Nondissipative gravitational turbulence

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The nonlinear stage of the development of the Jeans instability in a cold nondissipative gravitating gas is considered. It is shown that in a time exceeding the Jeans time a nondissipative gravitational singularity (NGS) (a stationary dynamical structure with a singularity at its core) is formed in the neighborhood of a local density maximum. The density of the gas at the center of the NGS (at $r \to 0$) becomes infinite: $\rho \propto r^{-\alpha}$, where $\alpha = 24/13$, and the potential Ψ of the field and the mean square velocity $\overline{V^2}$ of the captured gas have a power singularity: $\Psi \propto \overline{V^2} \propto r^{2-\alpha}$. A turbulent state arises as a result of the development of the instability when the initial density distribution is irregular. The turbulent state is a hierarchical structure, consisting of nested moving NGS of different scales, with smaller-scale ones trapped in the field of larger-scale ones. In each given NGS, scaling relations are then fulfilled both for the density of the gas and for the number density of the smaller-scale nondissipative gravitational singularities. A brief comparison with observational data shows that the actual hierarchical structure of the Universe on scales from spherical star clusters to rich clusters of galaxies apparently corresponds to developed gravitational turbulence.

Because of the action of the forces of universal gravitation a cold gravitating gas is unstable. The development of this instability leads to the appearance of regions of strong compression of the gas, and this has decisive significance for the formation of galaxies, galaxy clusters, superclusters, etc. It is usually assumed that the main role in this process is played by hidden mass that is nondissipative, i.e., interacting only through gravitational forces.[1,2] The dynamics of a stellar gas is also nondissipative in the first approximation.[3] Therefore, considerable attention is paid to the study of the development of the instability of a nondissipative gravitating gas in an expanding Universe and to the determination of the structures that arise in this process.[1,2]

In the present paper we consider the nonlinear stage of the gravitational instability in regions that are sufficiently small in comparison with the radius of the horizon, when the expansion is unimportant and the Newtonian approximation is valid. It is the development of this instability that leads to the onset of gravitational turbulence. Because of self-gravitation and the absence of dissipation, gravitational turbulence differs fundamentally from, e.g., the turbulence of an incompressible liquid (it differs not only in its structure, but also in its physical nature). In Kolmogorov turbulence the energy source of the pulsations is the macroscopic fluid flow, specified over large scales, while the spectra of the pulsations are determined by the flow of dissipated energy from large scales to small scales. In gravitational turbulence there is no dissipation and the stationary spectra arise here as a result of the development of an irregular, unstable initial state. The characteristic time of development of the turbulence is

$$t \gg t_g, \qquad t_g = (4\pi G \rho_0)^{-1/2}, \tag{1}$$

where t_g is the Jeans time, G is the gravitational constant, and ρ_0 is the initial density of the gas.

We shall consider strongly unstable (i.e., far-from-equilibrium) initial states, when the initial velocity $V_0 \cong 0$ and the density disturbances $\delta\rho(0) > \rho_0$, so that the nonlinear development of the Jeans instability leads to the onset of strong disturbances of the gas density: $\delta\rho \gg \rho_0$.[1)] In this case, as will be shown in Secs. 1–5, in the time (1) a stable stationary structure, with a singularity at its core, develops in the neighborhood of the local density maximum. We shall call this formation a nondissipative gravitational singularity (NGS). The density at the center of the NGS becomes infinite, and the field potential Ψ and mean square velocity $\overline{V^2}$ of the gas trapped in the NGS have a power singularity:

$$\rho \propto r^{-\alpha}, \qquad \Psi = \Psi_0 + \Psi_1 r^{2-\alpha}, \qquad \overline{V^2} \propto r^{2-\alpha}. \tag{2}$$

The parameter α here is found to be almost constant: $\alpha \cong 1.8$.

The nonlinear development of the Jeans instability for an isolated initial density peak leads, in this way, to the formation of an entirely regular dynamical structure. The turbulent state arises as a result of the development of the instability in the case when the initial density distribution is random and irregular. The turbulent state is investigated in Sec. 6 under the assumption that the spectrum of the fluctuations of the initial density distribution is homogeneous and isotropic. It is shown that in this case a hierarchical structure consisting of nested moving systems of NGS of different scales develops, with smaller-scale singularities trapped in the gravitational field of larger-scale scale singularities. The potential of the gravitational field is everywhere continuous, but the density becomes infinite at each singularity. The number of inhomogeneities of different scales depends on the initial spectrum of the density fluctuations. However, the scaling relations (2), both for the density of matter and for the number density of the smaller-scale NGS trapped in the field of a larger-scale NGS, and also for their velocities and field potential, remain the same for all scales. It is this structure that forms developed nondissipative gravitational turbulence. Naturally, it is realized only in a certain range of scales.

In the concluding part of Sec. 6 we give a brief comparison with observational data. This shows that the hierarchical structure that is actually observed on scales from spherical star clusters to rich clusters of galaxies corresponds, apparently, to developed gravitational turbulence.

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1. THE PRIMARY GRAVITATIONAL COMPRESSION. THE ONSET OF THE SINGULARITY

The dynamics of a gravitating gas in the absence of dissipation is described by the equations

$$\partial\rho/\partial t+\operatorname{div}(\rho\mathbf{V})=0,$$
$$\partial\mathbf{V}/\partial t+(\mathbf{V}\nabla)\mathbf{V}+\nabla\Psi=0,\quad \Delta\Psi=\rho. \tag{3}$$

Here $\mathbf{V}$ is the hydrodynamic velocity of the gas, ρ is the gas density, and Ψ is the potential of the gravitational field. For simplicity the density ρ is written in the system of units in which $4\pi G=1$ (G is the gravitational constant), or, in other words,

$$\rho=4\pi G\rho_1, \tag{4}$$

where ρ_1 is the ordinary gas density. The equations (3) are valid for a cold gas, of temperature $T\to 0$.

In this section we shall study nonlinear and nonstationary flows of the gravitating cold gas described by Eqs. (3). First we shall consider the simplest, spherically symmetric configuration, when the velocity has one component $V_r=V$ and all quantities depend only on r and t. In the following it will be shown that, because of the central character of the gravitational forces, it is possible to study the leading singularity of the structure of gravitational turbulence using the example of spherically symmetric motion.

In the spherically symmetric case Eqs. (3) take the form

$$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho V)=0,$$

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial r}+\frac{\partial\Psi}{\partial r}=0,\quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)=\rho. \tag{5}$$

We express ρ by means of the last equation, substitute it into the first equation, and integrate over r^2dr. We obtain

$$\frac{\partial y}{\partial t}+r^2V\rho=C(t), \tag{6}$$

where $C(t)$ is an arbitrary function of time, and

$$y=r^2\frac{\partial\Psi}{\partial r},\quad \rho=\frac{1}{r^2}\frac{\partial y}{\partial r}. \tag{7}$$

We shall consider natural conditions when the total mass of the matter is equal to M and there are no mass sources anywhere in space ($dM/dt=0$). Then for $r\to\infty$, as is clear from (7), $y_\infty=M/4\pi$, i.e., $dy_\infty/dt=0$, and from (6) it follows that $C(t)=0$. The system (5) then takes the simple form

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial r}+\frac{1}{r^2}y=0,\quad \frac{\partial y}{\partial t}+V\frac{\partial y}{\partial r}=0. \tag{8}$$

The solution of Eqs. (8) can be obtained by using a hodograph transformation.[5,6] In place of (8) we then obtain

$$-\frac{\partial r}{\partial V}+V\frac{\partial t}{\partial V}=0,\quad \left(\frac{\partial r}{\partial y}-V\frac{\partial t}{\partial y}\right)\left(1+\frac{y}{r^2}\frac{\partial t}{\partial V}\right)=0. \tag{9}$$

The latter equation decomposes into two equations: Either

$$\partial r/\partial y-V\partial t/\partial y=0, \tag{9a}$$

or

$$\partial t/dV=-r^2/y. \tag{10}$$

It is not difficult to convince oneself that the system (9), (9a) does not have physically interesting solutions. From (9) and (10) we obtain

$$\partial r/\partial V=-Vr^2/y,\quad \partial t/\partial V=-r^2/y. \tag{11}$$

Hence,

$$r^{-1}=V^2/2y+H(y), \tag{12}$$

where $H(y)$ is an arbitrary function. Integrating next the second equation (11), with (12) taken into account, we find $t(y, V)$. The final expressions have the form

$$t=(2y)^{-1/2}H^{-3/2}(y)[\operatorname{arctg} z+z/(1+z^2)]+\mathcal{M}(y),$$
$$r=1/H(y)(1+z^2),\quad z=-V[2yH(y)]^{-1/2}. \tag{13}$$

Here $\mathcal{M}(y)$ and $H(y)$ are two arbitrary functions, determined by the initial conditions of the problem. In the relativistic case the analogous solution was obtained by Tolman.[7] In particular, if the gas at the initial time $t=0$ is stationary, i.e.,

$$\rho=\rho_0(r),\quad V=V_0(r)=0, \tag{14}$$

then

$$\mathcal{M}(y)=0,\quad H(y)=\frac{1}{r},\quad y=\int_0^r r_1^2\rho_0(r_1)\,dr_1. \tag{15}$$

The last two relations in (15) with a specified initial density $\rho_0(r)$ determine the function $H(y)$. From (13), (7) we find after this, in implicit form, the functions ρ, V, and Ψ for all values of t and r. We note that the choice (14) of a zero initial condition for the velocity is related to a real problem of interest to us[2]; in addition, it is precisely in this case that the effects of gravitational compression are manifested most sharply.

