Exact solutions of equations describing an incommensurate phase in the absence of the Lifshitz invariant

V. A. Golovko

Evening Metallurgical Institute, Moscow

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The exact solutions of a differential equation for the order parameter are obtained for an incommensurate phase which forms because the coefficient of the square of the derivative in the thermodynamic potential is negative. These solutions describe an equilibrium state characterized by a certain ratio of the coefficients of the thermodynamic potential. Three types of solution are obtained, depending on the values of these coefficients. The first solution describes the properties of an incommensurate phase which are essentially the same as the incommensurate phase which appears in the presence of the Lifshitz invariant. In particular, a domain-like structure appears near the transition to the commensurate phase; this transition is of the second order and has some special properties. In the case of the second type of solution the transition to the commensurate phase gives clear indications that it is of the first order. The incommensurate phase described by the third type of solution has very unusual properties. In addition to a periodic part, the order parameter has a constant component. The influence of temperature on this phase seems to be the opposite to that on familiar types of incommensurate phase. It forms from the original phase as a result of a strongly nonlinear mechanism and near the transition its properties are those of a known incommensurate phase near the transition to a commensurate phase. The exact solutions are also obtained in the presence of an external stimulus coupled to the order parameter, which makes it possible to plot stimulus-temperature phase diagrams. Generalization of the method to the case of a thermodynamic potential of more complex nature is discussed.

In discussions of an incommensurate phase on the basis of the Landau theory of phase transitions, we need to distinguish two cases: those in which the symmetry of the system is such that the Lifshitz invariant can exist, and those in which the invariant is absent. Following Ref. 1, we shall call these incommensurate phases type I and II. The majority of theoretical treatments have been concerned with type I phases, although there have been several investigations of the type II phase, to which attention was drawn in Ref. 2. We shall consider a type II incommensurate phase. Such a phase appears in a number of substances, particularly in magnetic materials,[2–4] ferroelectrics,[1,5] and liquid crystals.[6]

Reverting to type I incommensurate phases, we note that they have certain characteristics near the transition to a commensurate phase. In this case an important role is played by higher harmonics of the Fourier expansion of the order parameter and the structure represents a regular sequence of domains of comparable size.[7,8] The order parameter for the incommensurate–commensurate phase transition is unusual: the parameter is represented by the density of domain walls (solitons), which gives rise to special characteristics of the transition differing from those that follow from the conventional Landau theory of second-order phase transitions.[1,7–10] This type of transition has stimulated a major discussion (see Refs. 1 and 11–13 as well as the literature cited there). The exact solution of complex differential equations for the order parameter was obtained in Ref. 12 for a special case, allowing unambiguous answers to some of the questions that have given rise to controversy.

The situation is quite different in the case of the theory of a type II incommensurate phase. Estimates of the first several harmonics of the order parameter were used in Ref. 3 to conclude that higher harmonics play a minor role in the whole range of existence of a type II incommensurate phase and its structure remains practically sinusoidal all the time. Hence the transition to a commensurate phase is clearly of the first order. These conclusions have been repeated many times by other authors[1,4,5,14–18] without providing a rigorous proof. A similar difference between incommensurate phases of types I and II was stressed specially in Refs. 1 and 18.

We shall find exact solutions of the equations for the order parameter describing a type II incommensurate phase under certain special conditions. From the first solution it follows that a type II incommensurate phase can undergo a second-order phase transition to a commensurate phase, and that such a transition has certain characteristics of a type I incommensurate phase. For certain values of the coefficients in the thermodynamic potential there is no qualitative difference between the effects of temperature on incommensurate phases of types I and II. There is also a second solution which gives rise to a first-order transition to a commensurate phase, which follows clearly from the structure of the solution. On the other hand, a new mechanism is revealed which gives rise to an incommensurate phase of different nature (third solution) in the phases known so far. In particular, the influence of temperature on such an incommensurate phase seems to be opposite to that on a type I incommensurate phase.

The first of these three solutions was obtained in Refs. 19–21 by a different method, and the thermodynamic potential typical of type II incommensurate phases was considered there. However, the question of whether the solution corresponds to an equilibrium state of an incommensurate phase (we shall show that this is only rarely true) is not answered in Refs. 19 and 20 and without considering this aspect it is incorrect to use the results in an analysis of incommensurate

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phases. The same question is tackled in Ref. 21 but only in the special case of the coefficients with the values $\tilde{\delta} = 5$ and $\tilde{\gamma} = 3/50$ (see below).

1. GENERAL ANALYSIS OF EQUATIONS

We shall consider phase transitions described by a single-component order parameter η and label as x the axis along which the incommensurate nature of the phase is manifested. The thermodynamic potential Φ per unit length along x has the following form in the case of a type II incommensurate phase:

$$\Phi = \frac{1}{d}\int_0^d \tilde{\Phi}(x)\,dx, \quad \tilde{\Phi}(x) = \frac{\alpha}{2}\eta^2 + \frac{\beta}{4}\eta^4 + \frac{\gamma}{6}\eta^6 + \frac{\delta}{2}(\eta')^2 + \frac{\lambda}{2}(\eta'')^2 + \frac{\varkappa}{2}\eta^2(\eta')^2, \qquad (1)$$

where d is the period of $\eta(x)$ and a prime denotes a derivative with respect to x. The exact solutions are obtained only if the last term in Eq. (1) is included, and this term can always be deduced from symmetry considerations.[5] It should be noted that in Refs. 17 and 22 this term is replaced with $(\eta')^4$, but the latter is of higher order than $\eta^2(\eta')^2$. The lower limit to the thermodynamic potential is set by assuming that $\gamma \geqslant 0$ and $\lambda > 0$. If $\varkappa < 0$, we can show that an additional condition must be satisfied:

$$\varkappa^2 < 12\gamma\lambda. \qquad (2)$$

In the case of the original (symmetric) phase we have $\eta = 0$ and $\Phi = 0$. For a commensurate phase, when $d\eta/dx = 0$, we have $\eta = \eta_c$ and $\Phi = \Phi_c$, where

$$\eta_c^2 = [-\beta + (\beta^2 - 4\alpha\gamma)^{1/2}]/2\gamma, \qquad (3)$$

$$\Phi_c = -[6\alpha\beta\gamma - \beta^3 + (\beta^2 - 4\alpha\gamma)^{3/2}]/24\gamma^2.$$

In the case of an incommensurate phase the equation for the function $\eta(x)$, which should ensure a minimum of Φ, can be obtained from the Euler equation

$$\lambda\eta'''' - \delta\eta'' - \varkappa[\eta^2\eta'' + \eta(\eta')^2] + \alpha\eta + \beta\eta^3 + \gamma\eta^5 = 0. \qquad (4)$$

This equation may have periodic solutions with a different period d. The equilibrium period is defined by the condition $\partial\Phi = \partial d = 0$, which in our case (compare with Ref. 12) becomes

$$\lambda\eta'''\eta' - \frac{\lambda}{2}(\eta'')^2 - \frac{\delta}{2}(\eta')^2 - \frac{\varkappa}{2}\eta^2(\eta')^2 + \frac{\alpha}{2}\eta^2 + \frac{\beta}{4}\eta^4 + \frac{\gamma}{6}\eta^6 = \Phi. \qquad (5)$$

The left-hand side of Eq. (5) is the first integral of Eq. (4), and Eq. (5) can be used instead of Eq. (4) to find $\eta(x)$. The equilibrium condition set by Eq. (5) can be modified if it is integrated (with respect to x) from zero to d and if Φ is substituted from Eq. (1):

$$\int_0^d [2\lambda(\eta'')^2 + \delta(\eta')^2 + \varkappa\eta^2(\eta')^2]\,dx = 0. \qquad (6)$$

Near the transition from the original phase, where $\eta \to 0$, Eq. (4) can be solved by means of standard expansions.[3–5] In the first approximation, we obtain

$$\eta = \rho_0 \cos qx, \quad \rho_0^2 = \frac{4(\alpha_0 - \alpha)}{3\beta + 2\varkappa q_0^2}, \quad q = q_0\left(1 + \frac{\varkappa}{8\delta}\rho_0^2\right),$$
$$\Phi = -\frac{(\alpha_0 - \alpha)^2}{2(3\beta + 2\varkappa q_0^2)}, \quad \alpha_0 = \frac{\delta^2}{4\lambda}, \quad q_0^2 = -\frac{\delta}{2\lambda}. \qquad (7)$$

This phase transition occurs also for $\delta < 0$ when $\alpha = \alpha_0$.

