Theory of the stability of shock waves; physical meaning of steady-state solutions for a perturbed shock front with weak outgoing waves

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(Submitted 4 October 1986)
Zh. Eksp. Teor. Fiz. **93**, 1986–1991 (December 1987)

The steady-state solutions of the linear theory of the stability of shock waves for small perturbations of a front, with weak waves moving away from the front (so-called sound generation), do not satisfy the causality principle for all possible orientations of these waves. This circumstance is manifested in particular in an instability of the solutions with respect to local perturbations. The existence of "resonances" does not produce enough feedback between the front and the flow behind the front to destabilize a plane shock front. Those results were derived without consideration of kinetic (dissipative) processes at the shock. The stability of the one-dimensional structure of a shock for which the parameter L has values in region III thus requires a special study.

The theory of the stability of shock waves[1–3] distinguishes among three regions of values of the parameter $L \equiv J^2(\partial v/\partial p)_H$ [J is the mass flux across the front of the shock wave, and $(\partial v/\partial p)_H$ is the derivative of the specific volume v with respect to the pressure along the shock adiabat], in which qualitatively different solutions are found. These regions are defined by the inequalities

$$\text{I. } -1<L<L_0,\quad L_0\equiv(1-M^2-\theta M^2)(1-M^2+\theta M^2)^{-1},$$
$$\text{II. } L<-1,\ \text{ or }\ L>1+2M,$$
$$\text{III. } L_0<L<1+2M,$$

where M is the Mach number of the shock wave with respect to the flow behind it, and θ is the degree of compression in the shock wave.

In case I, small perturbations of the surface of the shock front decay. In case II, according to linear stability theory, perturbations of the surface of the wavefront grow exponentially. In region II, however, the shock can always be decomposed into other (stable) elements in the manner of an arbitrary shock, and it apparently decays quickly (see Ref. 4 and the bibliography there). In the case of a decay of this sort, the linear analysis does not reflect the actual evolution of a perturbation even in the early stage, since that analysis rests on an unperturbed basis which does not exist.[4]

Under the inequalities III, the linear analysis yields steady-state solutions (steady-state in a moving coordinate system) which correspond to a rippled shock front from which sound waves are propagating (so-called sound generation by a shock wave[1]). The behavior of perturbations in the pressure and other properties as functions of the coordinates and the time in such solutions is determined by a factor

$$\exp[i(kx+ly-\omega t)] \tag{1}$$

with real values of k, l, and ω. (The x and y axes run along the shock front and normal to it, respectively.) From the standpoint of the theory for the reflection sound by a shock wave, these solutions correspond to an infinite reflection coefficient (a "resonance" in the terminology of Refs. 5 and 6). Corresponding to each value of the parameter L is a definite orientation of the sound waves, characterized by the angle $\gamma = \operatorname{arctg}(k_x/k_y)$ (this is the angle between the outward normals to the shock front and the front of an outgoing sound wave). In region III there is a one-to-one relationship between L and γ (see the equations given in Refs. 4 and 7). As L is varied from L_0 to $1+2M$, the angle γ correspondingly changes from $\gamma_0 \equiv \arccos M$ to π, spanning the entire range of values of γ for outgoing sound waves.

The stability of a shock wave in region III and the question of the physical meaning of the solutions (1) with outgoing waves are studied in Refs. 7 and 8. It was shown[7] that as the pressure amplitude (p_f) of the incident wave approaches zero for an angle of incidence near the resonant angle the pressure of the reflected wave, p_r, also approaches zero (in proportion to $p_f^{1/2}$). Small perturbations of the flow behind the front, including fluctuating (noisy) perturbations, thus do not destabilize the shock wave in region III when reflected once (we will have more to say about multiple reflections).

It was shown in Ref. 8 that the solutions (1) and their analog—three-wave configurations (an unperturbed shock wave, a perturbed shock wave, and a weak outgoing compression or rarefaction wave; Fig. 1)—do not satisfy the causality principle at angles

$$\pi/2<\gamma<\pi. \tag{2}$$

We can show that the solutions (1) and the corresponding solutions for three-wave configurations also fail to satisfy the causality principle in the region

$$\gamma_0<\gamma<\pi/2. \tag{3}$$

This region, combined with (2), spans the entire interval of

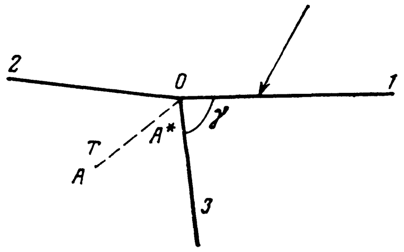

FIG. 1. Three-wave configuration with a weak outgoing wave 3 ($\gamma<\gamma_0$). 1—Unperturbed shock wave; 2—perturbed shock wave; T—tangential shock; A, A^* indicate sectors; arrow—direction of the streamlines ahead of the shock front in the coordinate system with a fixed point 0.

