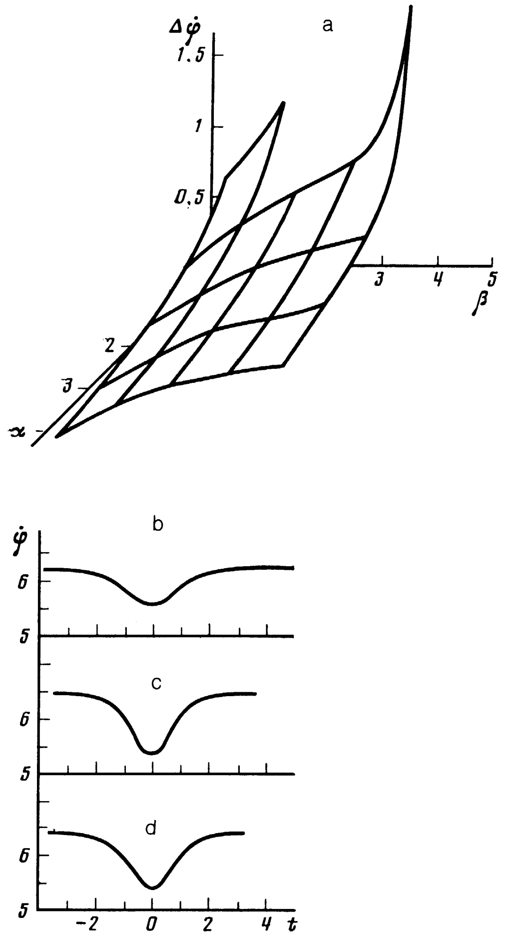

FIG. 5. Phase characteristics of soliton wave which is being formed, showing the dependence of the deviation of the frequency during a pulse $\Delta\dot{\varphi} = \dot{\varphi}_{\max} - \dot{\varphi}_{\min}$ on α and β (a) and the dependence of the frequency correction $\dot{\varphi}$ on t for $\alpha = 5, \beta = 3$ (b), $\alpha = 3, \beta = 5$ (c), and $\alpha = 3, \beta = 4$ (d).

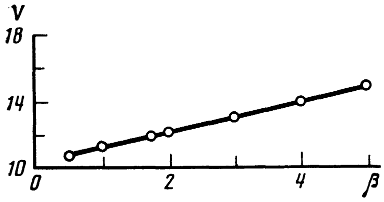

FIG. 6. Dependence of the velocity V of a solitary wave on the gain β of the medium. The ordinate gives the displacement for a wave with t per unit length z.

dences of φ calculated for different pairs of values of α and β. It is clear from Figs. 5b–5d that the dependence of $\dot{\varphi}$ on t (like the dependence of $|q|^2$ on t) is symmetric relative to the position of the maximum of the pulse, but in contrast to the latter it is in the form of an inverted bell-shaped curve with a minimum of $\dot{\varphi}_{\min}$ at the center of the pulse and a maximum at the edges of the pulse. Figure 5a shows the dependence of the maximum deviation of the frequency during a pulse $\Delta\dot{\varphi} = \dot{\varphi}_{\max} - \dot{\varphi}_{\min}$ on the values of α and β. This dependence is qualitatively similar to the dependence of $|q|^2_{\max}$ on α and β: an increase in β for a given value of α (and, conversely, a reduction in α for a given value of β) enhances $\Delta\dot{\varphi}$.

It is interesting to consider also the behavior of the velocity of propagation, which is another important parameter of a solitary wave, as a function of α and β. It is found that the velocity of such a solitary wave increases linearly on increase in the gain β, but is independent of α (Fig. 6). We recall that the reverse is true of a solitary secant wave of Eq. (3): the velocity rises linearly with α and is independent of β. Therefore, in spite of the apparent similarity, a solitary secant wave of Eq. (3) and a solitary wave discussed in the present section, found by numerical calculations of the case when β, $\alpha \gtrsim 1$, represent different types of solution of Eq. (1) which are distinguished by the nature of the dependence of the maximum amplitude and velocity on α and β, and by the time dependence of $\dot{\varphi}$.