It is convenient to introduce dimensionless coefficients $\tilde{\alpha}$, $\tilde{\delta}$, and $\tilde{\gamma}$, defined as follows:

$$\alpha = \frac{\beta^2\lambda}{\varkappa^2}\tilde{\alpha}, \quad \delta = \frac{\beta\lambda}{\varkappa}\tilde{\delta}, \quad \gamma = \frac{\varkappa^2}{\lambda}\tilde{\gamma}. \qquad (8)$$

We shall now consider whether the results obtained below can be applied to real physical systems. We can ignore further terms with derivatives in the potential (1) if the changes in $\eta(x)$ are sufficiently slow, i.e., if $q = 2\pi d$ is small. The exact solutions found below are valid mainly for $|\tilde{\delta}|$, which is not small. If we take q_0 from Eq. (7) as q, we find that the ratio $|\beta|/|\varkappa|$ should be small.

2. METHOD USED TO OBTAIN EXACT SOLUTIONS

We shall replace η with a new function $Z = (\eta')^2$. Using Eq. (5), we find that Z is described by

$$\frac{\lambda}{8}\left[4Z\frac{d^2Z}{d\eta^2} - \left(\frac{dZ}{d\eta}\right)^2\right] - \frac{\delta}{2}Z - \frac{\varkappa}{2}\eta^2 Z + \frac{\alpha}{2}\eta^2 + \frac{\beta}{4}\eta^4 + \frac{\gamma}{6}\gamma^6 - \Phi = 0. \qquad (9)$$

If the solution of this equation is known, the dependence $\eta = \eta(x)$ is then found from the relationship

$$\int [Z(\eta)]^{-1/2}\,d\eta = x - x_0, \qquad (10)$$

where x_0 is an arbitrary constant. For simplicity, we shall assume that $x_0 = 0$.

We shall try to find the solution of Eq. (9) in the form of a polynomial in η. Substitution in Eq. (9) shows that the degree of this polynomial is at most four; therefore, we shall seek the solution in the form

$$Z = \frac{\varkappa}{8\lambda}c\eta^4 + \frac{\beta}{\varkappa}g_1\eta^2 + \frac{\beta^2\chi}{\varkappa^3}g_0, \qquad (11)$$

where the unknown coefficients c, g_1, and g_0 are made dimensionless (the odd powers of η will be discussed below). Substituting Eq. (11) into Eq. (9), we obtain

$$3c(c-1) + 8\tilde{\gamma} = 0, \quad 2g_1(5c-4) = \tilde{\delta}c - 4,$$
$$g_0(2-3c) = 2(g_1^2 - \tilde{\delta}g_1 + \tilde{\alpha}), \quad \Phi = \beta^3\lambda^2 g_0(2g_1 - \tilde{\delta})/2\varkappa^4. \qquad (12)$$

The first three relationships define c, g_1, and g_0, and the last one yields Φ (if the condition $\partial\Phi/\partial d = 0$ is satisfied). Since $\tilde{\gamma} \geqslant 0$, it follows from the first relationship that $0 \leqslant c \leqslant 1$ and the real solutions for c exist only if $\tilde{\gamma} \leqslant 3/32$. For $\varkappa < 0$, we have to satisfy the inequality (2), i.e., $\tilde{\gamma} > 1/12$, and, therefore, we have

$$1/3 < c < 2/3. \qquad (13)$$

The relationship (10) yields real solutions only for $Z > 0$. Moreover, the range of values of η where $Z > 0$ should be limited because otherwise η becomes infinite. Consequently, depending on the sign of $\varkappa$ and the number of real roots of the equation $Z(\eta) = 0$, we can have three cases of interest: 1) $\varkappa > 0$ with four roots; 2) $\varkappa < 0$ with two roots; 3)

$\varkappa < 0$ with four roots. Each case yields its own form of solution $\eta = \eta(x)$ which will be discussed in turn below.

The solutions of Eq. (4) obtained in this way are not general because there is only one trivial constant x_0 (the cases $c = 4/5$ and $2/3$ are special—see Secs. 3 and 5). Therefore, the relationship (6) does not give the period d, but it links the coefficients of Φ for which the solution represents an equilibrium situation, which corresponds to a certain curve in the phase diagram. The following comment should be made in this connection. Numerical calculations[22] show that Φ considered as a function of d has a number of minima. Naturally, thermodynamic equilibrium corresponds to the absolute minimum of Φ. It is not possible to say whether the solutions obtained below subject to Eq. (6) represent an absolute minimum or some other extremum of Φ. However, if the curve described by Eq. (6) begins at $\alpha = \alpha_0$ and near this value of α the exact solution is identical with Eq. (7), it corresponds to an equilibrium right up to the phase transition point.

It should be added that if $\tilde{\gamma} = 3/50$ and $1/12$ (if $c = 4/5$ or $2/3$) then Eq. (9) has a solution in the form of a polynomial which in contrast to Eq. (11) contains odd powers of η. However, the expressions which are then obtained do not reduce to Eq. (7) and, in accordance with the comment made above, we shall not consider this case.

3. FIRST SOLUTION

We shall consider the case when $\varkappa > 0$ and the polynomial of Eq. (11) has four real roots, which implies that the condition $\beta g_1 < 0$ holds. Of the three ranges of the value of η, where $Z > 0$, we have to take (see Sec. 2) the region between two inner roots. Substituting Eq. (11) into Eq. (10) and calculating the integral, we obtain the first solution of Eq. (4):

$$\eta=\rho\,\mathrm{sn}(px,k),\quad \rho^2=-\frac{8\beta\lambda k^2 g_1}{\varkappa^2 c(1+k^2)},\quad p^2=-\frac{\beta g_1}{\varkappa(1+k^2)},\quad k^2=\frac{1-(1-cg_0/2g_1^2)^{1/2}}{1+(1-cg_0/2g_1^2)^{1/2}}, \tag{14}$$

where $\mathrm{sn}(z, k)$ is a Jacobi elliptic function with modulus k.

The conditions $0 \leqslant k \leqslant 1$ and $\beta g_1 < 0$ define the range of values of the coefficients of the potential Φ where the solution (14) exists. We shall determine when this condition represents an equilibrium state of an incommensurate phase. Substituting Eq. (14) into Eq. (6) and using Eq. (12), we can represent the condition (6) in the form

$$g_1=\frac{15}{10+(2-3c)u_1},\quad u_1=\frac{2(8+7k^2+8k^4)E-(1-k^2)(16+7k^2)K}{(1+k^2)[(1+k^2)E-(1-k^2)K]}, \tag{15}$$

where K and E are complete elliptic integrals with the modulus k and the condition $10 \leqslant u_1 \leqslant 11.5$ is satisfied. Equation (15) subject to Eq. (12) with k from Eq. (14) yields the relationship between $\tilde{\alpha}$, $\tilde{\delta}$, and $\tilde{\gamma}$ for which the solution described by Eq. (14) corresponds to an equilibrium.