We shall give an example. Suppose that, at the initial time $t=0$,

$$V_0(r)=0,\quad \rho_0(r)=\begin{cases}\rho_{10}(1-r^2/a^2), & r\leqslant a,\\ 0 & r>a,\end{cases} \tag{16}$$

where a is the characteristic size of the density inhomogeneity. For the following it is convenient to measure r in units of a, and y in units of $y_0=\rho_{10}a^3/3$. Then $r/a\to r$ and $y/y_0\to y$. It then follows from (15) that, at $t=0$,

$$y(r)=\begin{cases}r^3-{}^3/_5r^5, & r\leqslant 1,\\ 0, & r>1.\end{cases} \tag{17}$$

The function $r^{-1}=H(y)$ has the following asymptotic forms:

$$H(y)=y^{-1/3}(1-{}^1/_5y^{2/3})\quad\text{for}\quad y\to 0,$$
$$H(y)=1+3^{-1/2}({}^2/_5-y)^{1/2}\quad\text{for}\quad y\to{}^2/_5-0,$$
$$0\quad\text{for}\quad y\to{}^2/_5+0. \tag{18}$$

The dependence of y on r, as determined by (13), (7), and (17), is presented in Fig. 1 for different values of t. The functions $\rho(r)$, $\Psi(r)$, and $V(r)$ can be found analogously. It can be seen from the figure that the gas is rapidly compressed under the action of the gravitational force, and at time

$$t_0=3^{1/2}\cdot 2^{-3/2}\pi t_g \tag{19}$$

a singularity arises—the gas density at the center becomes infinite. In the latter expression t_0 is given in dimensional variables; it can be seen that the singularity arises in the characteristic time t_g (1) of development of the Jeans instability. The behavior of all quantities at the time $t=t_0$ in the vicinity of the singular point $r=0$, as follows from (13), (18), is given by the expressions (in dimensional variables)

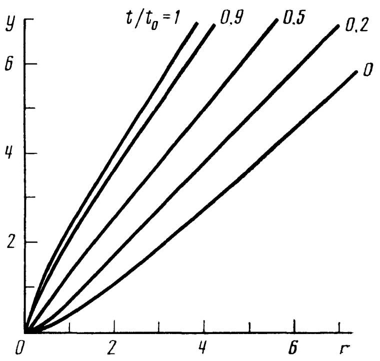

FIG. 1. Dependence of the quantity $y = r\partial\Psi/\partial r$ on r at different times t/t_0.

$$\rho={}^3/_7(40/9\pi)^{6/7}\rho_{10}(r/a)^{-12/7},\ \Psi=\Psi_m+{}^7/_2\beta(r/a)^{2/7},$$
$$V=(2\beta)^{1/2}(r/a)^{1/7},\quad \beta={}^4/_3\pi G\rho_{10}a^2(40/9\pi)^{6/7}. \tag{20}$$

It can be seen that the field potential, unlike the density, has a finite value at the singular point. This constitutes the principal difference from the equilibrium distribution of the gas, for which the particle density and the potential, which are connected by the Boltzmann relation, can become infinite only simultaneously.

A singularity of the form (19), (20) arises for any initial spherically symmetric density distribution $\rho_0(r)$ in (15) with a maximum at the center $r = 0$. The reason for this is that the motion of the gas in the region $r \sim 0$ is determined by the particle distribution $\rho_0(r)$ in the vicinity of the center, which, for $r \ll a$, has the form (16), with

$$\rho_{10}=\rho_0(0),\quad a^2=2\rho_{10}(d^2\rho_0(0)/dr^2)^{-1}. \tag{21}$$

Therefore, in the general case as well, the time t_0 of onset of the singularity and the behavior of all quantities in the vicinity of the center are described by the formulas (19)–(21).

Analogous singularities arise in a finite time t_0 upon compression of a gravitating gas even in the absence of spherical symmetry. For example, for planar motion in conditions when $V_0(x) = 0$, at $t = 0$ we have

$$t_0=(2\pi G\rho_{10})^{-1/2},\quad \rho=\frac{\beta_1(x/a)^{-4/3}}{12\pi Ga^2},$$
$$\Psi=\Psi_m+\frac{3}{4}\beta_1\left(\frac{x}{a}\right)^{4/3},$$
$$V=-\frac{\beta_1t_0}{a}\left(\frac{x}{a}\right)^{1/3},\quad \beta_1=4\pi G\rho_{10}\cdot 3^{1/3}a^2, \tag{22}$$

where ρ_{10} and a are given, as before, by (21). For cylindrically symmetric motion under the same conditions we have

$$t_0=\frac{1}{2}(G\rho_{10})^{-1/2},\quad \rho=\frac{\beta_2}{4\pi G}r^{-4/3},\quad \Psi=\Psi_m+\frac{3}{2}\beta_2r^{2/3},$$
$$V^2=2\beta_2r^{2/3}\ln[(2/\pi)^{1/2}t_0/\beta_2^{1/2}r^{2/3}],\quad \beta_2=2^{7/3}\pi^{4/3}\rho_{10}a^{4/3}.$$

We now consider the distribution of matter near an aribtrary density maximum. In the general case, such a distribution can be represented in the form

$$\rho=\rho_0-{}^1/_2\rho_0''(x^2/a^2+y^2/b^2+z^2/c^2). \tag{23}$$

We introduce the variable

$$\xi=(x^2/a^2+y^2/b^2+z^2/c^2)^{1/2} \tag{24}$$

and shall seek the solution of the system (3) with the initial conditions (23) in the form

$$\nabla\Psi=\frac{\mathbf{r}}{\xi}B(\xi,t),\quad \mathbf{V}=\frac{\mathbf{r}}{\xi}V(\xi,t),\quad \rho=\rho(\xi,t). \tag{25}$$

Substituting (25) into (3) and using arguments analogous to those employed in the derivation of (6), (7), (8), we obtain

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial\xi}+B=0,\quad \frac{\partial B}{\partial t}+\frac{V}{\xi^2}\frac{\partial}{\partial\xi}(\xi^2B)=0. \tag{26}$$

Introducing the variable $y = \xi^2\partial B/\partial\xi$, it is not difficult to reduce the system of equations (26) to the corresponding equations (8) obtained earlier for the spherically symmetric case, with the one difference that now the role of the variable r is played by the variable ξ. Because of this, the solution of the system (26) also turns out to be identical to that obtained above [(13), (19)–(21)], with the same replacement of r by ξ. Thus, the singularity that arises in the general case of the three-dimensional density distribution (23) has the same character as in the spherically symmetric case. The pattern of the motion becomes two-dimensional (cylindrical) or one-dimensional (planar) only when we set $c = \infty$ (or, respectively, $c = \infty$ and $b = \infty$) in (23) from the outset; the character of the behavior of the density near the singularity then changes [see (22)].

The singularities considered above arise on account of gravitational compression of peaks in the density of matter that is initially at rest or moving uniformly. In the presence of dispersion of the initial velocity the appearance of other singularities is possible, and these have been considered previously by Zel'dovich[8] and Arnol'd.[9] An essential point is that they have the same behavior of the parameters in the vicinity of the singularity as in the planar case (22). In this case [see (22)]

$$V^2\gg|\Psi-\Psi_m|,$$

so that the motion near the singularity has, essentially, a purely kinematic character, and the role of the field is of little importance.[9] On the other hand, in our case, for arbitrary compression, as can be seen from (20) and (26), near the singularity the virial relation

$$V^2\sim|\Psi-\Psi_m|$$

is fulfilled. In this case the decisive influence on the dynamics of the gas after the appearance of the singularity is exerted by the capture of the gas in the potential well that has been formed. The following sections of the paper are devoted to an analysis of the pattern of motion that arises in this case.

2. MANY-FLUX FLOWS

The applicability of the equations (3) of single-flow hydrodynamics to the description of the motions of a cold gravitating gas is limited. This can be seen, in particular, from the results of the preceding section: In a finite time t_0 a flux-breaking singularity arises. Beyond the singularity, in a nondissipative medium, many-flux flows appear: At a given point r at time t several fluxes, having different velocities V_i and densities ρ_i, are present simultaneously. By considering, in this case too, a cold gas interacting only through gravitational forces, we arrive at the equations [in place of (3)]

$$\frac{\partial\rho_i}{\partial t}+\operatorname{div}(\rho_i\mathbf{V}_i)=0,$$

$$\frac{\partial\mathbf{V}_i}{\partial t}+(\mathbf{V}_i\nabla)\mathbf{V}_i+\nabla\Psi=0,\quad \Delta\Psi=\sum_i\rho_i,\quad i=1,\dots n. \qquad (27)$$

With the development of the motion, because of multiple reflection and breaking, the number n of fluxes can become rather large. It is convenient, therefore, to introduce the distribution function

$$f(\mathbf{r},\mathbf{V},t)=\sum_i\rho_i(\mathbf{r},t)\,\delta(\mathbf{V}-\mathbf{V}_i(\mathbf{r},t)).$$

The distribution function $f(\mathbf{r},\mathbf{V},t)$ has the meaning of the matter density created at the spatial point $\mathbf{r}$ at time t by the fluxes whose velocities lie in the interval between $\mathbf{V}$ and $\mathbf{V}+d\mathbf{V}$. Using the system of equations (27), we obtain

$$\frac{\partial f}{\partial t}+\mathbf{V}\nabla f+\nabla\Psi\frac{\partial f}{\partial\mathbf{V}}=0,\quad \Delta\Psi=\rho,\quad \rho=\int f\,d\mathbf{V}. \qquad (28)$$

We emphasize that the kinetic equation (28) describes in our case the hydrodynamics of many fluxes of a cold continuous medium, and not the motion normally considered (see, e.g., Ref. 10) for a system of particles. The possibility of going over to the kinetic description (28) in the many-flux hydrodynamics (27) is based on the equivalence of the inertial mass and gravitational mass. Only because of this circumstance is it the case that the acceleration acquired by each flux at a given point $\mathbf{r}$ under the action of the gravitational force is independent of the flux density and is described by a single expression.

