Quantum energy levels of a double-junction dc interferometer

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The energy levels of a double-junction dc interferometer are calculated for conditions under which macroscopic quantum effects are important. The spectrum in a system of this sort, without damping and with a Josephson coupling energy comparable to the Coulomb energy $e^2/2C$, is noticeably different from the spectrum of a harmonic oscillator.

Macroscopic quantum effects are important in Josephson tunnel junctions of small dimensions. These effects arise because the phase-difference operator φ does not commute with the variable which is its canonical conjugate: the number of electron pairs. The quantum properties, in particular, the energy spectrum, of such a system depend on the ratio of the Josephson coupling energy $E_J = I_c/2e$ to the Coulomb energy $E_Q = e^2/2C$, where C is the capacitance of the junction, I_c is its critical current, and we are using a system of units with $\hbar = c = 1$. If $E_J \gg E_Q$, the spectrum is similar to an oscillator spectrum, even for remote levels n. In the opposite case, deviations from the spectrum of a harmonic oscillator should be taken into consideration.

Let us examine a double-junction quantum interferometer which is described by an action containing two variables, φ_1 and φ_2, which correspond to two coupled Josephson junctions. Specifically, we consider a dc SQUID. A system of this sort has been studied previously[1–4] for several purposes. Macroscopic quantum tunneling in a system of this sort from a metastable potential minimum was studied in some recent paper.[5,6]

In the present paper we use the semiclassical approximation to calculate the energy spectrum of a SQUID of this sort. An important point is that even under the condition $E_Q \lesssim E_J$ anharmonic effects are important, and the spectrum becomes noticeably different from an oscillator spectrum. We consider a symmetric SQUID with $I_{c1} = I_{c2} = I_c$ and $C_1 = C_2 = C$, and we assume that the external magnetic flux is zero: $\Phi_e = 0$. We introduce linear combinations (ν, θ) of Josephson phases φ_1 and φ_2:

$$\nu = {}^1/_2(\varphi_1+\varphi_2), \quad \theta = {}^1/_2(\varphi_1-\varphi_2).$$

The total Lagrangian in terms of the variables ν, θ then takes the form[1,3]

$$L(t) = 2E_J\left[\left(\frac{\Phi_0}{2\pi}\right)^2 \frac{C}{E_J}\left(\frac{\dot\nu^2}{2}+\frac{\dot\theta^2}{2}\right) - V(\nu,\theta)\right] \tag{1}$$

$$V(\nu,\theta) = -\cos\nu\cos\theta - j\nu + \theta^2/\beta_L, \qquad \beta_L = 2\pi L I_c/\Phi_0,\ j = I/I_c.$$

Here L is the total inductance of the interferometer ring, I is the total current, and Φ_0 is the quantum of magnetic flux.

To write an expression for the action S, it is convenient to switch to an integration over a dimensionless time $t \to (\Phi_0/2\pi)(C/E_J)^{1/2}\tau$:

$$S = \frac{1}{\gamma}\int d\tau\, L(\tau), \qquad \gamma = \left(\frac{2E_Q}{E_J}\right)^{1/2}. \tag{2}$$

The Bohr-Sommerfeld quantization conditions can be extracted conveniently from the Green's function

$$G(\varepsilon) = \mathrm{Sp}(\varepsilon - H)^{-1} = i\int_0^\infty dT\, e^{i\varepsilon T} G(T), \tag{3}$$

$$G(T) = \int d\mathbf{r}_0 \int D\mathbf{r}(\tau)\exp[iS(\mathbf{r}(\tau))], \tag{4}$$

where we have used the notation $\mathbf{r} = \{\nu,\theta\}$ in the functional integral (4), and where the path integration is carried out under the boundary condition