The corresponding value of Φ is found from Eq. (12):

$$\Phi=\frac{16\beta^3\lambda^2 g_1^3 k^2(2-3c)(10-u_1)}{15\varkappa^4 c^2(1+k^2)^2}. \tag{16}$$

Relationships similar to Eq. (13) of Ref. 12 yield

$$\frac{\partial\Phi}{\partial\alpha}=-\frac{4\beta\lambda g_1}{\varkappa^2 c(1+k^2)}\left(1-\frac{E}{K}\right). \tag{17}$$

In an analysis of the results obtained it is convenient to proceed as follows. First from Eq. (12), we find $\tilde{\delta}$ and $\tilde{\alpha}$ expressing g_0 in terms of k:

$$\tilde{\delta}=\frac{2}{c}[g_1(5c-4)+2],\quad \tilde{\alpha}=\frac{g_1^2}{c^2(1+k^2)^2}[3c(3+2k^2+3k^4)-8(1+k^2+k^4)]+\frac{4g_1}{c}. \tag{18}$$

Specifying c (i.e., in fact $\tilde{\gamma}$) and taking different values of k, we obtain from Eq. (15) the value g_1 subject to the condition $\beta g_1 < 0$. Then, Eq. (18) is used to calculate values of $\tilde{\alpha}$ and $\tilde{\delta}$ for which the solution (14) corresponds to an equilibrium state.

If $\beta > 0$, $g_1 < 0$ must hold, which is possible only if $c > 22/23$. The curves corresponding to Eq. (15) are shown dashed in Fig. 1. In agreement with the comments made at the end of Sec. 2, we shall not discuss these curves because they do not begin on the initial–incommensurate transition line, although they are characterized by $\Phi \leqslant 0$ and, according to Eq. (16), even by $\Phi < \Phi_c$ and the solution (14) may correspond to an equilibrium state of the incommensurate phase.

If $\beta < 0$, $g_1 > 0$ must hold and it is then meaningful to consider only the values of $c \geqslant 2/3$ when $\Phi \leqslant 0$, in accordance with Eq. (16). It should be noted that if $\beta < 0$, the transition from the initial to the commensurate phase is of the first order and, according to Eq. (3), it occurs when $\alpha = \alpha_h \equiv 3\beta^2/16\gamma$. In the limit $k \to 0$, Eqs. (14)–(16) reduce to Eq. (7); therefore, if $\beta < 0$, Eq. (14) for the curves satisfying Eq. (15) should correspond to an equilibrium state of the incommensurate phase. In the limit $k \to 1$ these curves terminate on the line BC in Fig. 1; if $c > 22/23$, the curves go to infinity. We can use Eqs. (16), (17), and (3) to show that $\Phi \to \Phi_c$, $\partial\Phi/\partial\alpha \to \partial\Phi_c/\partial\alpha$, and $\Phi \leqslant \Phi_c$ when $k \to 1$

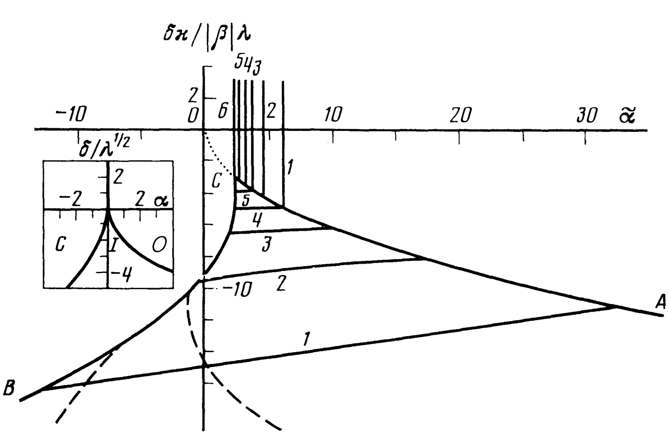

FIG. 1. Curves in the $(\tilde{\alpha}, \delta)$ plane where the solution (14) obtained for $\beta < 0$ corresponds to an equilibrium state of an incommensurate phase. The curve denoted by CA is the line of the original–incommensurate phase transition, whereas at the points on the curve BC the transition to a commensurate phase takes place (for different values of $\tilde{\gamma}$). The vertical straight lines represent the boundary between the original and commensurate phases. 1) $\tilde{\gamma} = 0.03$; 2) $\tilde{\gamma} = 0.04$; 3) $\tilde{\gamma} = 0.05$; 4) $\tilde{\gamma} = 0.06$; 5) $\tilde{\gamma} = 0.07$; 6) $\tilde{\gamma} = 1/12$. The inset shows the phase diagram calculated for $\beta = 0$, $\varkappa > 0$, and $\tilde{\gamma} = 33/2116$; here, I denotes an incommensurate phase, O is the original phase, and C is a commensurate phase.

(and this is true even if $\beta > 0$); therefore, if $k = 1$, there is a second-order transition to the commensurate phase. If $c \to 2/3$, the curves contract to a point characterized by $\tilde{\alpha} = 9/4$ and $\tilde{\delta} = 3$, which is a triple point where the incommensurate, initial, and commensurate phases meet. At the triple point the incommensurate phase is degenerate ($\Phi = 0$ is independent of k), which can be observed also in the case of a type I incommensurate phase.[12]

According to Eq. (14), in the limit $k \to 1$ an incommensurate phase has a domain-like structure with a period $d \sim |\ln(1 - k^2)| \to \infty$. The specific heat of the incomnensurate phase near the point $\alpha = \alpha_j$ of the transition to the commensurate phase diverges proportionally to $1/(\alpha - \alpha_j)[\ln(\alpha - \alpha_j)]^2$. All this is analogous to the properties of a type I incommensurate phase.[12]

These results make it possible to consider also the case when $\beta = 0$. The coefficients in Eq. (11) become finite in the limit $\beta \to 0$ if g_1 and g_0 tend to infinity. We can see from Eq. (15) that this occurs when the denominator of g_1 vanishes, i.e., when $c = (10 + 2u_1)/3u_1$. Such a relationship, defining the value of k for a given c, is in this case fixed and can be satisfied only if $22/23 \leqslant c \leqslant 1$. It follows from the first relationship in the system (18), subject to Eq. (8) in the limit $\beta \to 0$, that $\beta g_1 = \delta \varkappa c/2\lambda(5c - 4)$. The second relationship in the system (18) defines curves (parabolas $\alpha \propto \delta^2$) on which the solution obtained corresponds to an equilibrium. Such curves begin at the Lifshitz point $\alpha = \delta = 0$ (Ref. 2) and go to infinity in the limit $\delta \to -\infty$. If $c = 1$ ($k = 0$), such a curve coincides with the original–incommensurate phase transition line $\alpha = \delta^2/4\lambda$ and the equation for the curve in the case when $c = 22/23$ ($k = 1$) is $\alpha = -11\delta^2/108\lambda$. This equation describes the whole incommensurate–commensurate phase boundary for the specific case when $\tilde{\gamma} = 33/2116$. In this case the phase diagram is known completely (Fig. 1).

The value $c = 4/5$ when $\tilde{\gamma} = 3/50$ is worth special attention. We can see from Eq. (12) that in this case g_1 can have any value and we also have $\tilde{\delta} = 5$. Therefore, in this case the solution (14) contains a nontrivial arbitrary constant g_1, i.e., the period d is arbitrary. The relationship (15) now describes g_1 [it is equivalent to the condition $\partial\Phi/\partial p = 0$ of Ref. 21; if $c \neq 4/5$, this condition cannot be used because p is fixed in accordance with Eq. (14)]. It follows from Eq. (15) that in this case we have $g_1 > 0$, i.e., we should have $\beta < 0$. All these quantities depend only on the coefficient $\tilde{\alpha}$ which, as usual, can be regarded as a linear function of temperature. The temperature dependences of the various quantities are plotted in Fig. 2, where for the sake of comparison we have included also $\Phi_c(\alpha)$ and $\eta_c(\alpha)$ of Eq. (3).