An important role in Eq. (27) or (28) is played by the caustic surfaces on which the fluxes either coalesce or proliferate. This process is shown for different cases in Figs. 2 and 3. In Fig. 2, $x^+(t)$ and $x^-(t)$ are points of the caustic that arises upon kinematic breaking of the flux.[8,5] Between x^+ and x^- there are three fluxes, and for $x>x^+$ and $x<x^-$ there is one. At the point x^+ the fluxes V_1 and V_2 coalesce, and the derivatives of V have singularities:

$$\left(\frac{\partial V_1}{\partial x}\right)_{x\to x^+(t)}\to-\infty,\quad \left(\frac{\partial V_2}{\partial x}\right)_{x\to x^+(t)}\to+\infty. \qquad (29a)$$

The densities $\rho^{(1)}$ and $\rho^{(2)}$, like the second derivative $\partial^2\Psi/\partial x^2$

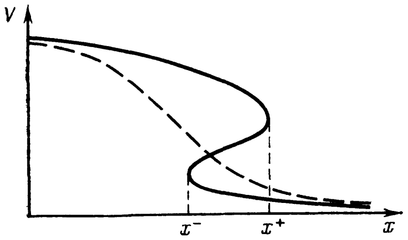

FIG. 2. Profile of the velocity $V(x)$. Dashed curve—before the breaking of the flux; solid curve—after the breaking of the flux; x^+ and x^- are points on the caustics.

of the field, also become infinite at the point $x^+(t)$. However, the gravitational potential Ψ itself and its first derivative remain continuous at points on the caustic:

$$\rho^{(1)}_{x\to x^+(t)}=\rho^{(2)}_{x\to x^+(t)}=\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial x^2}\right)_{x\to x^+(t)}=C(x^+,t)\,(x^+-x)^{-1/2},$$

$$\left(\frac{\partial\Psi}{\partial x}\right)_{x\to x^++0}=\left(\frac{\partial\Psi}{\partial x}\right)_{x\to x^+-0},\quad \Psi(x^++0)=\Psi(x^+-0). \qquad (29b)$$

Analogous conditions are fulfilled at the point x^- at which the fluxes V_2 and V_3 coalesce. A general classification of the types of caustic surfaces that arise upon kinematic breaking is given in papers by Arnol'd *et al.*[9,11]

Figure 3 shows the proliferation of fluxes in an oscillating gas captured by the field. If the frequency of the oscillations, as in the case considered in Fig. 3, varies as a function of their level, then the number of caustics and, consequently, the number of fluxes will increase continuously with time (see, e.g., Ref. 12). The boundary conditions on the caustics continue to have the form (29).

The singularities (20), (21) that we considered in Sec. 1, which arise at the center of a spherical or an elliptical density peak, are capture singularities. They are, however, distinctive: For $t>t_0$, as will be shown below, singular features containing an infinite sequence (converging toward the center) of caustic points arise in them. It is on the basis of these features that the stationary nondissipative gravitational singularity is subsequently formed.

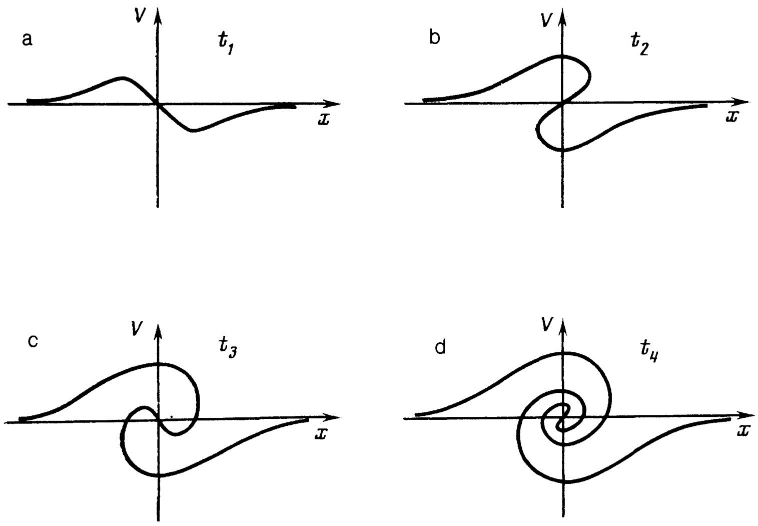

FIG. 3. Development with time of a many-flux flow $V(x)$ upon capture in a potential well: $t_1<t_2<t_3<t_4$.

3. SPHERICALLY SYMMETRIC GRAVITATIONAL SINGULARITY

We shall continue the analysis of the compression of a spherically symmetric density peak into regions beyond the singularity. For this we shall go over from (3) to the kinetic equation (28), which, in the case under consideration, can be written in the form

$$\frac{\partial f}{\partial t}+V\frac{\partial}{\partial r}f+\frac{\partial\Psi}{\partial r}\frac{\partial f}{\partial V}=0,\qquad \frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)=\tilde\rho,\quad \tilde\rho=\int f\,dV. \tag{30}$$

Here $f=f(V,r,t)$ is the distribution function of the fluxes with respect to the radial velocity $V_r=V$. In accordance with Sec. 1, it is assumed that other velocity components are absent at the initial time $t=0$, and, consequently, because of the spherical symmetry of the field potential, at all subsequent times. The effective density $\tilde\rho$ is written with allowance for the spherical accumulation of flux, and is connected with the true density ρ_1 by the relation [see (4)]

$$\tilde\rho=4\pi Gr^2\rho_1. \tag{31}$$

It follows from (13), (14), and (31) that in the primary-compression period $t<t_0$ we have $\tilde\rho=C(t)r^2$ in the vicinity of the center. At the time $t=t_0$ (19) of onset of the singularity the value of $\tilde\rho$ increases sharply: $\tilde\rho=C_0r^{2/7}$ (20). For $t>t_0$ in the case of purely kinematic breaking the quantity $\tilde\rho$ would have a finite value $\tilde\rho_0(t)$ at $r=0$, but in this case, as follows from (30), the potential of the field at $r\cong 0$ would change its form sharply and would acquire a singularity of the logarithmic type:

$$\Psi(r,t)_{r\to0}\approx\tilde\rho_0(t)\ln r. \tag{32}$$

In this case strong capture in the gravitational field (32) will occur, so that the motion of the gas, like the value of its density at $r\cong0$, should be determined specifically by the action of the field.

Another important point is that the logarithmic potential well at times t close to t_0 turns out to be very narrow. This implies that the frequency of the oscillations of the fluxes trapped in the well is large, and, as will be shown below, increases rapidly as one goes deeper into the well, i.e., as $r\to0$. Consequently, for $t>t_0$ an infinite set of caustic singularities, converging toward the point $r=0$, immediately arises in the vicinity of the center. Correspondingly, an infinite set of fluxes also arises here. We shall call such a structure a gravitational singularity.

To describe the gravitational singularity it is natural to use the theory of adiabatic capture (see Ref. 13 and Sec. 36 of Ref. 14), since the variation with time of the parameters of the potential well is slow in comparison with the frequency of oscillations of the fluxes trapped in it. This means that in the first approximation the oscillations occur as if in a stationary well. Neglecting, then, the term $\partial f/\partial t$ in Eq. (30), we obtain

$$f=f(\varepsilon),\quad \varepsilon={}^1/_2V^2+\Psi.$$

In other words, because of the frequent oscillations in the well the distribution function, in the first approximation, turns out to be mixed over the surface of constant "energy" ε. Taking into account now the slow variation of the well with time, we have

$$f=f(\varepsilon,t)=f(I),\qquad I=\int_0^{r_1}V\,dr=2^{1/2}\int_0^{r_1}(\varepsilon-\Psi)^{1/2}dr,\quad \Psi(r_1,t)=\varepsilon. \tag{33}$$

Here $I(\varepsilon,t)$ is an adiabatic invariant, $r_1=r_1(\varepsilon,t)$ is the reflection point, and it has been taken into account that $V=\pm2^{1/2}(\varepsilon-\Psi)^{1/2}$. The formulas (33) give the general solution of Eq. (30) in the adiabatic approximation. The conditions for applicability of this approximation for our case will be discussed later.