$$\mathbf{r}(0) = \mathbf{r}(T) = \mathbf{r}_0. \tag{5}$$

We will not reproduce the intermediate calculations, which are given in detail in the literature (e.g., Ref. 7), but we do wish to point out a way of constructing the basic quantization equations, and to emphasize some particular features of this system. In the semiclassical approximation, the integrals in (4) and (3) can be evaluated by the stationary-phase method. In this case the action $S(\mathbf{r}(\tau))$ should be expanded in a series around a nontrivial extremum. In our case, the necessary condition for the existence of such an extremum, $\delta S/\delta\mathbf{r}(\tau) = 0$, reduces to a system of Euler-Lagrange equations:

$$\ddot\nu + \sin\nu\cos\theta = j/2, \quad \ddot\theta + \sin\theta\cos\nu + 2\theta/\beta_L = 0. \tag{6}$$

Time-dependent solutions $\mathbf{r}_0(\tau) = \{\nu_0(\tau), \theta_0(\tau)\}$ follow from (6). In order to write them explicitly, we note that the functional integration in (4) under condition (5) has the result of selecting closed classical trajectories from among the solutions (6). A subsequent integration over $\mathbf{r}_0$ singles out periodic orbits. The Green's function $G(T)$ receives contributions not only from trajectories which have a period T but also from those which have a cycle T/n, where n is a measure of the number of crossings of the main orbit. We thus need periodic solutions of Eqs. (6), $\mathbf{r}_0(\tau)$, on which $S(\mathbf{r}_0(\tau))$ has an extremum. As a result we find for $G(T)$

$$G(T) = \sum_n \exp[inS(\mathbf{r}_0(T/n))]\Delta_n(T), \tag{7}$$

$$\Delta_n(T) = \int d\rho_0 \int D\rho(\tau) \times \exp\left\{-\frac{i}{2\gamma}\int_0^T d\tau\,\rho_i\left(\frac{d^2}{d\tau^2}\delta_{ij} + \frac{\partial^2 V}{\partial \mathbf{r}_i\,\partial \mathbf{r}_j}\right)_{\mathbf{r}=\mathbf{r}_0(\tau)}\rho_j\right\}. \tag{8}$$

Here $\Delta_n(T)$ is a quantity which is found by expanding the action S around an extremum in the small deviation $\rho(\tau) = \mathbf{r}(\tau) - \mathbf{r}_0(\tau)$, equal to $[\det(d^2/d\tau^2 + V'')]^{-1/2}$.

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An important point in the calculation of Δ_n is whether there are singular points on the periodic orbits. We thus first consider periodic solutions of (6). They are of the form

$$\mathbf{r}_0(\tau)=\{\nu_0(\tau),\ \theta_0(\tau)=0\},$$

where $\nu_0(\tau)$ and the basic period of the motion are determined from the relations

$$\int_0^{\nu_0(\tau)} d\nu\, D^{-1}=\tau,\qquad T(\varepsilon_k)=2\int_{\nu_-(\varepsilon_k)}^{\nu_+(\varepsilon_k)} d\nu\, D^{-1},$$

$$D=\{2[\varepsilon_k-V(\theta=0,\nu)]\}^{1/2}. \tag{9}$$

Here $\nu_+(\varepsilon_k)$, $\nu_-(\varepsilon_k)$ are the turning points for the energy of the classical motion; the ε_k are defined as the roots of the equation

$$\varepsilon_k+\cos\nu_0+j\nu_0=0.$$

This solution determines orbits which lie entirely on the axis and which have two singular points: turning points. The presence of two singular points leads to a factor of $(-1)^n$ in Δ_n, in a manner reminiscent of the appearance of an additional phase in a wave function because of the presence of a turning point in the ordinary WKB method.[8] We can thus write

$$\Delta_n=\left(-\frac{i}{2\pi}\right)^{1/2}(-1)^n\left(-\frac{dT(\varepsilon_k)}{d\varepsilon_k}\right)^{-1/2}\times\frac{T(\varepsilon_k)}{n}\left[2i\sin\frac{n\zeta(\varepsilon_k)}{2}\right]^{-1}. \tag{10}$$