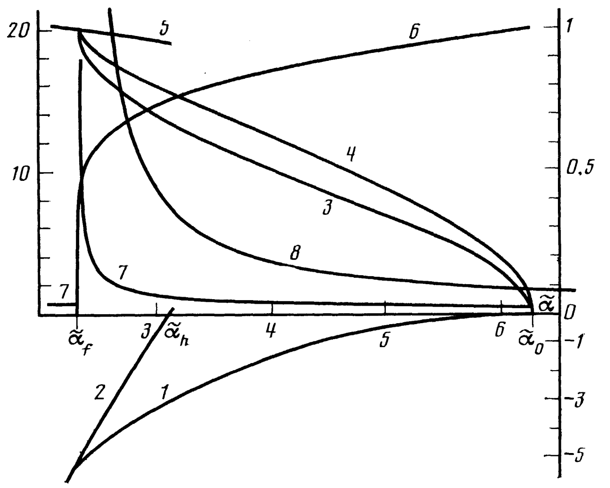

FIG. 2. Dependences on α of the thermodynamic potentials of an incommensurate phase (Φ) and of a commensurate phase (Φ_c), of the parameters k and ρ, of the wave vector $q = 2\pi/d$, of the reduced specific heat $\tilde{C} = -|\beta|\partial^2\Phi/\partial\alpha^2$ and of the susceptibility χ of an incommensurate phase in the case when $\tilde{\gamma} = 0.06$ ($c = 4/5$), $\tilde{\delta} = 5$, $\beta < 0$, $\varkappa > 0$. The scales for $\tilde{C}$ and χ are given on the left. Here, $\tilde{\alpha}_0 = 6.25$, $\tilde{\alpha}_f = 125/54$, $\rho_f^2 = 125|\beta|\lambda/9\varkappa^2$. 1) $\Phi\varkappa^4/|\beta|^3\lambda^2$; 2) $\Phi_c\varkappa^4/|\beta|^3\lambda^2$; 3) k; 4) ρ/ρ_f; 5) η_c/ρ_f; 6) q/q_0; 7) $\tilde{C}$; 8) $10\chi\beta^2\lambda/\varkappa^2$.

Since for $c = 4/5$ the solution (14) allows us to find the dependence of the potential Φ on the period d, the incommensurate–commensurate phase transition can be considered from the soliton density point of view. For each period of $\mathrm{sn}(z, k)$ there are two solitons, so that the soliton density is $n = 2/d = p/2K$. Substituting Eq. (14) into Eq. (1) and going to the limit $k \to 1$, we obtain in the case when $\alpha \to \alpha_f$ the expression

$$\Phi = \Phi_c + \frac{25}{3} \cdot 2^{1/2}\lambda^2|\beta|^{5/2}\varkappa^{-7/2} \cdot \left[(\tilde{\alpha}_f - \tilde{\alpha})n + \frac{4000}{81} n \exp\left(-\frac{5|2\beta|^{1/2}}{3n\varkappa^{1/2}}\right)\right]. \quad (19)$$

This expression is again fully analogous to that obtained in the theory of a type I incommensurate phase.[9,1] It can be used to find the equililbrium value of n.

It should be noted that the value $c = 2/3$ ($\tilde{\gamma} = 1/12$) is also singular because, according to Eq. (12), g_0 is arbitrary. However, when $c = 2/3$, solution (14) can represent an equilibrium state only at the triple point (as discussed above and shown in Fig. 1).

4. SECOND SOLUTION

We shall consider the second of the cases discussed in 2: we shall assume that $\varkappa < 0$ and that the polynomial of Eq. (11) has only two real roots, which is true if $g_0 < 0$. There is one region where $Z > 0$ and instead of Eq. (14), we then have

$$\eta = \rho\,\mathrm{cn}(px, k), \quad \rho^2 = \frac{8\beta\lambda k^2\tilde{g}_1}{\varkappa^2 c}, \quad p^2 = -\frac{\beta\tilde{g}_1}{\varkappa}, \quad k^2 = \frac{1}{2}\left[1 \pm \left(1 - \frac{cg_0}{2g_1^2}\right)^{-1/2}\right] \quad (20)$$

where $\tilde{g}_1 = g_1/(1 - 2k^2)$ and the sign for k^2 should be opposite to the sign of βg_1. It is clear from Eq. (20) that we should also have $\beta\tilde{g}_1 > 0$.

The condition for an equilibrium in the case of Eq. (6) now yields the relationship

$$\tilde{g}_1 = \frac{15}{10(1-2k^2) + (2-3c)u_2}, \quad u_2 = \frac{(1-k^2)(16-23k^2)K - 2(8-23k^2+23k^4)E}{(1-k^2)K - (1-2k^2)E}. \quad (21)$$

It should be noted that $10 \geqslant u_2 \geqslant -16$. Instead of Eq. (16), we find that Φ is now described by

$$\Phi = 16\beta^3\lambda^2\tilde{g}_1^3k^2(1-k^2)(2-3c)[u_2 - 10(1-2k^2)]/15\varkappa^4c^2. \quad (22)$$

Since $u_2 - 10(1 - 2k^2) \leqslant 0$, it follows that $\Phi \leqslant 0$ when $c < 2/3$, which corresponds to the interval (13) which should be regarded as the condition $\varkappa<0$. Instead of Eq. (18), we now obtain

$$\tilde{\delta}=\frac{2}{c}[(1-2k^2)(5c-4)\tilde{g}_1+2],$$
$$\tilde{\alpha}=\frac{\tilde{g}_1^2}{c}[(1-2k^2)^2(9c-8)+4k^2(1-k^2)(3c-2)]+\frac{4\tilde{g}_1}{c}(1-2k^2). \qquad (23)$$

These expressions can be analyzed in the same way as in 3. If $\beta>0$, we should then have $\tilde{g}_1>0$. This inequality is obeyed for all values of c of Eq. (13) if $0\leqslant k\leqslant k_0$, where k_0 is the root of the denominator in $\tilde{g}_1$ of Eq. (21). In the limit $k\to 0$, the curves obtained from Eq. (23) begin at the original–incommensurate phase boundary (Fig. 3) and Eqs. (20)–(22) reduce to Eq. (7); therefore, the solution of Eq. (20) for these curves describes an equilibrium incommensurate phase. This $\tilde{\gamma}$, apart from the limit $\tilde{\gamma}=3/32$, corresponds to two curves for different values of c. If $c>0.538$, then we always have $\Phi<\Phi_c$ for the curves; this is true right up to $\tilde{\alpha}\to-\infty$. If $c<0.538$, then at some points of these curves we obtain $\Phi=\Phi_c$ (curve MN in Fig. 3) and we then have $\Phi>\Phi_c$, indicating a transition to a commensurate phase.

If $\beta<0$, the curves deduced from Eq. (23) do not begin at the original–incommensurate phase boundary. We shall not consider this case, especially as initially we would then have $\Phi>\Phi_c$ (although for some of these curves we have $\Phi<\Phi_c$ in the limit $\tilde{\alpha}\to-\infty$).