The form of the function $f(I)$, according to Ref. 13, is determined by the conditions on the capture boundary, i.e., for $I=I_m(t)$, where I_m is the maximum adiabatic invariant, calculated for the capture boundaries. It is natural to take as a capture boundary the level of the first caustic $\Psi=\Psi_c(t)$; the energy in the well ($\varepsilon\leqslant0$) is reckoned from this level. The potential at a point on the caustic, according to (29), is continuous, together with its first derivative, and will also be reckoned from the level Ψ_c. In order to find the function $f(I_m)$ it is necessary to determine the total magnitude of the mass $\tilde M$ of matter trapped in the well:

$$\tilde M=\int f\,dV\,dr=\int_0^{I_m}f(I)\,dI\int\delta(I-I(\varepsilon))\,\delta\left(\varepsilon-\frac{V^2}{2}-\Psi\right)d\varepsilon\,dV\,dr. \tag{34}$$

It is not difficult to convince oneself that, since [see (33)]

$$\int\delta\left(\varepsilon-\Psi-\frac{V^2}{2}\right)dV\,dr=2^{1/2}\int\frac{d\varepsilon}{(\varepsilon-\Psi)^{1/2}}=\frac{dI}{d\varepsilon},$$

the second integral in (34) is equal to unity. Therefore,

$$\tilde M=\int_0^{I_m}f(I)\,dI$$

and, consequently,

$$f(I_m)=\frac{d\tilde M}{dt}\Big/\frac{dI_m}{dt}. \tag{35}$$

The value of the invariant I_m increases monotonically with time from $I_m=0$ at $t=t_0$ to $I_m=I_m(t)$. The quantity $f(I_m)$ varies correspondingly, in accordance with (35), and, according to (33), the value of f is subsequently conserved:

$$f(I)=f(I_m)_{I_m=I}. \tag{36}$$

The formula (36) expresses the distribution function $f(I)$ in terms of its value on the boundary I_m at the earlier time at which I_m was equal to I. Thus, in its general form the solution of the kinetic equation (30) in the region of capture is known and is given by (33), (35), and (36); it depends strongly on the potential $\Psi(r,t)$ of the field. The problem now reduces to the integration of just the potential equation (30), which takes the form

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial\Psi}{\partial x}=2^{1/2}\int_\Psi^0 f(\varepsilon,t)(\varepsilon-\Psi)^{-1/2}d\varepsilon,\quad x=\ln\frac{r}{r_0(t)},\qquad f(\varepsilon,t)=f(I),\quad I=2^{1/2}\int_0^{r_1}(\varepsilon-\Psi)^{1/2}dr,\quad \Psi(r_1,t)=\varepsilon. \tag{37}$$

Here the parameter $r_0(t)$ is determined by the conditions for joining with the solution (13), (20), and the function $f(I)$ is determined by the conditions (35), (36) on the boundary of the well. We shall consider the process of formation of the well immediately after the primary singularity:

$$\tau=(t-t_0)/t_0\ll 1. \tag{38}$$

For a point $r_c(t)$ on the first caustic, using the solution (13), (20) we obtain

$$r_c(t)=r_{c0}\tau^{7/6},\quad r_{c0}=ak, \tag{39}$$

where a is the scale characterizing the initial density distribution (16), (21); k is a constant of order unity that determines the exact position of the caustic, and will be determined later. Correspondingly, for the mass $\tilde{M}$ trapped in the well we have

$$\tilde{M}=\int_0^{r_c}\tilde{\rho}\,dr=M_0k^{9/7}\tau^{3/2},\quad M_0=\frac{4}{3}\pi\left(\frac{40}{9\pi}\right)^{6/7}\rho_1a^3. \tag{40}$$

Taking into account the power-law character of the dependence of r_c and $\tilde{M}$ on τ, we shall seek the solution of Eq. (37) in the self-similar form

$$\Psi(x,\ t)=\Psi_1\tau^{\alpha_0}\tilde{\Psi}(x), \tag{41}$$

where Ψ_1 is a normalization constant with the dimensions of the potential Ψ, x is defined in (37), and α_0 is an arbitrary constant; Ψ_1 and α_0 will be determined below [see (47)]. According to (37) we then obtain

$$r_0(t)=br_c(t)=br_{c0}\tau^{7/6},$$

$$I=\tau^{\alpha_0/2+7/6}I_1F(y),\quad F(y)=\int_{-\infty}^{x_1(y)}[y-\tilde{\Psi}(x)]^{1/2}e^x\,dx, \tag{42}$$

$$\tilde{\Psi}(x_1)=y,\quad y=\varepsilon/\Psi_1\tau^{\alpha_0},\quad I_1=2^{3/2}br_{c0}\Psi_1^{1/2}.$$

Here b is a constant of order unity. In particular, for the maximum invariant,

$$I_m=\tau^{\alpha_0/2+7/6}I_1F_0,\quad F_0=\int_{-\infty}^{0}[-\tilde{\Psi}(x)]^{1/2}e^x\,dx,\quad \tilde{\Psi}(0)=0. \tag{43}$$

It follows from (35), (40), and (42) that

$$f(I_m)=\tau^{1/3-\alpha_0/2}\frac{3}{(\alpha_0+7/3)}\frac{M_0k^{9/7}}{I_1F_0}. \tag{44}$$

In order to find now the distribution function $f(I)$ of the adiabatic invariants we must, according to (36), equate $I_m(\tau)=I$ and hence express τ as a function of I. We have

$$\tau(I)=(I/I_1F_0)^{2/(\alpha_0+7/3)}.$$

Substituting this quantity into (44) and taking (36) into account, we find

$$f(I)=I^{(4/3-\alpha_0)/(7/3+\alpha_0)}\frac{3}{\alpha_0+7/3}M_0k^{9/7}(I_1F_0)^{-3/(\alpha_0+7/3)}. \tag{45}$$

Correspondingly, for the energy distribution function, using (42), we obtain

$$f(\varepsilon,t)=\tau^{1/3-\alpha_0/2}[F(y)]^{(4/3-\alpha_0)/(7/3+\alpha_0)}\frac{3}{7/3+\alpha_0}\frac{M_0k^{9/7}}{I_1}F_0^{-3/(\alpha_0+7/3)}. \tag{46}$$

Substituting this expression into (37) we determine the parameters α_0 and Ψ_1:

$$\alpha_0=1/3,\qquad \Psi_1=3/16M_0k^{2/7}(ba)^{-1}F_0^{-3/8}. \tag{47}$$

Equation (37) is brought after this to its final form:

$$\frac{d^2\tilde{\Psi}}{dx^2}+\frac{d\tilde{\Psi}}{dx}=\int_{\Psi}^{0}F^{1/8}(y)(y-\tilde{\Psi})^{-1/2}\,dy,$$

$$F(y)=\int_{-\infty}^{x_1(y)}[y-\tilde{\Psi}(x)]^{1/2}e^x\,dx,\quad \tilde{\Psi}(x_1)=y. \tag{48}$$

We shall investigate the asymptotic behavior of the function $\tilde{\Psi}(x)$ in the vicinity of the singularity, i.e., for $x\to-\infty$ or $r\to 0$. We make the assumption,which will be justified later, that $F(y)$ falls off sufficiently rapidly as $y\to-\infty$, so that the integral

$$\int_{-\infty}^{0}F^{1/8}(y)\,dy=C_0 \tag{49}$$

converges. Equation (38) in the limit $x\to-\infty$ then takes the form

$$\frac{d^2\tilde{\Psi}}{dx^2}+\frac{d\tilde{\Psi}}{dx}=\frac{C_0}{(-\tilde{\Psi})^{1/2}}. \tag{50}$$

Its solution that goes to $-\infty$ as $x\to-\infty$ has the form

$$\tilde{\Psi}(x)=-\left(-\frac{3}{2}C_0x\right)^{2/3}\left[1+\frac{2}{9}\frac{\ln(-x)}{-x}+\dots\right]. \tag{51}$$

Thus, as $r\to 0$ the gravitational-field potential $\Psi\sim-[\ln(1/r)]\to-\infty$. The effective density $\tilde{\rho}$ (30) then vanishes logarithmically as $r\to 0$:

$$\tilde{\rho}\propto(-x)^{-1/3}\propto[\ln(1/r)]^{-1/3}. \tag{52}$$

Comparing (52) with (32), we see that the distributions of $\tilde{\rho}$ and $\tilde{\Psi}$ near the singularity are completely determined by the action of the gravitational field. Finally, from (51), (48), and (46) it is not difficult to obtain also the asymptotic form of the energy distribution function as $\varepsilon\to-\infty$:

$$f(\varepsilon,\ \tau)=C\tau^{1/6}(-y)^{-1/32}\exp[-1/8(-y)^{3/2}],\quad y=\varepsilon/\Psi_1\tau^{1/3}. \tag{53}$$

It can be seen that the distribution function falls off exponentially as $\varepsilon\to-\infty$, and this, in particular, is the justification for (49).