 0038-5646/87/121132-04$04.00

angles ($\gamma_0<\gamma<\pi$) in which waves 3 (Fig. 1) are outgoing.

Formal proof follows immediately from the fact that according to (1) and the corresponding solutions for three-wave configurations[4] an acoustic signal would be able to propagate along the shock front at a velocity

$$V_t=c(1-M\cos\gamma)/\sin\gamma, \quad (4)$$

which would satisfy the inequalities

$$c(1-M^2)^{1/2}<V_t<c$$

for angles (3); here c is the velocity of sound behind the shock front. However, we know that the sound propagation velocity along the surface of a shock front is $c(1-M^2)^{1/2}$. Consequently, even for angles (3) a weak perturbation (a signal) would actually propagate at a supersonic velocity according to (1), in contradiction of the laws of linear gasdynamics.

More specifically, the violation of the causality principle in the case of angles (3) is seen in the circumstance that although the normal drawn to front 3 through point 0 in Fig. 1 does fall in sector A, in contrast with the situation in case (2) (Ref. 8), it does not pass through the region of the flow behind the front for front 3: the region which lies between front 3 and the contact discontinuity (sector A * in Fig. 1). Perturbations from sector A * do not reach point 0, so the motion of this point and the state of the matter at it are not causally related to the parameters of the flow in sector A *. The normal to the perturbed shock front 2 drawn through point 0 does not pass through the region of the flow behind the front for shock 2, regardless of the orientation of front 3. In other words, it does not pass through the region which lies between front 2 and the tangential discontinuity. The reason is that wave 2 is always outgoing. It arises from the effect of the external perturbation source on shock 1.

Some phenomena consistent with causality associated with the solutions (1) were pointed out in Ref. 8 for angles (3). These effects are flows which arise from the interaction of the shock wave with (a) sound waves arriving from the region behind the front (or ahead of it) with a pressure amplitude which is quadratic in the amplitude of wave 3 and (b) an infinitely thin piston (or a system of such pistons) which pushes against the point 0 (Fig. 1) and slides along the tangential discontinuity.

There is no mechanism for the effect in the linear approximation (1), but in these cases [(a) and (b)] such a mechanism does indeed exist. If the mechanism vanishes, then its consequences—the flow described by (1) or by the analogous solution for a three-wave configuration—also vanishes.[8]

All these phenomena that are consistent with causality corresponding to (1), apply equally to angles (3) also; i.e., they are pertinent to the entire angular interval $\gamma_0<\gamma<\pi$ in which weak waves are outgoing.

Since the order of magnitude of the small perturbations decreases when they are reflected near a resonance point, however, it is also legitimate to pose the question of the stability of a shock wave in the following way: As a small perturbation—either a random perturbation or one deliberately produced by some one-shot external agent—of the flow behind the front (with a pressure p_f) overtakes the shock front at an angle close to the resonant angle, it is reflected from it in the form of acoustic and entropy waves whose small values are of a lower order ($\sim p_f^{1/2}$; Ref. 7). The interaction of these waves generates an incoming wave which is a small quantity of higher order ($\sim p_f$). As this wave is reflected, the order of the small quantities can again decrease to $p_f^{1/2}$; etc. Would a process of multiple reflections of this sort not be divergent in the perturbation amplitude? To answer this question we need to take a look at two different types of initial incoming perturbations.

1) The perturbation (with a pressure p_f) is a wave which is oriented at an angle so close to the resonant angle that the reflected acoustic wave and the reflected entropy wave are quantities of a lower order, $p_f^{1/2}$. The new incoming wave generated by the interaction of these waves is a small quantity of order p_f. However, this new wave is oriented in a direction far from the resonant direction. (The orientation is determined by the law of specular reflection of the sound wave from the nearly plane entropy inhomogeneity, and the coincidence of the angle γ and the resonant angle γ_L would be an unlikely chance event.) The reflection of such a wave from the front of the shock wave would not change the order of magnitude of the small perturbation, so this process of successive reflections of the perturbation would die out.