If $\beta=0$, we can proceed as in Sec. 3 and assume that $\tilde{g}_1\to\infty$. Equating the denominator of $\tilde{g}_1$ to zero, we find that $c=[10(1-2k^2)+2u_2]/3u_2$, which defines the value of k for a given c such that $0.454<k^2<0.5$ subject to the condition (13). The curves (parabolas $\alpha\propto\delta^2$), where the solution corresponds to an equilibriun, begin at the Lifshitz point $\alpha=\delta=0$ and go to infinity for $\delta\to+\infty$. However, for these curves we have $\Phi<\Phi_c$ only if $c>0.538$. If $c=0.538$, so that $k^2=0.473$, then the whole curve corresponds to

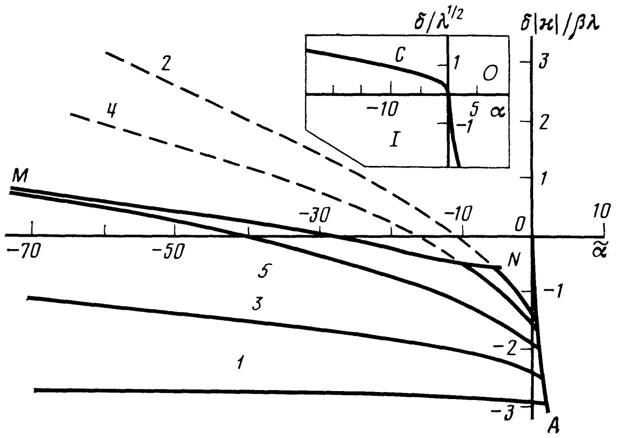

FIG. 3. Curves in the $(\tilde{\alpha},\delta)$ plane where the solution (20) obtained for $\beta>0$ corresponds to an equilibrium state of an incommensurate phase. The dashed curves extend the continuous curves to the range of metastability of an incommensurate phase. The MN curve corresponds to the transition to a commensurate phase and the $0A$ curve represents the transition from the original to incommensurate phase. 1) $\tilde{\gamma}=0.084$, $c=0.661$; 2) $\tilde{\gamma}=0.084$, $c=0.339$; 3) $\tilde{\gamma}=0.09$, $c=0.6$; 4) $\tilde{\gamma}=0.09$, $c=0.4$; 5) $\tilde{\gamma}=3/32$, $c=0.5$. The inset shows the phase diagram calculated for $\beta=0$, $\varkappa<0$, and $\tilde{\gamma}=0.0932$.

$\Phi=\Phi_c$ and such a curve together with $\alpha=-10.63\delta^2/\lambda$ defines the incommensurate–commensurate phase transition line for the appropriate value $\tilde{\gamma}=0.0932$. In this case the phase diagram can be determined completely (inset in Fig. 3).

In the case of the solution (20), when $\varkappa<0$, an incommensurate phase exists also at $\delta>0$, and cooling may result in a consecutive series of transitions from the original to a commensurate and then an incommensurate phase, which differs from the usual form. The incommensurate–commensurate phase transition is of the first order, which follows directly from the structure of the solution (20). This is because the function $\mathrm{cn}(z,k)$ differs from $\mathrm{sn}(z,k)$ since its value is zero for a major part of the period in the limit $k\to 1$. Therefore, in the case of the solution described by Eq. (20) there are no domains with a commensurate structure characterized by $\eta=\eta_c\neq 0$ even in the form of nuclei and a continuous transition to a commensurate phase is impossible.

It should be noted that it is pointless to consider the special values $c=4/5$ and $2/3$ in the case of the solution (20) because of the limitations imposed by Eq. (13).

5. THIRD SOLUTION

We shall now consider the case when $\varkappa<0$ and the polynomial of Eq. (11) has four real roots. As in Sec. 3, we must have $\beta g_1<0$. Both ranges of values of η where $Z>0$ give the same results because of the symmetry of $Z(\eta)$, and instead of Eq. (14) we now have the third solution:

$$\eta=\rho\,\mathrm{dn}(px,k),\quad \rho^2=-\frac{8\beta\lambda g_1}{\varkappa^2c(2-k^2)},\quad p^2=\frac{\beta g_1}{\varkappa(2-k^2)},$$
$$k^2=\frac{2(1-cg_0/2g_1^2)^{1/2}}{1+(1-cg_0/2g_1^2)^{1/2}}. \qquad (24)$$

This solution differs qualitatively from those given by Eqs. (14) and (20) because $\mathrm{dn}\,z>0$, whereas $\mathrm{sn}\,z$ and $\mathrm{cn}\,z$ exhibit periodic variation of the sign with z. The incommensurate phase corresponding to the solution (24) can be called an incommensurate phase with a constant component. For example, in the case of ferroelectrics, when η represents the polarization, such an incommensurate phase will be polar. In Refs. 17 and 22 such an incommensurate phase is called a rippled commensurate phase.

The condition (6) assumes the following form after the substitution of the solution (24):

$$g_1=\frac{15}{10+(2-3c)u_3},$$
$$u_3=\frac{2(23-23k^2+8k^4)E-23(1-k^2)(2-k^2)K}{(2-k^2)[(2-k^2)E-2(1-k^2)K]}. \qquad (25)$$

We note that $17.5\geqslant u_3\geqslant 16$. Instead of Eqs. (16) and (17), we obtain

$$\Phi=\frac{16\beta^3\lambda^2g_1^3(1-k^2)(2-3c)(10-u_3)}{15\varkappa^4c^2(2-k^2)^2},$$
$$\frac{\partial\Phi}{\partial\alpha}=-\frac{4\beta\lambda g_1E}{\varkappa^2c(2-k^2)K}. \qquad (26)$$

Hence, if we allow for Eq. (13), we find that $\Phi\geqslant 0$ and, therefore, in this case the solution (24) cannot correspond to an equilibrium state. Negative values of Φ are obtained for

$c > 2/3$ when Eq. (24) can give only a relative minimum of the functional (1), because the absolute minimum does not exist. Inclusion of further terms in Eq. (1) may convert the relative minimum into an absolute minimum, but the qualitative nature of the solution should not change. Bearing in mind that very little is known about incommensurate phases with a constant component, it would be of interest to analyze the solution described by Eq. (24) in the case when $c > 2/3$ in order to find any special features of the phase described by this solution.

We shall first consider the general case of the transition from an incommensurate phase with a constant component to a commensurate phase in order to compare this with the consequences of Eq. (24). If we seek the solution of Eq. (4) in the form $\eta = \eta_c + \eta_1$, where η_1 is a small correction to Eq. (3), we obtain

$$\eta_1=\rho_1 \cos q_1 x, \quad \rho_1^2 = \frac{1}{4\lambda\mu}[(\delta+\varkappa\eta_c^2)^2 - 8\lambda\eta_c^2(\beta+2\gamma\eta_c^2)],$$

$$q_1^2 = -\frac{\delta+\varkappa\eta_c^2}{2\lambda}, \quad \Phi=\Phi_c+\frac{\mu}{8}\rho_1^4, \tag{27}$$

$$\mu = \frac{\eta_c^2}{9\lambda q_1^4}\Big\{2(6\beta+17\gamma\eta_c^2)^2 + 102\gamma^2\eta_c^4 + 27\frac{\varkappa^2}{\lambda}\eta_c^2(\beta+2\gamma\eta_c^2) + 6\varkappa(9\beta+32\gamma\eta_c^2)\Big[\frac{2}{\lambda}\eta_c^2(\beta+2\gamma\eta_c^2)\Big]^{1/2}\Big\}.$$

The expression for ρ_1^2 is valid to first order in $\alpha - \alpha_1$ and $\delta - \delta_1$, where (α_1, δ_1) is a point on a curve defined by the equation

$$\delta+\varkappa\eta_c^2+[8\lambda\eta_c^2(\beta+2\gamma\eta_c^2)]^{1/2}=0. \tag{28}$$

The transition from an incommensurate phase with a constant component to a commensurate phase is a conventional second-order transition which occurs for $\alpha = \alpha_1$ and $\delta = \delta_1$, when $\mu < 0$, which is possible only if $\varkappa < 0$.