The complete solution of Eq. (48) has been found numerically by the method of iterations. The result of the calculation is presented in Fig. 4. The constants b and k, determined by the conditions for joining of the potential $\tilde{\Psi}$ and its derivative $\partial\tilde{\Psi}/\partial r$ with the solution (13), (20), were found to be equal to

$$k\approx 1.5,\quad b\approx 1.0.$$

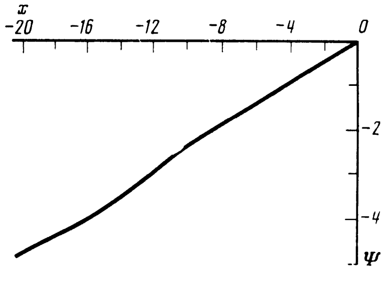

FIG. 4. Dependence of the potential Ψ on $x=\ln(r/r_0)$.

The condition for applicability of the adiabatic approximation is

$$P=\left|\frac{T}{2\pi\Psi}\frac{\partial\Psi}{\partial t}\right|\ll 1. \qquad (54)$$

Here T is the period of the oscillations. Taking into account that, according to (41), (47),

$$\frac{1}{\Psi}\frac{\partial\Psi}{\partial t}=\frac{1}{3t_0}, \quad T=2\frac{\partial I}{\partial\varepsilon},$$

from (33), (42) we obtain for $-y\gtrsim 1$

$$P=P_0(-y)^{1/4}\exp[-(-y)^{3/2}],$$

where P_0 is a constant of order unity and y is defined as in (53). For $y\to 0$ it is found that $P\cong 0.3$. Thus, the conditions for applicability of the adiabatic approximation are always well fulfilled for sufficiently large values of $|y|$.

We have considered the process of the onset and initial development of a nondissipative gravitational singularity. In time, for $t\gg t_0$, mixing occurs throughout the volume of the gas, and as a result a distribution that rapidly approaches the stationary distribution $f=f(\varepsilon)$ is established. An important point is that the distribution function preserves in this process its exponential decay in the region of the singularity, where $\varepsilon\to-\infty$. Therefore, the logarithmic laws (51) and (52) describing the decays of the potential and density $\tilde\rho$ always remain valid, and this leads, in dimensional variables, to the expressions

$$\rho=C/r^2[\ln(1/r)]^{1/3}, \quad \Psi=6\pi GC[\ln(1/r)]^{2/3}. \qquad (55)$$

An important point is that, because of the conservation of the total mass M and total energy $\mathscr{E}$ of the system in the process of the gravitational compression:

$$M=4\pi\int_0^\infty\rho_0(r)r^2\,dr,$$
$$\mathscr{E}=4\pi\cdot 2^{1/2}\int_\Psi^0\frac{\varepsilon f(\varepsilon)}{(\varepsilon-\Psi)^{1/2}}\,d\varepsilon, \qquad (56)$$

the stationary density distribution established turns out to depend extremely weakly on the concrete form of the distribution function $f(\varepsilon)$. This can be seen from Fig. 5, in which, as an example, stationary distributions of the density and potential are shown for the same values of M and $\mathscr{E}$ and substantially different functions $f(\varepsilon)$:

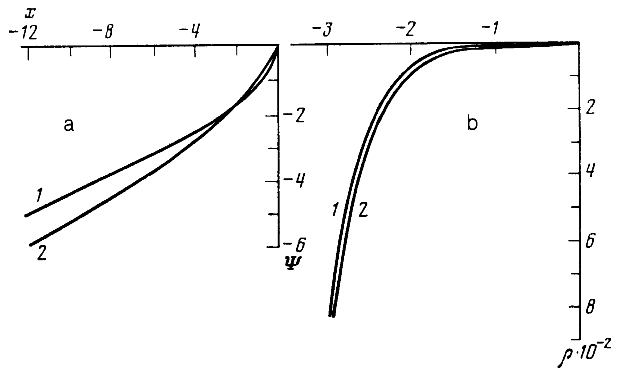

FIG. 5. Stationary distributions of (a) the field potential Ψ and (b) the density ρ as functions of $x=\ln(r/r_0)$ for the energy distribution functions (57a) (curve 1) and (57b) (curve 2).

$$f(\varepsilon)=\begin{cases} f_0, & \varepsilon_0\leqslant\varepsilon\leqslant\varepsilon_1,\\ 0, & \varepsilon<\varepsilon_0, \quad \varepsilon>\varepsilon_1,\end{cases} \qquad (57a)$$

$$f(\varepsilon)=\begin{cases} f_0\exp(\varepsilon/\varepsilon_0), & \varepsilon<\varepsilon_1,\\ 0, & \varepsilon>\varepsilon_1.\end{cases} \qquad (57b)$$

The constants f_0, ε_1, and ε_0 appearing in (57) are determined uniquely from the values of the total mass M and energy $\mathscr{E}$. We note that in the expression (55) the constant $C\cong|\mathscr{E}|/MG$.

Thus, for spherically symmetric gravitational compression a nondissipative gravitational singularity arises at the center in the Jeans time (1) and subsequently develops into a stationary singularity. The field potential and density in the NGS always have a singularity of the form (55).

4. CONSERVATION OF THE SQUARE OF THE ANGULAR MOMENTUM

Above, we considered spherically symmetric compression. In the general case, however, as was shown in Sec. 1, the three-dimensional gravitational compression of an arbitrary initial density peak has an ellipsoidal character [see (23)–(26)]. In this case an important role in the formation of the three-dimensional singularity is played by the angular momentum. It is necessary, therefore, to consider the question of the conservation of the angular momentum in a many-flux flow in a nondissipative medium.

As can be seen without difficulty, the total angular momentum of the gas is conserved in the presence of an arbitrary number of fluxes. In fact, by defining the angular-momentum density $\mathbf{l}$ of the gas by the relation

$$\mathbf{l}=\int[\mathbf{r}\mathbf{V}]f\,dV, \qquad (58)$$

starting from (28) we obtain for the components l_i the equations

$$\frac{\partial l_i}{\partial t}=\nabla_\mu\Big\{\varepsilon_{ikl}r_k\Big[-\int V_lV_\mu f\,dV+\Big(\nabla_\mu\Psi\nabla_l\Psi-\frac{\delta_{\mu l}}{2}(\nabla\Psi)^2\Big)\Big]\Big\}. \qquad (59)$$

Here ε_{ikl} is the unit antisymmetric tensor, and we have summation over repeated indices. By integrating (59) over d^3r in a closed system, we find that

$$L_i=\int l_i(\mathbf{r},t)\,d^3r=\text{const}.$$

In particular, in our formulation (14) of the problem the initial velocities

$$\mathbf{V}_0(\mathbf{r})=0 \qquad (60)$$

and, consequently, the total angular momentum is equal to zero: $L_i=0$.

The fundamental difference between a many-flux flow and a single-flux flow lies in the existence of independent higher moments of the velocity. From the vanishing, e.g., of the average velocity (the velocity of the directed motion) it does not by any means follow that the mean square velocity (the velocity of the thermal, chaotic motion) is also equal to zero. The mean square velocity, as is well known, determines the pressure, which is a very important characteristic of the dynamics of the gas. The situation with the angular momentum is entirely analogous. Besides the mean angular momentum, in a many-flux system we can also introduce the mean square angular momentum

$$\overline{L^2}=\int l_2\, d^3r,\quad l_2=\int [\mathbf{r}\mathbf{V}]^2\, d\mathbf{V}. \qquad (61)$$

This quantity, like the ordinary pressure, is a scalar, and it is this quantity which determines the internal angular-momentum pressure, as it were, that opposes the compression of the system.

We now elucidate how the square of the angular momentum evolves in time. We introduce $\overline{L^2}$ by the relation (61). Differentiating (61) with respect to time and using (28), we obtain

$$\frac{d\overline{L^2}}{dt}=-2\int \Psi(\mathbf{r},t)r^2\left[\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 j_r)\right]d\mathbf{r}, \qquad (62)$$

where ρ is the average gas density (28) and $j_r=\int V_r f\, dV$ is the flux of matter in the radial direction. Starting from the continuity equation and Poisson's equation, it is not difficult to obtain the relation

$$\frac{\partial}{\partial t}(\nabla\Psi)+\mathbf{j}_{\rm div}=0, \qquad (63)$$

where $\mathbf{j}_{\rm div}$ is the divergent part of the matter flux ($\mathrm{curl}\,\mathbf{j}_{\rm div}=0$).

Henceforth, we shall confine ourselves to considering initial conditions corresponding to an elliptically symmetric matter distribution (23) at time $t=0$. Using then the expression (63), and also the symmetry of the problem under replacement of x by $-x$, y by $-y$, or z by $-z$, we obtain

$$\frac{1}{2}\frac{d\overline{L^2}}{dt}=\int r^2\Psi\left\{\frac{\partial}{\partial t}\Delta_{\theta,\varphi}\Psi\right\}d\mathbf{r}, \qquad (64)$$

where $\Delta_{\theta,\varphi}$ is the angular part of the Laplacian in spherical coordinates. After integration of the expression (64) by parts and introduction of the variable $\mu=\cos\theta$ we find, finally,

$$\frac{d}{dt}(\overline{L^2}+\overline{L_\Psi^2})=0,\quad L_\Psi^2=\int\left[(1-\mu^2)\left(\frac{\partial\Psi}{\partial\mu}\right)^2+\frac{1}{1-\mu^2}\left(\frac{\partial\Psi}{\partial\varphi}\right)^2\right]r^2\, dr\, d\varphi\, d\mu. \qquad (65)$$

Here L_Ψ^2 is the field part of the total squared angular momentum.