The last factor in this expression stems from the null modes of the operator in the exponential function in (8). The corresponding equation for the eigenvectors $\hat{Q}_j(\tau)$ is the same as the linear stability equation of a periodic trajectory:

$$\left[\delta_{ij}\frac{d^2}{d\tau^2}+\begin{pmatrix}\cos\nu_0(\tau) & 0\\ 0 & 2/\beta_L+\cos\nu_0(\tau)\end{pmatrix}_{ij}\right]\hat{Q}_j(\tau)=0. \tag{11}$$

The periodicity of the potential in (11) gives rise to Bloch solutions:

$$\hat{Q}_j(\tau+T)=\exp[i\zeta_j(\varepsilon_k)]\hat{Q}_j(\tau).$$

The quantities $\zeta_j(\varepsilon_k)$ depend on the energy of the classical motion and are called "stability angles." For the two independent solutions of (11), one of the angles ζ_j is zero and corresponds to the solution $\hat{Q}_j=d\mathbf{r}_0(\tau)/d\tau$; the second is nonzero and appears in (10).

We write the last factor on the right side of (10) as the sum

$$\sum_{p=0}\exp[-in(p+{}^1/_2)\zeta(\varepsilon_k)].$$

We then substitute Δ_n into (7) and calculate $G(\varepsilon)$ in (3). The stationary-phase approximation in the integration over the time T leads to a condition which must be satisfied by the classical orbit for the given ε_{mp} and p:

$$\varepsilon_k+(p+{}^1/_2)\gamma d\zeta(\varepsilon_k)/dT(\varepsilon_k)=\varepsilon_{mp}. \tag{12}$$

Summing over n, we then find the complete Green's function $G(\varepsilon)$. Its poles determine the Bohr-Sommerfeld quantization condition:

$$2\int_{\nu_-(\varepsilon_k)}^{\nu_+(\varepsilon_k)} d\nu D+(\varepsilon_{mp}-\varepsilon_k)T(\varepsilon_k)-\gamma\zeta(\varepsilon_k)(p+{}^1/_2)=\pi\gamma(2m+1). \tag{13}$$

Let us calculate the stability angle for the case $j=0$. We find the energy levels ε_{mp} (ε_{mp} and ε_k are normalized to $2E_J$) for a potential $V(\nu,\theta)$ with a relief which has minimum at the point $\theta=\nu=0$ ($j=0$). Equation (12) remains unchanged, while Eq. (13) reduces to

$$4K(\varkappa)(\varepsilon_{mp}+\varepsilon_k-2)+16E(\varkappa)-\zeta(\varepsilon_k)\gamma(p+{}^1/_2)=\pi\gamma(2m+1),\qquad 2\varkappa^2=\varepsilon_k+1. \tag{14}$$

For a classical periodic solution $\nu_0(\tau)$ and a period $T(\varepsilon_k)$ we find from (9)

$$\nu_0(\tau)=2\arcsin[\varkappa\sin(\operatorname{am}\tau)],\qquad T(\varepsilon_k)=4K(\varkappa).$$

Here am τ is the amplitude of the (Jacobi) elliptic function,[9] and $K(\varkappa)$ and $E(\varkappa)$ are elliptic integrals of respectively the first and second kinds.