2) The initial perturbation is not a "single-angle" perturbation; it is an integral superposition of plane waves in various orientations (a Fourier integral over the wave vector **k**). The pressure amplitude of partial waves with wave vectors whose magnitudes lie in the infinitesimal interval dk is proportional to dk in this case (i.e., it is also proportional to the differential of the angle, $d\gamma$). It is thus infinitely small. Everywhere except at the one resonant point, $\gamma=\gamma_L$, the amplitude of the wave reflected from the shock front will be a small quantity of the same order as the incident wave, and it can be written in the form[7]

$$dp_r=\frac{-\psi(\gamma_r)+[\psi^2(\gamma_r)-4\psi(\gamma_f)a\,dp_f]^{1/2}}{2a}=-\frac{\psi(\gamma_f)}{\psi(\gamma_r)}dp_f, \quad (5)$$

where γ_f and γ_r are the values of the angle γ for the incident and reflected waves, respectively, and the value of the coefficient a is of no importance to the discussion below. We refer the reader to Ref. 7 regarding the behavior of ψ as a function of γ and the parameters characterizing the thermodynamic properties of the medium and the shock adiabat. The important point regarding this function for the discussion below is that it vanishes only at the resonant reflection angle: $\gamma_r=\gamma_{r,L}$. Near the point $\gamma_{r,L}$ we have $\psi(\gamma_r)=\mathrm{const}(\gamma_r-\gamma_{r,L})$. The pole $\gamma=\gamma_{r,L}$ in (5) does not cause the total pressure p_r, of the reflected wave, to diverge because of their interference. Accordingly, in calculating p_r we should understand the integral in its principal-value sense:

$$p_r=-\int_0^{\gamma_0}\frac{\psi(\gamma_f)}{\psi(\gamma_r)}\frac{dp_f}{d\gamma_f}d\gamma_f.$$

Consequently, again in case (2) a perturbation reflected from the shock front remains a small quantity of the same order, so the process of multiple reflections of the perturbations dies out.

According to Ref. 9, incoming waves generated by nonlinear interaction of outgoing waves give rise to infinite amplification of perturbations of a shock front if the shock adiabat from region III has a certain shape.[1)]

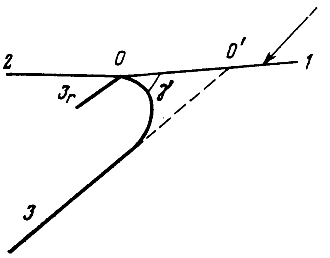

FIG. 2. Perturbed three-wave configuration with a weak outgoing wave 3. 1—Unperturbed shock wave; 2—perturbed shock wave; 0′—point at which the front 1 intersects the continuation of the unperturbed rectilinear section of front 3; 3_r—weak reflected wave; γ—the angle formed by line 1 and the tangent to line 3 at point 0; arrow—direction of the streamlines ahead of the shock front in the coordinate system with fixed point 0′.

That result, however, refers to waves of finite amplitude incident at the resonant angle and all of the same harmonic (it does not refer to a fluctuational "white noise"). Nonlinear analysis[7] shows that the picture of the interaction of weak waves of finite amplitude which are incident at the resonant angle and which are of alternating sign in terms of p_f, on the one hand, with a shock front, on the other, does not reduce to a simple four-wave configuration (the incident and reflected weak waves and the unperturbed and perturbed shock waves).

These arguments essentially complete the proof that spontaneous steady-state perturbations (i.e., perturbations not driven by an external agent) of a shock front, with sound waves of the type (1) propagating away from it, cannot exist anywhere in the entire angular interval $\gamma_0 < \gamma < \pi$.

It is interesting, however, to examine the problem of the existence of the solutions (1) or of the equivalent solutions for three-wave configurations (Fig. 1), again from the standpoint of the stability of such solutions. We see that the results of such an analysis are fundamentally different in cases III and I, i.e., in the cases of outgoing ($\gamma_0 < \gamma < \pi$) and incoming ($0 < \gamma < \gamma_0$) sound waves. It is convenient to carry the analysis out in the terminology of three-wave configurations.

We consider a three-wave configuration with an outgoing wave 3, which is perturbed in a small neighborhood of the point at which the fronts intersect, in such a way that front 3 takes the form of a convex curve (Fig. 2), while wave 3 would become an incoming wave near the point at which the fronts intersect. According to (4), the velocity V_t at which the point 0′ moves along the surface of the shock front is a minimum in the case $\gamma = \gamma_0$. This minimum value is

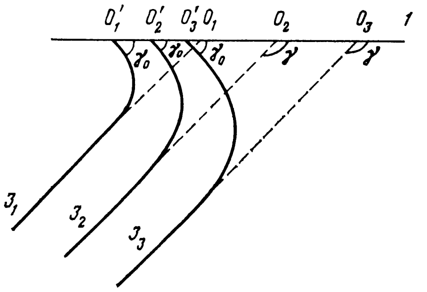

FIG. 3. Position of front 3 at the times $t_1, t_2 = t_1 + \Delta t$, and $t_3 = t_1 + 2\Delta t$. The points 0 and 0′ have the same meaning as in Fig. 2. The subscripts 1–3 correspond to the times t_1, t_2, and t_3. 1—Unperturbed shock wave. Deformations of the shock front are not shown.