It follows from the conservation law (65) that if at the initial time $t=0$ the gravitating gas had, by virtue of (60), zero angular momentum, then, as a result of complete relaxation to the mixed spherically symmetric state, although $\bar{L}_i$ remains equal to zero, the matter acquires a squared angular momentum $\overline{L^2}$ exactly equal to the field part L_Ψ^2 at $t=0$:

$$\overline{L^2}=L_{\Psi_0}^2. \qquad (66)$$

We note that, in the case of elliptical compression, before the time of formation of the caustic, as can be seen from (25), $L^2=0$. After the formation of the caustic, because of reflection from the field potential, the radial character of the motion of the particles is destroyed and the fluxes acquire a squared angular momentum L^2 on account of the squared angular momentum L_Ψ^2 of the field. In (66), $\Psi_0(r,\mu,\varphi)$ is the initial distribution of the field potential at $t=0$. Calculating the value of the quantity $L_{\Psi_0}^2$ for an initial ellipsoidally symmetric matter-density distribution (23), we find

$$L_{\Psi_0}^2=\frac{16}{105}\frac{G\rho_0 r_0^7}{a^{-2}+b^{-2}+c^{-2}}\times\left\{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}-\frac{1}{a^2b^2}-\frac{1}{a^2c^2}-\frac{1}{b^2c^2}\right\},$$
$$r_0=\max\{a,b,c\}. \qquad (67)$$

Here a, b, and c are the semiaxes of the initial ellipsoid. It can be seen that the squared angular momentum $\overline{L^2}$ acquired is proportional to the difference of the squares of the semiaxes. If all the axes are equal, the compression is spherically symmetric and, naturally, $L_{\Psi_0}^2=0$. If the initial ellipsoid differs little from a sphere, the mean density of the squared angular momentum is equal to

$$l_2(r_0)=m_0^2r_0^4,\quad m_0^2=G\rho_0^2(\varepsilon_1^2+\varepsilon_2^2-\varepsilon_1\varepsilon_2),$$
$$\varepsilon_1=(a-b)/a,\quad \varepsilon_2=(a-c)/a,\quad \varepsilon_1,\varepsilon_2\ll 1. \qquad (68)$$

Here ρ_0 is the gas density at the maximum of the initial distribution, and $r_0\cong a$ is the initial radius of the density peak [the formula (68) is written in dimensional variables].

5. GRAVITATIONAL SINGULARITY OF GENERAL FORM

We consider now the process of mixing and of the establishment of a stationary many-flux distribution after a hydrodynamic singularity of general form, with the ellipsoidal character (23), (26).

The main difference from the spherically symmetric case considered in Sec. 3 lies in the appearance of a nonzero squared angular momentum of matter about the center $r=0$. In the absence of angular momentum all characteristics of the kinetic equation (28) pass through the center and are bounded only by the energy level ε (32). Because of this, a contribution to the density $\bar\rho$ at a given point r is made by all trajectories with $\varepsilon>\Psi(r)$ [see (37)]. But the appearance of a nonzero squared angular momentum forbids penetration of the characteristics into the region close to the center. A result of this is the "reflection" of trajectories, which, as is well known,[10,14] is described by

$$\varepsilon_{\rm eff}=V^2/2+\Psi(r)+m^2/2r^2=\mathrm{const}. \qquad (69)$$

It leads to the appearance of two points of reflection of trapped trajectories ($r_{\min}$ and $r_{\max}$). The density of the trapped gas turns out to be smeared out, as it were, over the finite region between $r_{\min}$ and $r_{\max}$.

The process of the smearing out of the distribution function of the trapped fluxes over the energies near the singularity for $t>t_0$ (19), as in the spherically symmetric case, occurs in a time $t-t_0\ll t_0$ (38). But the isotropization of the system over the angular variables in a strictly spherical potential occurs on account of ergodic occupation of trajectories of the region between $r_{\min}$ and $r_{\max}$ (Ref. 10). This process is slower—its characteristic time is of order t_0. An important feature here, however, is the deviation of the field potential from spherical symmetry; this deviation leads to a considerable acceleration of the mixing and to relaxation of the system over the angular variables (the analogous process of so-called fast relaxation was considered for a gravitating gas in Ref. 16). Therefore, we can suppose that the mixing process, even in the presence of a nonzero squared angular momentum and with allowance for the fast relaxation, can be described in the adiabatic approximation.

Also of great importance is the fact that, because of the conservation of the squared angular momentum, the trajectories that emerge at time t_0 from the peripheral regions never fall close to the center and, consequently, do not affect the structure of the central region. Therefore, the stationary distribution near the singularity is established very rapidly in comparison with the total relaxation time t_0 of the initial density peak. In other words, the process of adiabatic mixing near the singularity (in contrast to the case of a spherically symmetric singularity—see Sec. 3) leads immediately to the formation of a stationary distribution function $f(\varepsilon, m^2)$. Correspondingly, the steady-state effective density $\tilde{\rho}(r)$ (31) can be represented in the form

$$\tilde{\rho}(r)=2^{1/2}\int\limits_0^\infty dm^2 \int\limits_{\Psi+m^2/2r^2}^{0} f(\varepsilon, m^2)\left(\varepsilon-\Psi-\frac{m^2}{2r^2}\right)^{-1/2} d\varepsilon\, dm^2. \tag{70}$$

In the following we shall consider the case of ellipsoidal compression that does not differ strongly from spherically symmetric compression [see (68)]. As the adiabatic invariants in such a system it is natural to choose the squared angular momentum m^2 and the radial-motion invariant I, analogous to (33):

$$I=2^{1/2}\int\limits_{r_{min}(\varepsilon)}^{r_{max}(\varepsilon)} \left[\varepsilon-\Psi(r_1)-\frac{m^2}{2r_1^2}\right]^{1/2} dr_1,$$

$$\varepsilon-\Psi(r_m)-\frac{m^2}{2r_m^2}=0. \tag{71}$$

The latter relation in (71) determines the turning points r_{min} (ε) and $r_{max}(\varepsilon)$.

The distribution function f near the singularity in the approximation of fast adiabatic mixing depends only on the invariants I and M^2. The concrete form of $f(I, m^2)$ is then determined from the boundary conditions specified in the region of the hydrodynamic-compression caustic. We shall consider a matter element $d\tilde{M}$ near the caustic surface r_0. In our approximation, when this element intersects the caustic surface it is rapidly redistributed and mixed in the many-flux zone in accordance with the conservation of the adiabatic invariant I and with the transformation of the field part $l_2(r_0)$ of the squared angular momentum into the squared angular momentum of the matter, as described by (66), (68). (Before the caustic surface, as follows directly from the solution of (25), (26), the squared angular momentum of the matter remains equal to zero.) Thus, near the caustic surface r_0, in accordance with (35), (68), the boundary conditions have the form

$$d\tilde{M}=\rho_h(r_0)dr_0=f(I)dI,\ f(I, m^2)=f(I)\delta(m^2-l_2(r_0)). \tag{72}$$

Here I is the adiabatic invariant (71) on the caustic surface r_0, when

$$\varepsilon=\varepsilon(r_0)={}^1/_2V_0^2(r_0)+\Psi(r_0)+m^2/2r_0^2={}^9/_7\Psi_0r_0^{2/7}+m^2/2r_0^2, \tag{73}$$

where $\rho_h(r_0)$, $V_0(r_0)$, and $\Psi(r_0)$ are the density, velocity, and gravitational-field potential in the hydrodynamic solution (20), (23) at the point r_0; it has been taken into account that $V_0^2=(4/7)\Psi_0r_0^{2/7}$ (20).

We now substitute the distribution function (72) into the expression for the effective density (70). The integration over the angular momenta dm^2 here is performed immediately. In addition, it is convenient to integrate over dr_0 instead of $d\varepsilon$. We have

$$d\varepsilon=(d\varepsilon/dr_0)_I dr_0, \tag{74}$$

where the conservation of the adiabatic invariant I is taken into account by the fact that the quantity $(d\varepsilon/dr_0)_I$ is calculated at constant I:

$$\left(\frac{d\varepsilon}{dr_0}\right)_I=\frac{dI}{dr_0}\Big/\frac{dI}{d\varepsilon}. \tag{75}$$

Substituting (74), (75) into (70), we find, with allowance for (72),

$$\tilde{\rho}(r)=2^{1/2}\int\limits_{r_{min}(r)}^{r_{max}(r)} \rho_h(r_0)\left(\frac{dI}{d\varepsilon}\right)_{r_0}^{-1}\left[\varepsilon(r_0)-\Psi(r_0)-\frac{l_2(r_0)}{2r^2}\right]^{-1/2} dr_0. \tag{76}$$

As is clear from (71), (73),

$$\left(\frac{dI}{d\varepsilon}\right)_{r_0}=2^{1/2}\int\limits_{r_{min}(\varepsilon)}^{r_{max}(\varepsilon)}\left[\varepsilon-\Psi_0r_1^{2/7}-\frac{l_2(r_0)}{2r_1^2}\right]^{-1/2} dr_1\approx 7\cdot 2^{1/2}\frac{\varepsilon^3}{\Psi_0^{7/2}}B_1,$$

$$B_1=\int\limits_0^1\frac{x^6\,dx}{(1-x^2)^{1/2}}=\frac{5\pi}{32}. \tag{77}$$

In the integration it has been taken into account that the main contribution to the integral is made by the region in which the hydrodynamic asymptotic form for the potential is valid (see below); in addition, in the last term in the integrand, $l_2(r_0)\propto r_0^4$ [see (68)], and for sufficiently small r_0 this term is unimportant.