The stability equation follows from (11):

$$\ddot{\theta}+[\omega^2-2\varkappa^2\operatorname{sn}^2(\varkappa,\tau)]\theta=0,\qquad \omega^2=2/\beta_L+1. \tag{15}$$

To determine $\zeta(\varepsilon_k)$ we need to examine the solutions of this equation which satisfy the condition

$$\theta(\tau+T)=e^{i\zeta}\theta(\tau). \tag{16}$$

If $\varkappa\leqslant0.9$, we can make use of the circumstance that the coefficients in the trigonometric series for the elliptic sine $\operatorname{sn}(\varkappa,\tau)$ are numerically small[9] and replace the latter by the ordinary sine. We then find from (15) a Mathieu equation ($\varkappa_1^2=1-\varkappa^2$)

$$\ddot{\theta}+\left[\omega^2-4\pi^2 f\left(1-\cos\frac{\pi\tau}{K(\varkappa)}\right)\right]\theta=0,$$

$$f=\frac{1}{K^2(\varkappa)}\exp\left[-\frac{\pi K(\varkappa_1)}{K(\varkappa)}\right]. \tag{17}$$

The value of $\zeta(\varepsilon_k)$ can be found easily by examining the semiclassical solutions of (17). The applicability of the semiclassical approximation requires[8] satisfaction of the inequality

$$\pi\left|\sin\frac{\pi\tau}{K(\varkappa)}\right|\Big/K(\varkappa)\left[\omega^2-8\pi^2 f\sin^2\frac{\pi\tau}{2K(\varkappa)}\right]^{3/2}\ll1. \tag{18}$$

This condition can be satisfied even at $\beta_L\leqslant1/2$. The change of variables $\tau\to2K(\varkappa)x/\pi$ reduces Eq. (17) to a standard form with coefficients having a period of π:

$$d^2\theta/dx^2+(a-2b\cos2x)\theta=0, \tag{19}$$

where

$$a=\bar{a}(\omega^2-4\pi^2 f),\qquad 2b=-4\bar{a}\pi^2 f,\qquad \bar{a}=[2K(\varkappa)/\pi]^2.$$

The angles $\zeta(\varepsilon_k)$ are found by using some fundamental solution of (19). It is convenient to choose a solution which is determined by the initial conditions:

$$\theta_1(0)=1,\qquad (d\theta_1/dx)_{x=0}=0.$$

By virtue of the periodicity of the potential, we can write $\theta_1(x)$ as a sum of Bloch solutions of the type $\exp(i\mu x)P(x)$, where $P(x)$ is periodic: $P(x+\pi)=P(x)$. As a result we find

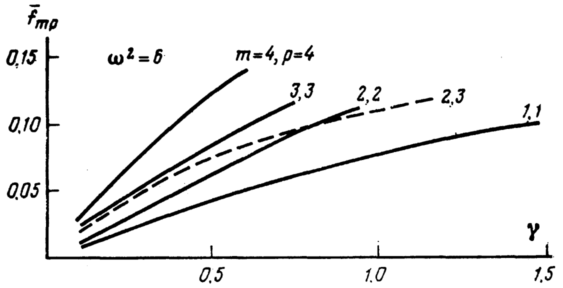

FIG. 1.

$$\cos \mu\pi=\theta_1(\pi). \tag{20}$$

Using the semiclassical approximation (19) as the Bloch solutions, we can determine a stability angle and its derivative with respect to the period in the case $\varkappa \leqslant 0.9$, working from Eqs. (16) and (20):

$$\zeta(\varepsilon_k)=\frac{8}{\pi}\omega K(\varkappa)E(\varkappa_0)=\frac{2}{\pi}\omega T(\varepsilon_k)E(\varkappa_0),$$

$$\varkappa_0=\frac{2\pi}{\omega}(2f)^{1/2}, \tag{21}$$

$$\frac{d\zeta(\varepsilon_k)}{dT(\varepsilon_k)}=\frac{2}{\pi}\omega K(\varkappa_0)+\frac{\pi}{2}\omega\frac{E(\varkappa_0)-K(\varkappa_0)}{K(\varkappa)[E(\varkappa)-\varkappa_1^2K(\varkappa)]}.$$

