Classical analog of superradiance in a system of interacting nonlinear oscillators

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(Submitted 19 June 1987)
Zh. Eksp. Teor. Fiz. **91**, 171–174 (January 1988)

We consider a classical analog of Dicke superradiance in a classical system of nonlinear oscillators interacting via the intrinsic radiation field. At the initial instant, the oscillators are not phased—their phases are random. The dependence of the frequency of the nonlinear oscillators on the amplitude leads to phasing of the oscillators and to an exponential growth of the radiation intensity. Eventually the nonlinearity limits this growth and forms a superradiant pulse.

Collective spontaneous emission (Dicke superradiance in an inverted system of interacting two-level atoms is quite well known.[1] The quantum-mechanical model of a two-level system describes well the spontaneous emission of atoms and molecules in electronic and vibrational transitions, and is widely used in quantum optics. To describe spontaneous and induced emission in the millimeter and submillimeter bands it becomes necessary to use the model of classical oscillators, since the emitted-photon energy in this range is low and to obtain high-power radiation the energy reserve of each oscillator must exceed by many times the energy of one photon. In addition, for emission of weakly relativistic electrons in a constant magnetic field in cyclotron-resonance masers, the oscillating-electron anharmonicity due to the relativistic correction to the Hamiltonian can be much less than the constant of the interaction with the radiation field. Therefore many levels participate in the emission and absorption processes even in the resonance approximation, and the two-level approximation cannot be used. It is therefore of undisputed interest to answer the question: can collective spontaneous emission occur in classical systems or it is in principle a quantum effect?

We consider in the present paper a classical analog of superradiance in a system of nonlinear oscillators that interact via the intrinsic radiation field. The anharmonicity of the oscillators (more accurately, the dependence of the oscillation frequency on the amplitude) leads in to phasing of initially unphased oscillators and to appearance of an average macroscopic polarization. The possibility of phasing oscillators by an external coherent field in the presence of nonlinearities was noted in Refs. 2–4; it is the operating principle of cyclotron-resonance masers.

The increase of the average polarization leads during the initial stage to an exponential increase of the radiation intensity. Eventually, at sufficiently high values of the average polarization, this growth becomes limited by nonlinearity and a superradiance pulse is produced.

A classical analog of superradiance is considered in Ref. 5 under conditions of the anomalous Doppler effect, but in this case the onset of superradiance is not related to the nonequidistant character of the spectrum.

Thus, let N nonlinear oscillators interact via the intrinsic radiation field and let x_i be the coordinate of the ith oscillator. We put

$$x_i=n_0^{1/2}(a_i e^{i\omega_0 t}+a_i^+ e^{-i\omega_0 t}),\quad i=1,\dots,N, \tag{1}$$

where n_0 is the initial excitation of the oscillators. The following simplified equations are then valid if $\gamma n_0 \ll \omega_0$ and $\nu_0 N \ll \omega_0$:

$$\dot a_i = i\gamma |a_i|^2 a_i - \frac{1}{N}\sum_j a_j, \tag{2}$$

where

$$m=1,\quad \gamma=3\gamma_0 n_0/4\omega_0^2\nu N,\quad \tau=\omega_0\nu N t,\quad \nu=\nu_0/\omega_0,$$

ω_0 is the frequency of the oscillators, γ_0 their anhramonicity, and $\nu_0=2e^2\omega_0^2/13c^2$ the natural linewidth. The initial conditions are

$$a_i(0)=e^{i\varphi_i}. \tag{3}$$

If the oscillators are not phased at the initial instant, φ_i can be regarded as random quantities uniformly distributed in the interval from 0 to 2π. We have then $\langle\langle^{-1}\Sigma_j a_j\rangle\rangle=0$ (the averaging is over all possible realizations of the random phases of each of the oscillators. In each specific realization, however, the quantity $N^{-1}\Sigma_j a_j$ can also differ from zero,

$$\left\langle\!\!\left\langle \frac{1}{N^2}\Big(\sum_j a_j\Big)\Big(\sum_{j'} a_{j'}^+\Big)\right\rangle\!\!\right\rangle \sim \frac{1}{N}.$$

Let

$$a_i(0)=e^{i\varphi_i}+\delta_i,\quad \delta_i\sim N^{-1/2}. \tag{4}$$

If $\delta_1=0$, then $a_i^{(0)}=\exp(i\varphi_i+i\gamma t)$ and

$$\frac{1}{N}\sum_i a_i^{(0)}=\langle a^{(0)}\rangle=0.$$

We seek a solution of (2) with initial conditions (4) in the form

$$a_i=a_i^{(0)}+a_i'.$$

Linearizing (2) in the vicinity of $a_i^{(0)}$ we get

$$\dot a_j'=2i\gamma a_j'+i\gamma a_j'^{*}\exp[2i\gamma t+2i\varphi_j]-\frac{1}{N}\sum_{j'} a_{j'}'. \tag{5}$$