Integrating now in (76), we find, finally,

$$\tilde{\rho}(r)=\tilde{\rho}_{10}r^{2/13},$$

$$\rho_{10}=2^{1/13}(7/9)^{19/26}\tilde{\rho}_0B_2\Psi_0^{1/13}/13B_1m_0^{2/13},$$

$$B_2=\int\limits_0^1 z^{-11/13}(1-z^2)^{-1/2}dz={}^1/_2\pi^{1/2}\Gamma({}^1/_{13})/\Gamma({}^{15}/_{26}). \tag{78}$$

Here we have taken into account the expression (68) for l_2 (r_0), and $\tilde{\rho}_h(r_0)=\rho_0r^{2/7}$ [the quantity $\tilde{\rho}_0$ is defined as in (20)]. Correspondingly, the total density

$$\rho=\rho_{10}r^{-24/13},\ \rho_{10}=\tilde{\rho}_{10}/4\pi G. \tag{79}$$

The potential and the modulus of the mean velocity are equal to

$$\Psi(r)=-\Psi_0+\Psi_1r^{2/13},\ \Psi_1={}^{338}/_{15}\pi G\rho_{10},\ \Psi_0\approx\Psi_1a^{2/13}.$$

$$\bar{V}=\overline{(V^2)}^{1/2}=V_0r^{1/13},\ V_0=({}^4/_{13}\Psi_1)^{1/2}, \tag{80}$$

where a is the initial size of the matter density peak. From this it can be seen that the field potential at the center of the NGS remains finite for all values of $m_0^2\neq 0$, and the dependence of ρ and Ψ on r near the singularity is a power-law dependence. However, as the spherically symmetric initial condition is approached ($m_0^2\to 0$) the depth of the potential well increases, and becomes infinite in the spherical case ($m_0^2=0$). The density ρ increases analogously, and for m_0^2 $=0$ loses its meaning. It is in this way that one passes to the limit of the distinct, strictly spherically symmetric case.

We note that, as can be seen from (76), the main contribution to the asymptotic form (80) is made by the region of values $r_0 \sim r^{13/24} \gg r$. Performing the calculation for the region $r_0 \gtrsim r_1 \gg r$ [a calculation entirely analogous to (76)–(78)], it is not difficult to convince oneself that the field potential Ψ and density ρ preserve here the dependence $r_1^{2/7}$, and, to within a constant, coincide with (20); this fact was used in (77).

To conclude this section we shall consider the important question of the stability of the spherically symmetric singularity that has formed—a singularity at which the field potential $\Psi(r) = \Psi_1(r/a)^{2/13}$, while the distribution function $f_0(\varepsilon, m^2)$ possesses the property (see the dependence of f_0 on m^2) that in the vicinity of the singularity at $r \ll r_0$ we have $\partial f_0/\partial V = V_r df_0/d\varepsilon$. For the perturbation δf of the distribution function in the linear approximation we have

$$\frac{\partial}{\partial t}\delta f+\mathbf{V}\nabla\delta f+\nabla\Psi\frac{\partial}{\partial\mathbf{V}}\delta f+\nabla\delta\Psi\frac{\partial}{\partial\mathbf{V}}f_0=0. \qquad (81)$$

We shall compare the third and fourth terms of Eq. (81) near the singularity $r \to 0$. Assuming that for $r \to 0$ the perturbation δf is an analytic function, we obtain from (81) the following estimates:

$$\nabla\Psi\frac{\partial}{\partial V}\delta f\sim\frac{2}{13}\frac{\tilde{\Psi}_1}{a}\left(\frac{r}{a}\right)^{-11/13}\frac{\delta\rho}{\Delta V},$$
$$\nabla\delta\Psi\frac{\partial}{\partial V}f_0\sim\frac{\delta\Psi}{\lambda}\frac{\rho_0}{\tilde{\Psi}_1^{1/2}}. \qquad (82)$$

Here ΔV is the characteristic width of the perturbation of the distribution function in velocity space, and λ is the characteristic size of the density perturbation.

Taking into account that, according to Poisson's equation, $(2/13)(1+2/13)\tilde{\Psi}_1 = a^2\rho_0$, and $\delta\Psi \sim \lambda^2\delta\rho$, we find from (82)

$$\left|\nabla\Psi\frac{\partial}{\partial\mathbf{V}}\delta f\right|\gg\left|\nabla\delta\Psi\frac{\partial}{\partial\mathbf{V}}f_0\right|. \qquad (83)$$

More precisely, the relation (83), according to the above estimates, takes the form

$$\frac{r}{a}\ll\left(\frac{13}{15}\frac{a}{\lambda}\frac{\tilde{\Psi}_1^{1/2}}{\Delta V}\right)^{13/11}. \qquad (84)$$

The inequality (84) is always fulfilled for $r \ll r_0$, since for perturbations localized inside the singularity we have $\Delta V < \tilde{\Psi}_1^{1/2}$ and $\lambda < a$.

Thus, near the singularity the last term in Eq. (81) is always small. Neglecting it, we rewrite (81) in the form

$$\frac{\partial}{\partial t}\delta f+\mathbf{V}\nabla\delta f+\nabla\Psi\frac{\partial}{\partial\mathbf{V}}\delta f=0.$$

This equation describes the motion of the captured particles in a specified potential $\Psi(r)$, and, as is well known,[12] does not have growing solutions. In time, the distribution function δf is merely smeared out over the phase volume.

Thus, the solution that we have obtained in the vicinity of the singularity is stable against small perturbations, at least over a period of time many times greater than the oscillation time.

6. THE HIERARCHICAL STRUCTURE OF THE DEVELOPED TURBULENCE

It was shown above that, because of the nonlinear development of the Jeans instability, initial density peaks of a nondissipative gravitating gas arrange themselves, in a time exceeding the Jeans time (1), into stationary and stable dynamical NGS structures. These are characterized by a well defined scaling law (79), (80) for the density distribution, velocity and field potential that depends only weakly on the initial conditions. However, the scale of each of these formations remains arbitrary—it is determined by the spatial dimension L of the initial density peak, or, equivalently, by the ratio of the squared mass of the density peak to the total energy: $L \sim GM^2/\mathscr{E}$ (56).

We now consider the case when the initial density of the gravitating gas has a random distribution containing a broad spectrum of fluctuations, which are assumed to be homogeneous and isotropic. The dynamics of the system is described as before by Eqs. (3) with the initial conditions (14), where, as usual, $\mathbf{V}_0(\mathbf{r}) = 0$ (60), and $\rho_0(\mathbf{r})$ is now a random function with a broad spectrum of scales L. We introduce the distribution function averaged over the scale L:

$$f_L(\mathbf{R},\mathbf{V},t)=\frac{1}{\pi^{3/2}L^3}\int f(\mathbf{R}+\boldsymbol{\xi},\mathbf{V},t)\exp(-\xi^2/L^2)\,d\boldsymbol{\xi}. \qquad (85)$$

Performing the corresponding averaging in Eqs. (28), we obtain

$$\frac{\partial f_L}{\partial t}+\mathbf{V}\frac{\partial}{\partial\mathbf{R}}f_L+\nabla\Psi\frac{\partial}{\partial\mathbf{V}}f_L+S_L=0,$$
$$S_L=\int\nabla(\delta\Psi)\frac{\partial}{\partial\mathbf{V}}f(\mathbf{R}+\boldsymbol{\xi})\,e^{-\xi^2/L^2}d\boldsymbol{\xi}, \qquad (86)$$
$$\Delta\Psi=\int f_L\,d\mathbf{V},\quad \delta\Psi=\Psi(\mathbf{R}+\boldsymbol{\xi})-\Psi(\mathbf{R}).$$

Below it will be shown that the correlation integral S_L is small, and can be neglected in the first approximation. In this approximation Eqs. (86) essentially coincide with the starting equations (28). The only difference is that the initial density distribution $\rho_0(r)$ is averaged:

$$\rho_{0L}(R)=\frac{1}{\pi^{3/2}L^3}\int\rho_0(\mathbf{R}+\boldsymbol{\xi})\exp(-\xi^2/L^2)\,d\boldsymbol{\xi}. \qquad (87)$$

This means that all the peaks of the initial density on scales smaller than L are smoothed out. Therefore, the solution of Eqs. (86) with the initial distribution (87) leads to the appearance of NGS that are only of scale L or larger.

We now consider a scale $L_1 \gg L$. The equations for f_{L_1} have the same form (86), but, by virtue of the averaging (87), the initial inhomogeneities can contain density maxima only with scales $\tilde{L} \gtrsim L_1$. The dynamical development of such systems will lead to the formation of stationary gravitational singularities with scale L_1, which contain singularities (describable by the function f of scale L) in the form of small elements trapped in the field of the NGS of scale L_1. The same also applies to the singularities of scale $L_2 \ll L$ captured in the field of the NGS with scale L and describable by the function f_{L_2}.