Substituting (21) into (12) and (14), we find a system of algebraic equations, which can easily be solved by numerical methods. We introduce the quantities $\bar{f}_{mp}$, which measure the deviation of the spectrum from that of a harmonic oscillator:

$$\bar{f}_{mp}=1-(\varepsilon_{mp}+1)/\omega_{mp}, \quad \omega_{mp}=\gamma[(p+{}^1/_2)\omega+m+{}^1/_2],$$

where ω_{mp} are the energy levels of the harmonic potential for a potential well $V(\theta,\nu)$. Figure 1 shows $\bar{f}_{mp}$ as a function of γ for $\omega^2=6$. It can be seen from this figure that with $m=p=4$ and $\gamma=0.5$ the anharmonic effects amount to $\approx 13\%$. We note, however, that the number of levels, n, is usually small,[10] and for $\gamma=0.5$ we would have $m\approx 5$, as follows from expression (29) below. The curves in Fig. 1 terminate in accordance with the condition $\varkappa\leqslant 0.9$. The point at which each curve terminates essentially tells us the value of γ for which the given level is one of the last possible levels in the $V(\theta,\nu)$ potential well.

In the case $\varkappa\geqslant 0.9$, we can use a perturbation theory in the parameter $\varkappa_1^2=1-\varkappa^2\ll 1$. From (15) we then find

$$\ddot{\theta}+[\omega^2-2+\varkappa_1^2+(2-\varkappa_1^2)/\mathrm{ch}^2\,\tau]\theta=0. \tag{22}$$

The potential in this equation, $u(\tau)=-(2-\varkappa_1^2)/\mathrm{ch}^2\tau$, is continued periodically from the interval $\{-T(\varepsilon_k)/4, T(\varepsilon_k)/4\}$ (Fig. 2).

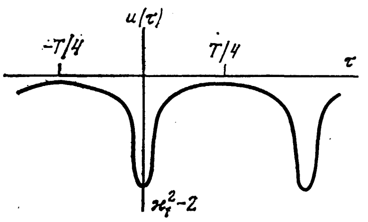

FIG. 2.

Using the method described above for the Mathieu equation, we write the stability angle in terms of those solutions of this equation which satisfy the boundary conditions

$$\theta_1(-T(\varepsilon_k)/4)=1, \quad \theta_1'(-T/4)=0.$$

As a result we find

$$\cos[\zeta(\varepsilon_k)/2]=\theta_1(T/4). \tag{23}$$

The solution $\theta_1(\tau)$ is given by[8]

$$\theta_1(\tau)=AP_s^\varepsilon(\mathrm{th}\,\tau)+BP_s^\varepsilon(-\mathrm{th}\,\tau), \tag{24}$$

where the constants A and B are found from the given initial conditions,

$$s={}^1/_2[-1+(1+8(1-\varkappa_1^2))^{1/2}], \quad \varepsilon=i\varepsilon_0, \quad \varepsilon_0^2=\omega^2-2+\varkappa_1^2,$$

and the spherical Legendre functions P_s^ε are expressed in terms of the hypergeometric function:

$$P_s^\varepsilon(\mathrm{th}\,\tau)$$

$$=\frac{1}{\Gamma(1-\varepsilon)}\left(\frac{1+\mathrm{th}\,\tau}{1-\mathrm{th}\,\tau}\right)^{\varepsilon/2}F\left(-s,s+1,1-\varepsilon,\frac{1-\mathrm{th}\,\tau}{2}\right). \tag{25}$$

Near the saddle point of the potential $V(\theta,\nu)$ the period $T(\varepsilon_k)$ is large. Making use of this circumstance, we substitute expression (25) for $\tau=T/4$ and the corresponding expression for $P_s^\varepsilon(-\mathrm{th}\,\tau)$ into (23), and we find the stability angle $\zeta(\varepsilon_k)$:

$$\zeta(\varepsilon_k)=\varepsilon_0T(\varepsilon_k)-\xi, \quad \xi=2\varphi-\pi, \quad \varphi=\mathrm{arctg}\,\varepsilon_0. \tag{26}$$

Retaining small terms on the order of $\varkappa_1^2\ \ln\varkappa_1$, we find from (26)

$$\frac{d\zeta(\varepsilon_k)}{dT(\varepsilon_k)}=\varepsilon-\frac{\varkappa_1^2}{\varepsilon_0}\ln\frac{4}{\varkappa_1}, \tag{27}$$

where $\bar{\varepsilon}_0^2=\omega^2-2$.