We shall now go back to the solution described by Eq. (24). If we express g_0 in terms of k, we find from Eq. (12) that

$$\tilde{\delta} = \frac{2}{c}[g_1(5c-4)+2],$$

$$\tilde{\alpha} = \frac{g_1^2}{c(2-k^2)^2}[k^4(9c-8) - 24(1-k^2)(1-c)] + \frac{4g_1}{c}. \tag{29}$$

The inequality $\Phi \leqslant \Phi_c$ should hold only in the range $2/3 \leqslant c \leqslant 9/11$. For these values of c it follows from Eq. (25) that $g_1 > 0$, i.e., we should have $\beta < 0$.

In the limit $k \to 0$ the curves deduced from Eq. (29) terminate at points obtained from Eq. (28) with suitable values of $\tilde{\gamma}$, and Eqs. (24)–(26) reduce to Eq. (27). It follows that the solution of the (24) type may represent an equilibrium state. The inequality $\mu < 0$ is satisfied if $c < 4/5$. When $c > 4/5$ the transition to a commensurate phase is of the first order and occurs at the points on the MN curve in Fig. 4. This figure shows also parts of the curves corresponding to Eq. (28) with suitable values of $\tilde{\gamma}$, in order to show how the curves described by Eq. (29) approach the former curves.

If $k = 1$, the curves of Eq. (29) terminate at points $\tilde{\alpha} = \tilde{\alpha}_p$ and $\tilde{\delta} = \tilde{\delta}_p$, where a second-order transition takes place in the initial phase since—according to Eq. (26)—we then have $\Phi = 0$ and $\partial\Phi/\partial\alpha = 0$, and the commensurate

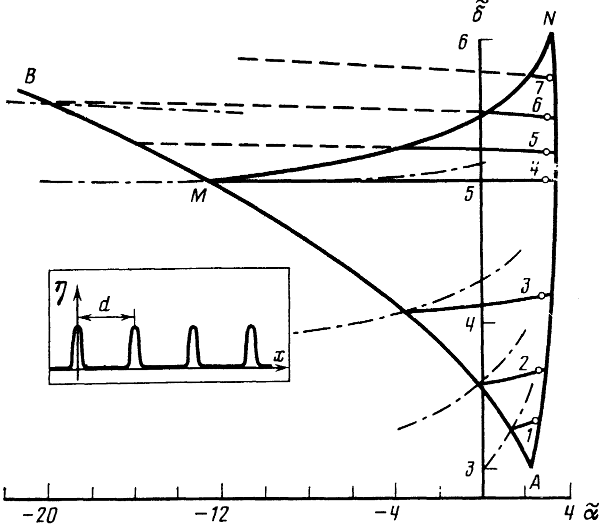

FIG. 4. Curves in the $(\tilde{\alpha}, \tilde{\delta})$ plane where the solution (24) obtained for $\beta < 0$ corresponds to a minimum of the thermodynamic potential. The AB curve corresponds to $k = 0$ and the portion AM represents transition of the second order to a commensurate phase. The MN curve represents the first-order transition to a commensurate phase. At the points on the AN curve the transition is to the original phase. The circles are the points where $\Phi_c = 0$. The chain curves are obtained from Eq. (28). 1) $\tilde{\gamma} = 0.075$; 2) $\tilde{\gamma} = 0.07$; 3) $\tilde{\gamma} = 0.065$; 4) $\tilde{\gamma} = 0.06$; 5) $\tilde{\gamma} = 0.059$; 6) $\tilde{\gamma} = 0.058$; 7) $\tilde{\gamma} = 0.057$. The inset shows schematically the function $\eta(x)$ of Eq. (24) in the limit $k \to 1$.

phase is unstable ($\Phi_c > 0$, Fig. 4). There are two triple points in Fig. 4. The point N is a conventional triple point. However, at the point A, where the curves under discussion contract in the limit $c \to 2/3$, an incommensurate phase with a constant component is degenerate (compare with Sec. 3).

In the limit $k \to 1$ the structure of an incommensurate phase with a constant component is very distinctive: there are fairly narrow layers (solitons) where $\eta \neq 0$, separated by wide intermediate layers (where $\eta \approx 0$) with the structure of the original phase (inset in Fig. 4, $d = 2K/p \to \infty$). Therefore, the solution (24) shows that the incommensurate phase state with a soliton structure can form directly from the original phase, whereas up to now such an incommensurate phase structure has been known to occur only close to the transition to a commensurate phase. It should be noted that the transition from the original phase to an incommensurate phase with a constant component occurs when $\delta > 0$, i.e., when—according to Eq. (7)—a conventional incommensurate phase cannot form from the original phase. The mechanism by which an incommensurate phase with a constant component forms is nonlinear and is related to those terms in Eq. (1) which can be written as follows: $(\delta + \varkappa\eta^2)(\eta')^2/2$. If $\delta > 0$ and $\varkappa < 0$, such an expression becomes negative only for high values of η^2, but this is still insufficient: we obtain $\Phi < 0$ only if the quantity $(\eta')^2$ is large, which gives rise to the structure of an incommensurate phase with a constant component described above. Moreover, it is not trivial that such a mechanism can give rise to a second-order phase transition.

However, we can argue that in this case the transition from the original phase should still be of first order. The transition is continuous in the case of repulsion between solitons. However, if in the case of every second soliton (Fig. 4) we reverse the sign of η, we obtain a structure with attraction between solitons. The transition to this structure occurs earlier than to a structure with repulsion. However, a structure with alternating signs of η [and of $\mathrm{cn}(px, k)$] cannot transform continuously into a commensurate phase (compare with Sec. 4). Therefore, according to Eq. (27), such a transition should occur if preceded by a transition to an incommensurate phase with a constant component described by a solution of the type given by Eq. (24). In either case the structure of the incommensurate phase should be of the soliton type sufficiently far from the commensurate phase structure. It should be noted that a second-order transition from the original phase to an incommensurate phase with a constant component may occur in a suitable external field (an electric field in the case of ferroelectrics), since the field imposes conditions such that solitons with the opposite sign of η are no longer favored by energy considerations.

It follows from the solution (24) that the specific heat of an incommensurate phase with a constant component close to the point $\alpha = \alpha_p$ of the transition to the original phase diverges proportionally to $1/(\alpha_p - \alpha) \times [\ln(\alpha_p - \alpha)]^2$, as in the case of the incommensurate–commensurate phase transition discussed in Sec. 3.

For the same reasons as in Sec. 3, it is worth considering particularly the value $c = 4/5$. The dependences of the various quantities on α are shown for this case in Fig. 5, where ρ_c is the value of ρ at $\alpha = \alpha_1$, whereas q_1 has the same meaning as in Eq. (27). It should be noted that the straight line corresponding to $c = 4/5$ terminates at a singular point M (Fig. 4), where there is also a minimum of the $\delta = \delta(\alpha)$ curve in Eq. (28) and $\mu = 0$ applies in the case of Eq. (27). It follows from the usual Landau thermodynamic potential $\Phi = A'(T - T_0)\eta^2 + \beta\eta^4 + C\eta^6$ that this is the point where $A' = B = 0$. Therefore, the behavior of the various quantities in the limit $\alpha \to \alpha_1$ (Fig. 5) may be atypical of an incommensurate phase with a constant component. In particular, a specific heat discontinuity vanishes on transition to a commensurate phase. Typical behavior of the quantities in the case of this transition follows from Eq. (27).