As a result, the complete solution is a hierarchical structure consisting of nested NGS of different scales, smaller-scale singularities being trapped in the field of a larger-scale NGS and moving along finite trajectories in this field. The potential of the gravitational field in this case is everywhere continuous, but the density of the gas becomes infinite at each singularity. The number of NGS of different scales that are formed depends essentially on the distribution of the initial fluctuations. At the same time, the scaling relations (79), (80) for the gas density, potential, and velocity do not depend on the initial fluctuations, and remain the same on all scales. Moreover, by virtue of the homogeneity and isotropy of the initial fluctuations the number density ρ_L of NGS of each scale is proportional to the total density of the gas: $\rho_L \sim \rho$. It follows from this that the number density of the smaller-scale NGS trapped in the field of a larger-scale NGS, like the gas density ρ, always obeys the scaling law (79):

$$\rho_L \propto \rho \propto r^{-\alpha}, \quad \alpha = {}^{24}/_{13} \approx 1.85. \tag{88}$$

We shall call such a hierarchical structure developed isotropic and homogeneous nondissipative gravitational turbulence. It develops simultaneously on all scales. However, the time τ of its development can be different for different scales—it is determined by the magnitude of the initial density ρ_{0L} on the given scale L: $\tau(L) > t_g(\rho_{0L})$. The maximum scale $L_{\max}$ of the turbulence is determined, generally speaking, by the size of the entire region occupied by the gravitating gas at the initial time. It can, however, be limited by the time t that has elapsed since the initial time, if the instability has not had time to develop on all scales by the given time t. In this case, the scale $L_{\max}$ is determined by the relation

$$t = \tau(\rho_{0L_{max}}) \approx k t_g(\rho_{0L_{max}}), \quad k \sim 3\text{—}5. \tag{89}$$

The minimum scale $L_{\min}$ is determined by the thermal motion in the gas: $L_{\min} \sim c_s t_g$, where c_s is the velocity of sound. Developed turbulence is established on scales

$$L_{min} \ll L \ll L_{max}. \tag{90}$$

We now estimate the magnitude of the correlation integral S_L (86). Taking into account the existence of NGS of scale L, their motion in the field of an NGS of larger scale $L_1 \gg L$, and the Coulomb-law interaction between the NGS, we find that

$$S_L \sim f/t_L, \quad t_L = p t_g,$$
$$p \sim (L_1/L)^3 (r/L_1)^{3/14} \Lambda^{-1}. \tag{91}$$

Here Λ is the Coulomb logarithm. It can be seen that the parameter p is always large: $p \sim 10^2$–10^5 for $L_1/L \sim 10$–10^2. Therefore, the correlation integral S_L in Eq. (86) is indeed small, and in the first approximation it can be neglected. This is valid only over a limited time $t \ll t_L$. For $t \gtrsim t_L$ the interaction S_L between NGS of the given scale becomes important, and this should lead to distortion of the distribution (79), (80). The first distortions, as can be seen from (91), should appear in the region of the singularity.

We also recall that in the solutions considered above we completely neglected the possibility of the presence of dispersion of the initial velocities in the density peaks of the gravitating gas [see (60)]. It should be expected that the presence of a small dispersion $V_0^2 \ll GM/L$ will not change the general structure of the steady-state NGS, but can influence the magnitude of the scaling parameter α [see (79), (88)].

In conclusion we note that the above picture of turbulence in an ideal gravitating medium has, apparently, a direct relation to the actually observed structure of the distribution of matter in the Universe. Indeed, the condition (89) determines a maximum scale $L_{\max}$, which turns out to be

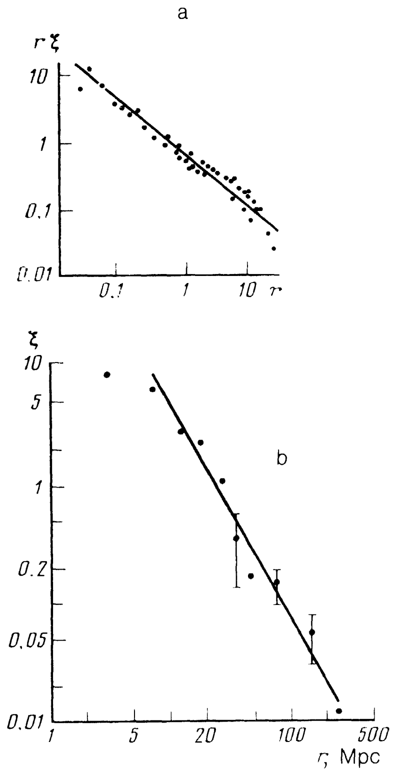

FIG. 6. The pair correlation function $\xi(\mathbf{r})$: a) for the distribution of galaxies in clusters[18]; b) for the distribution of clusters of galaxies in Abell clusters.[19] The solid curve corresponds to the theory (88).

equal to $L_{\max} \sim 20$–40 Mpc; the minimum scale $L_{\min} \sim 1$–10 pc, on the other hand, corresponds to the average distance between stars. It is on these scales 10 pc $< L <$ 10 Mpc that one observes the hierarchical pattern of the "crowding" of matter[1,2] (galaxy clusters, galaxies, and spherical star clusters inside galaxies)—a pattern which corresponds fully to the above-described hierarchical structure of gravitational turbulence.

A specially important point is that the density distribution of smaller-scale objects captured by larger-scale objects that is observed in these structures can be characterized by a single law:

$$\rho_L = C r^{-\alpha}. \tag{92}$$

For the distribution of galaxies in clusters we have the value $\alpha \cong 1.8$ (Refs. 1, 2), while for the distribution of spherical clusters in galaxies we have $\alpha = 1.7$–2.4 (Ref. 17).[3]

As an example, in Fig. 6 we show the pair correlation functions $\xi(r)$ for the distribution of galaxies in clusters[18] and for the distribution of clusters of galaxies in rich Abell clusters of class D-4 (Ref. 19). In the hierarchical structure the functions $\xi(\mathbf{r}) \propto \rho(\mathbf{r})$ (Ref. 2), and therefore these correlation functions are compared with the theoretical formula (88). Their good agreement with the theory is evident. We note that in the previously developed correlation theory of gravitational turbulence[2] the distribution law (92) for galaxies in clusters was related uniquely to the spectrum of the initial fluctuations. According to the results obtained here, the law (79), (88) is the same for all structures of nondissipative turbulence [over the limited time (91)] and does not depend on the spectrum of the initial fluctuations. The reason for the discrepancy is that the correlation theory always assumed the presence of random phases of the Four-

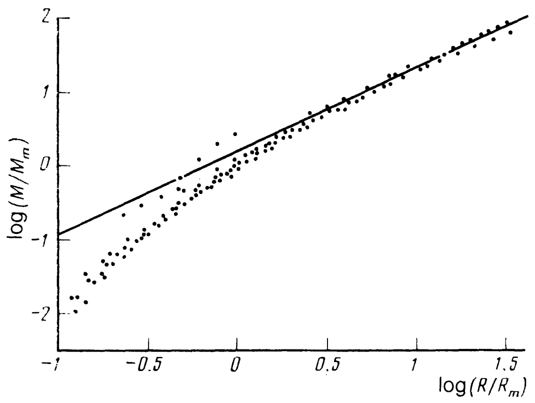

FIG. 7. The distribution of dynamical mass M in the vicinity of spiral galaxies; R_m and M_m are normalization constants.[20] The solid curve corresponds to the theory (88).

ier components, which is essentially equivalent to the assumption of the absence of dense dynamical formations (see Ref. 2, Sec. 57). The basis of the present theory, on the other hand, includes the fact of the appearance of dense stationary dynamical NGS formations, which have a decisive influence on the structure of the turbulence.

Furthermore, observations of rotations curves make it possible to establish the distribution of dynamical mass on the periphery of galaxies and thereby to determine the presence of hidden mass. The density of dynamical mass in the vicinity of spiral galaxies is found to be distributed in accordance with the law (92) with $\alpha \cong 1.8$–2.0 (Ref. 20), while in the vicinity of elliptical galaxies it is distributed by the same law with $\alpha \cong 1.5$–2.0 (Refs. 21, 17). As an example, observational data are compared with the formula (88) in Fig. 7, and their fairly good agreement is evident. The law (92) with $\alpha \cong 1.7$–2.0 also characterizes the distribution of matter density in the central part of a galaxy.[22,23]

The existence of a single power law (92) both for the matter-density distribution and for the distribution of trapped objects in the hierarchical structures, and the rather good agreement of the observed magnitude of the scaling parameter α with its theoretical value (88), (79), are evidence that the structure of the Universe on the scales under consideration may be developed gravitational turbulence.

It should also be noted that, starting from the scaling law (79) obtained, one can show that under gravitational compression of a nondissipative gas with a distribution of initial perturbations of the general form (23) black holes are not formed. This question, however, requires a more detailed treatment with allowance for relativistic effects.

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[1)]Initial states close to equilibrium lead to weak gravitational turbulence with $\delta\rho \ll \rho_0$, which has been considered, e.g., in Ref. 3. The linear theory of the Jeans instability was developed in the well-known paper of Lifshitz.[4]

[2)]In a uniformly expanding Universe the relative velocities within small volumes of matter that was previously in a state of thermodynamic equilibrium with radiation are small.[1,2]

[3)]We have in mind spherical clusters in elliptical galaxies and in the central part of spiral galaxies (in the bulge).

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