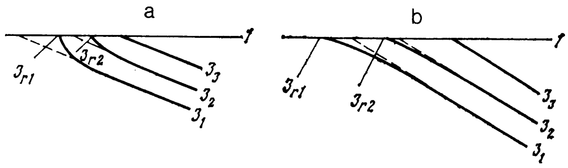

FIG. 4 Damping of various perturbations (a and b) in region I of values of the parameter L (the notation is explained in Figs. 2 and 3).

$c(1 - M^2)^{1/2}$. The angle γ_0 is the boundary between incoming and outgoing waves. If the unperturbed three-wave configuration is to be reconstructed, point 0 must overtake point 0′ (Fig. 2), and the angle γ must pass through the value γ_0. Front 3 propagates through the medium at the sound velocity c, and this front is convex according to the Huygens principle. An element of front 3 approaches the shock front more slowly, the larger the value of the angle γ_s for this element (γ_s is the angle formed by the outward normals to the elements of front 3 and to the shock front). The angle γ thus increases over time, approaching γ_0. However, γ cannot become larger than γ_0, since those elements of the front of wave 3 which are oriented at angle $\gamma_s > \gamma_0$ do not overtake the shock front (for $\gamma_s > \gamma_0$, the elements of the front wave 3 are outgoing). Figure 3 gives a qualitative idea of the subsequent evolution of front 3, for which the relation $\gamma \approx \gamma_0$ holds.

The initial perturbation in the case of a three-wave configuration with an outgoing wave 3 (case III for the parameter L) thus occupies a progressively larger region as time elapses, and the orientation of the fronts deviates more and more from the unperturbed orientation.

It is important to note that the intensity of the perturbation does not increase without bound over time in this case. If an initial rectilinear front 3 and, correspondingly, a rectilinear segment of front 2 are of bounded size, then the rarefaction waves which propagate out of the region of the unperturbed flow behind the front (from left to right in Fig. 3) lead to complete damping of wave 3—the perturbation of the original shock wave—as time elapses.

In the case of a three-wave configuration with an incoming wave 3 (case I for the parameter L), it is simple to show (by arguments analogous to those presented above) that a perturbation of the initial configuration which is localized near the front intersection point evolves in time in such a way that the configuration of waves with large rectilinear sections in their fronts approaches the original configuration. Figure 4 shows versions of these perturbations and their qualitative development.

These results of a qualitative analysis of the stability of three-wave configurations, i.e., of solutions of type (1), also agree completely with the conclusions reached above on the basis of the causality principle regarding the conditions for the existence of steady-state perturbations of a shock front.

However, the stability of the shock waves was discussed above without consideration of the structure of the front (without considering dissipative processes at it). The stability of a one-dimensional structure of a shock in region III of the values of the parameter L requires further study.

We can thus draw the following conclusions.

1. Solutions describing steady-state perturbations of a shock wave with outgoing weak (sound) waves do not satisfy the causality principle for all possible orientations of such

waves ($\gamma_0 < \gamma < \pi$). They thus do not correspond to physical reality. This circumstance is manifested, in particular, in the instability of these solutions with respect to local perturbations.

2. These solutions acquire a real meaning only if the perturbations moving away from the shock front are generated by an external source: by weak waves coming in at an angle close to the resonant angle from the side of the flow behind the front or ahead of the front (or generated by some sort of "piston").[8] The pressure amplitude of such incoming waves is quadratic in the pressure of the reflected waves, p_r, and the cause of the effect thus remains "out of the picture" in the approximation linear in p_r.

3. The presence of "resonances" does not give rise to feedback between the front and the flow behind the front strong enough for the onset of an instability of a plane shock front.

4. The results presented above were derived without consideration of the structure of the shock front (without consideration of the dissipative processes in it).

[1)] It was assumed in Ref. 9 that the original perturbation is an outgoing wave. The autonomous motion of such a wave along a shock front does not satisfy the causality principle. A physically well-posed formulation of the question requires the specification of the original perturbation in the form of incoming waves. From the mathematical standpoint, however, in a study of the nature of the feedback between the shock front and the flow behind it, the particular direction of the initial perturbation (toward the front or away from it) is of no fundamental importance.

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Translated by Dave Parsons