Since for $c = 4/5$ the period of the solution (24) is arbitrary, the transition from an incommensurate phase with a constant component to the original phase can be considered

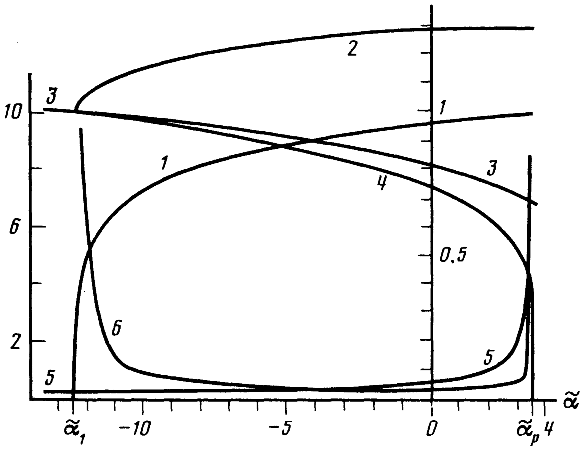

FIG. 5. Dependences on $\tilde{\alpha}$ of the parameters k and ρ, of the wave vector q, of the specific heat $\tilde{C}$, and of the susceptibility χ of an incommensurate phase with the constant component in the case when $\gamma = 0.06$ ($c = 4/5$), $\tilde{\delta} = 5$, $\beta < 0$, $\varkappa < 0$. The scales for $\tilde{C}$ and χ are on the left; $\alpha_p = 125/36$, $\tilde{\alpha}_1 = -12.5$. 1) k; 2) ρ/ρ_c; 3) η_c/ρ_c; 4) q/q_1; 5) $\tilde{C}$; 6) $\chi\beta^2\lambda/\varkappa^2$.

from the point of view of the soliton density. Here, the soliton density (inset in Fig. 4) is $n = 1/d = p/2K$. If we follow the same procedure as in the derivation of Eq. (19) we find that, in the limit $k \to 1$ ($\alpha \to \alpha_p$), the potential is

$$\Phi = 25(2/3)^{1/2}|\beta|^{5/2}\lambda^2|\varkappa|^{-7/2}[(\tilde{\alpha} - \tilde{\alpha}_p)n + (1000/9)n\exp(-5|\beta|^{1/2}/n|6\varkappa|^{1/2})]. \quad (30)$$

The complete analogy with Eq. (19) is worth noting, although in the present case the soliton structure of the incommensurate phase is different and the transition is to the original phase and not to a commensurate phase.

6. EFFECTS OF AN EXTERNAL FIELD ON AN INCOMMENSURATE PHASE

The exact solutions can be obtained also in the presence of an external field E_η coupled to the order parameter η. In this case the potential of Eq. (1) should be supplemented by a term $-E_\eta\eta$; it appears on the left in Eqs. (5) and (9). We shall now rewrite the solution of Eq. (9) in the form

$$Z = \frac{\varkappa}{8\lambda} c\eta^4 + a_3\eta^3 + \frac{\beta}{\varkappa} g_1\eta^2 + a_1\eta + \frac{\beta^2\chi}{\varkappa^3} g_0. \quad (31)$$

The first equation in the system (12) still applies to the coefficient c. The equations for the remaining coefficients in Eq. (31) can be solved only if we assume that $c = 4/5$ or $2/3$. We shall first consider the case $c = 4/5$, because if $E_\eta = 0$, only this case is interesting and it is discussed in detail above. We then have $a_3 = 0$, the value of g_1 is arbitrary, and as before the solution corresponds to $\tilde{\delta} = 5$. The remaining quantities then become

$$a_1 = \frac{2\varkappa E_\eta}{\beta\lambda(g_1 - 5)}, \quad g_0 = 5(5g_1 - g_1^2 - \tilde{\alpha}),$$
$$\Phi = \frac{\beta^3\lambda^2}{2\varkappa^4} g_0(2g_1 - 5) - \frac{\varkappa^2 E_\eta^2}{2\beta^2\lambda(g_1 - 5)^2}. \quad (32)$$

Out of the three cases discussed in Sec. 2, we shall consider only two, when the polynomial of Eq. (31) has four real roots, which occurs if $\beta g_1 < 0$.

In the integral of Eq. (10) if $\varkappa > 0$, we have to take the values of η between two inner roots of the polynomial of Eq. (31) (compare with Sec. 3), whereas for $\varkappa < 0$, we have to take values between the two largest roots if for the sake of simplicity we shall assume that $E_\eta > 0$ (the region between two other roots yields a solution which is unfavorable from the energy point of view). The solution of both $\varkappa \gtrless 0$ cases can be written in the form

$$\eta = \frac{m + \rho\,\mathrm{sn}(px, k)}{1 + l\,\mathrm{sn}(px, k)}. \quad (33)$$

The quantities occurring above can be expressed in terms of g_1, k and l:

$$m = (1 + k^2 - 2l^2)\, l \left(\frac{-5\beta\lambda g_1}{\varkappa^2 N}\right)^{1/2},$$
$$\rho = (2k^2 - l^2 - k^2 l^2)\left(\frac{-5\beta\lambda g_1}{\varkappa^2 N}\right)^{1/2}, \quad (34)$$
$$p^2 = \frac{2\beta g_1(1 - l^2)(l^2 - k^2)}{\varkappa N},$$

where $N = 2(1 + k^2)(k^2 + l^4) + l^2(1 - 10k^2 + k^4)$. The coefficients a_1 and g_0 can also be expressed in terms of g_1, k, and l; instead of a_1, we can write down directly E_η in accordance with Eq. (32):

$$E_\eta=\frac{|\beta|^{5/2}\lambda^{3/2}}{|\varkappa|^3}\tilde{E}_\eta,\quad \tilde{E}_\eta=2\sqrt{5}\,N^{-3/2}\,l(k^2-l^4)(1-k^2)^2 g_1^{3/2}(5-g_1),$$
$$g_0=\frac{5g_1^2}{2N^2}[16k^6-4l^2(1+k^2)(k^2+l^4)(1+6k^2+k^4)$$
$$+l^4(1+28k^2+38k^4+28k^6+k^8)+16k^2l^8]. \qquad (35)$$

Here we are allowing for the fact that $\beta<0$.

The solution of Eq. (33) is bounded for any value of x only if $|l|<1$, which corresponds to $N\geqslant 0$. The case $\varkappa>0$ corresponds to $k^2\geqslant l^2$, whereas the case $\varkappa<0$ corresponds to $l\geqslant 0$ and $l^4\leqslant k^2\leqslant l^2$ (if $E_\eta>0$). For $l=0$, Eqs. (33)-(35) reduce to Eq. (14), whereas for $k^2=l^4$ they reduce to Eq. (24) if we modify k.

The values of g_1 corresponding to an equilibrium state can be found from the condition (6), the form of which does not change even for $E_\eta\neq 0$:

$$g_1=25G_1/3G_2,\quad G_1=[3(1-k^2)^2(k^2-l^4)\Pi$$
$$-(1-k^2)(k^2-l^2)(1-3k^2+2l^2)K-NE]N, \qquad (36)$$
$$G_2=5(1-k^2)^2(k^2-l^4)N\Pi-(1-k^2)(k^2-l^2)[5(2$$
$$+l^2)(1-k^2)^3-2(23-8l^2)(1-l^2)(1-k^2)^2+$$
$$+4(15-4l^2)(1-l^2)^2(1-k^2)-24(1-l^2)^3]K-[(8$$
$$+4l^2+3l^4)(1-k^2)^4-4(16-5l^2-l^4)(1-l^2)(1-k^2)^3$$
$$+8(19-9l^2+l^4)(1-l^2)^2(1-k^2)^2-48(3-l^2)(1-l^2)^3(1$$
$$-k^2)+48(1-l^2)^4]E,\quad \Pi=\Pi(-l^2,k),$$

where $\Pi(n,k)$ is the complete elliptic integral of the third kind in the notation of Ref. 23. It follows from Eq. (36) that $g_1>0$, so that $\beta<0$. The phase diagrams obtained for this case are plotted in Fig. 6. The transition from a commensurate phase to the original phase in the case when $\varkappa>0$ and from an incommensurate phase with a constant component to a commensurate phase when $\varkappa<0$ occurs in the limit $k\to 0$, $l\to 0$. When $k^2\to 1$ and $l^2\to 1$, a transition takes place from an incommensurate to a commensurate phase for $\varkappa>0$

