# Change in domain-wall energy in a simple model of a structural transition

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A completely integrable system of Landau-Lifshitz equations is used to analyze a model of orientational phase transitions. The energy of the domain walls between angular phases is derived as a function of the anisotropy energy, a structural parameter. One reason for the change in the domain structure between angular phases may be an interchange bifurcation of a separatrix contour.

The purpose of the present paper is to use a simple model to show that representations found previously[2] for the anisotropy energy in the form of functions which depend on an arbitrary number of structural parameters and which lead to steady-state Landau-Lifshitz equations that are completely integrable in the spatially one-dimensional case can be used to study several questions in the theory of magnetic orientational phase transitions.[1]

As an orientational phase transition takes place, a change occurs in the number of magnetic phases; e.g., angular phases are created or disappear.[1] From the standpoint of the dynamic system of Landau-Lifshitz equations, such a transition involves bifurcations of the number and type of the singular points which correspond to spatially uniform solutions (magnetic phases). It also involves a change in the structure of the separatrix solutions corresponding to the various interfacial domain walls. Examples of such bifurcations for various types of domain structures in magnetically ordered media are given in Ref. 1.

In this paper we will use a simple example to show that the situation arising when the point of an orientation phase transition is crossed is of such a nature that two distinct types of domain walls correspond to a selected pair of angular phases. The two types of domain walls correspond to two different separatrix solutions of the Landau-Lifshitz equations. One of the possible domain walls corresponds to a rotation of the magnetic moment along one of the angular variables, while the other wall corresponds to a rotation in two angular variables. We previously noted[2] the possibility of analyzing this situation in detail, but did not carry out the analysis. In particular, we did not study how the domain-wall energy depends on the structural parameter during phase transitions.

It is clear that a detailed analytic study of the situation would be possible only when the corresponding Landau-Lifshitz equations are completely integrable, since in deciding among possible alternatives one needs not only an explicit expression for the corresponding separatrix solutions but also the explicit dependence of the energies of the domain walls between angular phases on the anisotropy energy, a structural parameter. For the simple model adopted here, the explicit solutions derived show that the domain-wall energy which corresponds to the case of two angular degrees of freedom is smaller than the energy of a domain wall with a single angular degree of freedom, between the same angular phases.

The class of multiparameter potentials of the anisotropy energy which was found previously[2] can thus serve as a foundation for constructing models of orientational phase transitions for an explicit study of the structural changes in walls between phases and of the behavior of the energy and its derivatives as functions of the structural parameters of the medium.

**1.** We recall[2] that in the spatially one-dimensional case the steady-state Landau-Lifshitz equations

$$\left[\mathbf{m}, \frac{d^2\mathbf{m}}{dx^2} - \frac{\partial U}{\partial \mathbf{m}}\right] = 0, \quad m^2 = 1 \tag{1.1}$$

are completely integrable for an isotropy energy of the type

$$U(\mathbf{m}, \mathbf{C}) = \frac{1}{2}(u_2 - u_1)^{-1} \sum_{n\geq 1} C_n (u_2^{n+1} - u_1^{n+1}). \tag{1.2}$$

Here the $C_n$ are arbitrary constants, while $u_1$ and $u_2$ are spheroconical variables on the surface of a unit sphere, which are related to the spherical variables $\theta$, $\varphi$ by[3]

$$\begin{gathered}\sin^2\theta\cos^2\varphi = (1+\varepsilon+u_1)(1+\varepsilon+u_2)/\varepsilon(1+\varepsilon),\\ \sin^2\theta\sin^2\varphi = -(1+u_1)(1+u_2)/\varepsilon,\\ \cos^2\theta = u_1u_2/(1+\varepsilon), \qquad -1-\varepsilon < u_1 < -1 < u_2 < 0\end{gathered} \tag{1.3}$$

($\varepsilon > 0$ is a parameter of the spheroconical coordinate system). Since Eqs. (1.1) are completely integrable in the case of an anisotropy energy of the type in (1.2), it is possible to completely classify all types of domain walls and to carry out a detailed study of all of their possible structural changes which would occur as the parameters $C_1$, $C_2$, $C_3$, ... pass through the bifurcation values.

For the case in which the energy of a multiaxial anisotropy is of the form

$$U(\mathbf{m}, C_2) = {}^1/_2(u_1+u_2) + {}^1/_2 C_2(u_1^2 + u_1u_2 + u_2^2), \tag{1.4}$$

an orientational phase transition corresponds to a bifurcation of equilibrium states $\theta = 0$ and $\theta = \pi$. This bifurcation occurs when the structural parameter $C_2$ crosses the value

$$C_{\text{ph}} = 1/(3+2\varepsilon). \tag{1.5}$$

Four angular phases are created:

$$\begin{gathered}M_1(\theta_{\text{ph}}, \varphi=0), \quad M_2(\pi-\theta_{\text{ph}}, \varphi=0),\\ M_3(\pi-\theta_{\text{ph}}, \varphi=\pi), \quad M_4(\theta_{\text{ph}}, \varphi=\pi),\end{gathered} \tag{1.6}$$

where $\theta_{\text{ph}}$ is defined by[2]

$$\sin^2\theta_{\text{ph}} = \frac{C_2 - C_{\text{ph}}}{2(1+\varepsilon)C_2C_{\text{ph}}} = \frac{(3+2\varepsilon)C_2 - 1}{2(1+\varepsilon)C_2}. \tag{1.7}$$

From the standpoint of a dynamic system, the angular phases (1.6) correspond to four saddle-saddle singular

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points, which exist under the condition

$$C_{\rm ph}<C_2<1, \tag{1.8}$$

and the possible types of domain walls between the angular phases correspond to a separatrix contour of singular points (1.6). It was mentioned in Ref. 2 that when the structural parameter passes through the value $C_2 = 1/3$ a structural change occurs in the separatrix contour. We turn now to a description of this structural change.