6. CONCLUSIONS

1. Our numerical experiments demonstrate that in an amplifying dispersive medium with a Kerr nonlinearity and exhibiting both gain saturation and linear losses a solitary wave is formed and all its parameters (energy, profile, pulse duration, etc.) are governed solely by the parameters of the amplifying medium α and β, but are independent of the parameters of the pulse being amplified. The gain β moreover determines the actual nature of the resultant solitary wave. In media with the gain greater than or of the order of the reciprocal of the dispersion length ($\beta \gtrsim 1$) the resultant solitary wave [like a solitary wave in the form of a secant pulse of Eq. (3)—see Ref. 2] is described by a smooth bell-shaped curve, but in contrast to Eq. (3), it is phase-modulated (Fig. 5) and is described by a different dependence of the wave amplitude and its propagation velocity on α and β see [Eq. (4)]. The propagation velocity of such a solitary wave increases linearly with β and is determined uniquely by the gain (Fig. 6). Moreover, the values of α and β determine uniquely also the phase parameters (dependence of $\dot{\varphi}$ on t) of the resultant solitary wave.

2. These numerical experiments confirm the earlier[2] analytic results on the formation of a solitary wave in the form of a secant pulse of Eq. (3) in media with values of the gain that are small compared with the reciprocal of the dispersion length ($\beta \ll 1$); these analytic results were obtained using a perturbation theory of solitons based on the nonlinear Schrödinger equation. It is sufficient that the initial pulse satisfies the condition of formation of a one-soliton pulse in accordance with the nonlinear Schrödinger equation [when $\alpha = \beta = 0$ in Eq. (1)]. Numerical calculations show that a solitary secant wave of Eq. (3) is unstable in media with $\beta \gtrsim 1$.

3. This effect may be observed, for example, in a single-mode fiber waveguide activated with neodymium ions. A negative dispersion at a wavelength of 1.06 μm can be attained because of the contribution of the resonant part of the dispersion associated with the ions where the amplification occurs. Selection of the detuning within the limits of the luminescence line width can make the resonant part of the dispersion negative.[2] When the population inversion density of the ions is 10^{19} cm^{-3}, it is found that $k'' = -1.1\times 10^{-26}$ s^2/cm for typical values of the cross section of the relevant transition and for n_2 representing a glass activated with neodymium. The dispersion length for a typical pulse duration of 10 ps amounts to approximately 100 m and $\alpha \approx \beta \approx 5$ for a loss factor ≈ 0.27 cm^{-1}. We learn from Eqs. (4) and (5) that the duration of a pulse of the resultant solitary wave is $\tau_b \approx 12$ ps and the power density in this wave is ≈ 42 MW/cm^2. We note in conclusion that this effect provides another example of a synergetic process in which stochastic radiation is transformed into a coherent wave.

Possible directions of further studies of the evolution of light pulses in an amplifying nonlinear dispersive medium, on the basis of the proposed model, can be suggested. In particular, it is necessary to establish more accurately the limits of α and β in the range where a secant pulse of Eq. (3) retains its stability. It would also be interesting to find whether there are analytic methods for solving Eq. (1), particularly, ways of obtaining an analytic expression for the envelope of a solitary wave of a new type found in our numerical experiments.

The authors are grateful to A. M. Prokhorov for valuable discussions of the results. One of the authors is deeply indebted to V.M. Arutyunyan for his encouragement.

[1]V. E. Zakharov and A. B. Shabar, Zh. Eksp. Teor. Fiz. **61**, 118 (1971); **64**, 1627 (1973) [Sov. Phys. JETP **34**, 62 (1972); **37**, 823 (1973)]; A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[2]V.S. Grigor'yan, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 447 (1986) [JETP Lett. **44**, 875 (1986)].
[3]M. J. Ablowitz and T. R. Taha, J. Comput. Phys. **9**, 13 (1984).
[4]G. L. Lamb Jr., *Elements of Soliton Theory*, Wiley, New York (1980).
[5]A. M. Prokhorov (ed.), Handbook on Lasers [in Russian], Vol. 1, So-vet-skoe Radio, Moscow (1978), p. 332.

Translated by A. Tybulewicz