We sum (5) over all $j=1,\dots N$ and assume that $N^{-1}\Sigma_j a_j=\langle a\rangle$, where the averaging is carried out over the initial phase φ. We then obtain

$$\langle \dot a'\rangle=2i\gamma\langle a'\rangle+i\gamma e^{2i\gamma t}\langle e^{2i\varphi}a'^{*}\rangle-\langle a'\rangle. \tag{6}$$

We put

$$\langle e^{2i\varphi}a'^{*}e^{i\gamma t}\rangle=y,\quad \langle a' e^{-i\gamma t}\rangle=z.$$

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Equations (5) and (6) and their conjugates lead then to equations for z and y:

$$\dot{z}=i\gamma z+i\gamma y-z,\quad \dot{y}=-i\gamma y-i\gamma z. \tag{7}$$

In this case

$$z(0)=\frac{1}{N}\sum_i \delta_i=\delta,$$

$$\dot{z}(0)=(i\gamma-1)\delta+i\gamma\delta',\quad \delta'=\frac{1}{N}\sum_i \delta_i^* e^{-2i\varphi_i}. \tag{8}$$

At $\delta_1=\delta$ we have $\delta'=0$.

From Eqs. (7) follows a characteristic equation whose roots are

$$\alpha_{1,2}=i\gamma-{}^1/_2\pm({}^1/_4-i\gamma)^{1/2}=ia_{1,2}+b_{1,2},$$

$$b_{1,2}=-{}^1/_2\pm{}^1/_2\{{}^1/_2[1+(1+16\gamma^2)^{1/2}]\}^{1/2}. \tag{9}$$

It is seen from (9) that at $\gamma\neq 0$ the growth rate $b_1>0$, i.e., z and $\langle a\rangle$ increase exponentially and the oscillators become automatically phased. The radiation intensity is

$$I\sim 2N\langle a\rangle\langle a^+\rangle,$$

or else

$$I\sim 2N^2\nu_0\langle a\rangle\langle a^+\rangle,\quad \text{since}\quad \tau=\nu_0 Nt,$$

$${}^1/_2 I(0)\sim 1,\quad \text{since}\quad \langle a(0)\rangle\sim\delta\sim N^{-1/2}. \tag{10}$$

Over times $\tau\gtrsim b_1^{-1}$, but such that $\langle a\rangle\ll 1$, we have

$$I(\tau)\propto e^{2b_1\tau}. \tag{11}$$

The nonlinearity limits the exponential growth of the intensity and produces a superradiant pulse. Equations (5)–(7) are valid only so long as the condition

$$\langle a\rangle\ll 1. \tag{12}$$

is satisfied. An analytic investigation of the set of equations (2) with the condition (12) violated is very complicated, since the mean-field approximation is not valid. Results of a numerical solution of the set (2) are shown in Fig. 1 for $\gamma=1, 2, 5$ and $N=100$.

For $\gamma=2$ we have $2b_1\sim 1.12$ If $\langle a\rangle\lesssim 0.2$ and $2<\tau<3.2$, the radiation intensity increases exponentially. With decrease of the nonlinearity the pulse duration decrease and the maximum pulse intensity increases. At $\gamma=2$ the maximum intensity $I=I_{\max}$ is reached at $\langle a\rangle\sim 0.34$. At $I=I_{\max}$ we have $\langle|a|^2\rangle\approx 0.62$. As the oscillators become phased their energy dispersion increases.

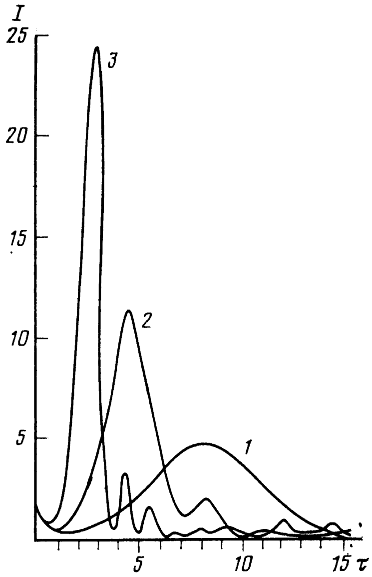

FIG. 1. Plot of $\frac{1}{2}I(\tau)$ for different values of γ: 1—$\gamma=1$; 2—$\gamma=2$; 3—$\gamma=5$.

The additional maxima are due to repeated phasing of the oscillators. It can thus be stated that the collective spontaneous emission can be produced also in a purely classical system of nonlinear oscillators. Totally unphased anharmonic oscillators excited at the initial instant of time and interacting via the intrinsic radiation field are in an unstable state. A small fluctuation of the average polarization initiates the development of this instability and leads to an exponential growth of the radiation intensity, whereas in a system of linear oscillators the initial fluctuations of the average polarization are exponentially damped. The nonlinearity eventually limits this growth and forms a superradiant pulse.

The authors are deeply grateful to L. V. Keldysh for valuable remarks in the course of the work.

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Translated by J. G. Adashko