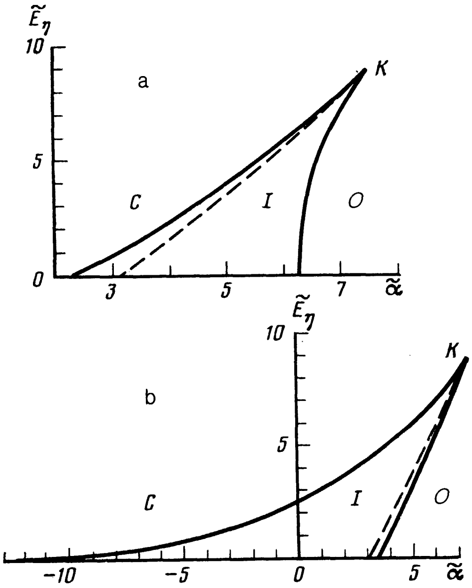

FIG. 6. Phase diagrams in terms of the variables $\tilde{\alpha}$ and $\tilde{E}_\eta$ plotted for $\tilde{\gamma}=0.06$ ($c=4/5$), $\tilde{\delta}=5$, $\beta<0$; a) $\varkappa>0$; b) $\varkappa<0$. The dashed curves show the incommensurate–commensurate boundary in the absence of the incommensurate phase (it terminates at the point K).

and from an incommensurate phase with a constant component to the original phase for $\varkappa<0$. In weak fields the shift of the location of the transitions from an incommensurate to a commensurate phase and from an incommensurate phase with a constant component to the original phase is respectively,

$$\tilde{\alpha}=125/54-(3/10\sqrt{5})|\tilde{E}_\eta|\ln|\tilde{E}_\eta|,$$
$$\tilde{\alpha}=125/36+(\pi\sqrt{3}/5\sqrt{5})|\tilde{E}_\eta|. \qquad (37)$$

It should be noted that the shift of the point of the transition from an incommensurate phase with a constant component to the original phase is linear in the field. All the transitions are second-order even when $E_\eta\neq 0$.

We shall now consider the spatial average value of the order parameter $\bar{\eta}$. Integrating the solution (33), we obtain

$$\bar{\eta}=\frac{1}{d}\int_0^d\eta(x)\,dx=\frac{\rho}{l}\left[1-\frac{2(1-l^2)(k^2-l^2)\Pi(-l^2,k)}{(2k^2-l^2-k^2l^2)K(k)}\right]. \qquad (38)$$

We can now calculate the linear susceptibility $\chi=d\bar{\eta}/dE_\eta$. This is easily done in the case when $\varkappa>0$, when we have to go to the limit $l\to 0$ in Eqs. (35) and (38):

$$\chi=\frac{\varkappa^2(1+k^2)}{\beta^2\lambda g_1(5-g_1)(1-k^2)^2}\left(\frac{2E}{K}-1+k^2\right). \qquad (39)$$

This expression, together with those in Sec. 3, describes the dependence $\chi(\alpha)$ of Fig. 2. The susceptibility has a kink at the transition from the original phase to an incommensurate phase and diverges proportionally to $1/2(\alpha-\alpha_j)$ on approach to the transition to a commensurate phase. The behavior of the susceptibility is the same as in the case of an incommensurate type I phase.[10]

If $\varkappa<0$, much more complicated calculations yield

$$\chi=\frac{\varkappa^2(2-k^2)}{8\beta^2\lambda k^2(1-k^2)K^2g_1(5-g_1)}$$
$$\cdot\left\{\frac{5\pi^2(2-k^2)^2(35-2u_3)}{75k^2+(2-k^2)[5(2-k^2)^2-2(6-6k^2+k^4)g_1]u_3'}\right.$$
$$\left.+\frac{8(1-k^2)}{k^2}K[(2-k^2)K-2E]\right\}, \qquad (40)$$

where $u_3'=du_3/dk^2$ and the value of k is the same as in Sec. 5. Together with the expressions in Sec. 5, Eq. (40) gives the dependence $\chi(\alpha)$ of Fig. 5. The divergence of χ in the limit $\alpha\to\alpha_1$ is untypical and is associated with the singularity of the point $\alpha=\alpha_1$ when $c=4/5$ (see Sec. 5). In general, the susceptibility exhibits a discontinuity at the transition from an incommensurate phase with a constant component to a commensurate phase. On approach to the transition to the original phase the susceptibility diverges proportionally to $\pi^2/(\alpha_p-\alpha)[\ln(\alpha_p-\alpha)]^2$. Note that there is some difference between the nature of the divergence of the susceptibility at the incommensurate–commensurate and incommensurate-with-constant-component–original phase transitions.

7. GENERALIZATION OF THE METHOD

The method discussed above can be used to find the exact solutions also in the case of a thermodynamic potential of more general nature than that described by Eq. (1). By way of example, we shall consider

$$\Phi(x)=\frac{\alpha}{2}\eta^2+\frac{\beta}{4}\eta^4+\sum_{i=3}^{2n-1}\frac{\gamma_i}{2i}\eta^{2i}+\frac{\delta}{2}(\eta')^2+\frac{\lambda}{2}(\eta'')^2+\frac{1}{2}(\eta')^2\sum_{i=1}^{n-1}\varkappa_i\eta^{2i}. \tag{41}$$

Proceeding as before, instead of Eq. (9) we now obtain

$$\frac{\lambda}{8}\left[4Z\frac{d^2Z}{d\eta^2}-\left(\frac{dZ}{d\eta}\right)^2\right]-\frac{\delta}{2}Z-\frac{1}{2}Z\sum_{i=1}^{n-1}\varkappa_i\eta^{2i}+\frac{\alpha}{2}\eta^2+\frac{\beta}{4}\eta^4+\sum_{i=3}^{2n-1}\frac{\gamma_i}{2i}\eta^{2i}-\Phi=0. \tag{42}$$

The solution of this equation may be sought in the form

$$Z=\sum_{k=0}^{n}a_k\eta^{2k}. \tag{43}$$

Substituting Eq. (43) into Eq. (42), we obtain $2n$ algebraic equations for $n+2$ quantities a_k and Φ. The additional equation represents generalization of the equilibrium condition of Eq. (6). We thus find that between $3n$ coefficients of Eq. (41) there should be $n-1$ relationships which make it possible to obtain the exact solutions. Equation (41) may be supplemented also by terms of other types, such as $(\eta')^4$.

The presence in the thermodynamic potential of terms containing high powers of the order parameter should not affect significantly the results and this may be useful. If we assume the desired values for the important terms in Eq. (41) and these values correspond to specific physical systems, we can select the high-order terms so as to obtain the solution of the type given by Eq. (43) in the case of Z. The relationships between the coefficients of high-order terms which are then needed do not have a significant effect on the physical results if the terms are selected successfully, but nevertheless the exact mathematical solutions can be obtained.

Such a method can be used also in a study of the influence of an external field on an incommensurate phase. In particular, the exact solutions corresponding to $E_\eta\neq 0$ can be obtained not only for $c=4/5$ or $2/3$, as in Sec. 6, but for any value of c if the potential Φ of Eq. (1) is supplemented not only by the term $-E_\eta\eta$, but also by $-\zeta E_\eta\eta^3$, where $\zeta=\varkappa^2(5c-4)/4\beta\lambda(g_1-\tilde{\delta})$, whereas Z still has the form given by Eq. (31) with $a_3=0$. The term $\zeta E_\eta\eta^3$ is permissible from the point of symmetry and its presence should not affect significantly the physical results (in fact the quantity which is linked to E_η is not η and $\eta+\zeta\eta^3$).

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