Equations (26) and (27) also hold for $\beta_L>1/2$ (but $\beta_L<2$). The only restriction is the requirement $\varkappa_1^2\ll\bar{\varepsilon}_0$. For values of β_L close to 2, the stability angle decreases, and if we formally set $\beta_L>2$ this angle becomes imaginary. The reason is that the classical periodic trajectory looses its stability, and there is a change in the relief of the potential energy $V(\theta,\nu)$.

Using (26) and (27), we find the following equation from our basic quantization equations, (12) and (14):

$$4\bar{\varkappa}_1^2\left(\ln\frac{4}{\bar{\varkappa}_1}\right)^2\left[\frac{\gamma}{\bar{\varepsilon}_0}\left(p+\frac{1}{2}\right)-\frac{2}{\ln(4/\bar{\varkappa}_1)}\right]$$

$$=[(2m+1)\pi-\xi]\gamma-16. \tag{28}$$

Here

$$2\bar{\varkappa}_1^2=1-\varepsilon_k^0=1-\varepsilon_{mp}+(p+{}^1/_2)\gamma\bar{\varepsilon}_0.$$

In deriving (28) we assumed

$$\varepsilon_k=\varepsilon_k^0+\varepsilon_k', \quad \varepsilon_k'=(\bar{\varkappa}_1^2/\bar{\varepsilon}_0)\gamma(p+{}^1/_2)\ \ln\,(4/\bar{\varkappa}_1)\ll 1.$$

The last inequality imposes a restriction on the numbers p. From (28) we find that the total number of levels, m, is on the order of the greatest integer in m_0 (i.e., $m\approx[m_0]$), where

$$m_0=8/\pi\gamma+{}^1/_2(\xi/\pi-1). \tag{29}$$

We solve Eq. (28) in the two limiting cases

$$\text{a)}\ (\gamma/2\bar{\varepsilon}_0)(p+{}^1/_2)\ln(4/\bar{\varkappa}_1)\equiv N\gg1,\quad \text{b)}\ N\ll1. \tag{30}$$

In case a) we find the following expression for the energy levels ε_{mp}:

$$\varepsilon_{mp}-1\approx(p+{}^1/_2)\gamma\bar{\varepsilon}_0-{}^1/_2\delta(1-\alpha)[\ln(8/\delta^{1/2})]^{-2}, \tag{31}$$

where

$\delta = 2\pi\bar{\varepsilon}_0|m-m_0|/(p+1/2)\ll1$

(i.e., m is equal to or close to $[m_0]$),

$$\alpha = 2[\ln\ln(8/\delta^{1/2})][\ln(8/\delta^{1/2}) + \ln\ln(8/\delta^{1/2})-1]^{-1}.$$

In limit b) we have (γ is small)

$$\varepsilon_{mp}-1\approx(p+{}^1/_2)\gamma\bar{\varepsilon}_0-{}^1/_2\delta_0(1-\alpha_0)[\ln(64/\delta_0)]^{-1}. \tag{32}$$

Here

$$\delta_0 = 2\pi\gamma|m-m_0|\ll1,$$
$$\alpha_0 = [\ln\ln(64/\delta_0)][\ln(64/\delta_0) + \ln\ln(64/\delta_0)-1]^{-1}.$$

The first terms on the right sides of (31) and (32) correspond to the energy spectrum of a harmonic oscillator which is oscillating around a saddle point of the potential $V(\theta,\nu)$. They determine the energies of fluctuations which are transverse with respect to the classical orbit.

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