Over the entire range (1.8) of parameter values there are separatrices which correspond to domain walls with a single angular degree of freedom ($\varphi = 0$ or $\varphi = \pi$) and which relate singular points of the following types: $\mathbf{M}_1\leftrightarrow\mathbf{M}_2$, $\mathbf{M}_2\leftrightarrow\mathbf{M}_3$, $\mathbf{M}_3\leftrightarrow\mathbf{M}_4$ and $\mathbf{M}_4\leftrightarrow\mathbf{M}_1$. In addition, there are two other pairs of separatrices, which correspond to domain walls with two angular degrees of freedom, $\theta$, $\varphi$, and which correspond to motion of the imaging point over a unit sphere along one of the contour lines $u_i(\theta,\varphi) = \text{const}$ of the spheroconical coordinate system. Specifically these two pairs of separatrices undergo changes in the region of parameter values in (1.8). For

$$C_{\rm ph}<C_2<{}^1/_3 \tag{1.9}$$

these two pairs relate the singular points $\mathbf{M}_1\rightleftarrows\mathbf{M}_2$ and $\mathbf{M}_3\rightleftarrows\mathbf{M}_4$, while at

$${}^1/_3<C_2<1 \tag{1.10}$$

they relate the singular points $\mathbf{M}_1\rightleftarrows\mathbf{M}_4$ and $\mathbf{M}_2\rightleftarrows\mathbf{M}_3$. Finally, at $C_2 = 1$, a bifurcation occurs and results in the disappearance of the angular phases and interfacial walls discussed above.

However, it was found in Ref. 2 that two separatrix paths exist in the range (1.8) of values of the structural parameter $C_2$. These two paths couple the same pair of singular points. The existence of these paths implies that there are two structurally different types of domain walls between the same pair of angular phases, so that the energies of these walls must be compared.

**2.** According to Landau-Lifshitz equations (1.1), the energy of a domain wall is

$$E=\int_{-\infty}^{\infty} dx\left[\frac{1}{2}\left(\frac{d\mathbf{m}}{dx},\ \frac{d\mathbf{m}}{dx}\right)+U(\mathbf{m})-U(\mathbf{M})\right]. \tag{2.1}$$

The last term in (2.1) arises because the energy of the domain wall is measured from the energy of spatially uniform states (magnetic phases).

For all values of the parameter $C_2$, in region (1.8), the energies of the domain walls with a single angular degree of freedom are given by

$$E[\mathbf{M}_1\leftrightarrow\mathbf{M}_2,\ \varphi=\text{const}]=C_2^{1/2}\{[-w(w+1+\varepsilon)]^{1/2} -{}^1/_2(2w+1+\varepsilon)\arccos[(2w+1+\varepsilon)/(1+\varepsilon)]\}, \tag{2.2}$$

$$E[\mathbf{M}_1\leftrightarrow\mathbf{M}_4,\ \varphi=\text{const}]=C_2^{1/2}\{[-w(w+1+\varepsilon)]^{1/2} +{}^1/_2(2w+1+\varepsilon)\arccos[-(2w+1+\varepsilon)/(1+\varepsilon)]\}, \tag{2.3}$$

where

$$w=-(1-C_2)/2C_2<0.$$

Furthermore, the energies for domain walls with two angular degrees of freedom, $\theta$, $\varphi$, are

$$E[\mathbf{M}_1\Leftrightarrow\mathbf{M}_2,\ u_1=\text{const}] =C_2^{1/2}\{\varepsilon^{1/2}-{}^1/_2(2w+1+\varepsilon)\arccos[(\varepsilon-1)/(\varepsilon+1)]\} \tag{2.4}$$

in the range of parameter values (1.9) and

$$E[\mathbf{M}_1\Leftrightarrow\mathbf{M}_4,\ u_2=\text{const}] =C_2^{1/2}\{\varepsilon^{1/2}+{}^1/_2(2w+1+\varepsilon)\arccos[(1-\varepsilon)/(1+\varepsilon)]\} \tag{2.5}$$

in the range (1.10). It can be shown that in regions (1.9) and (1.10) the following inequalities hold, respectively:

$$E[\mathbf{M}_1\leftrightarrow\mathbf{M}_2,\ \varphi=\text{const}]-E[\mathbf{M}_1\Leftrightarrow\mathbf{M}_2,\ u_1=\text{const}]\geqslant0,$$
$$E[\mathbf{M}_1\leftrightarrow\mathbf{M}_4,\ \varphi=\text{const}]-E[\mathbf{M}_1\Leftrightarrow\mathbf{M}_4,\ u_2=\text{const}]\geqslant0. \tag{2.6}$$

The energies of the corresponding domain walls in (2.6) are strictly equal only for the bifurcation value $C_2 = 1/3$. At this point, the first derivatives of the energies of the domain walls with respect to the parameter $C_2$ are also equal. However, we do not have to go above the second derivatives of the energy to find a difference. Figure 1 shows domain-wall energies (2.2)–(2.5) as functions of the parameter $C_2$ (curves 1–4, respectively). Of the pair of separatrix trajectories which link the same pair of singular points (angular phases), that which leads to the lower energy is the trajectory which corresponds to the domain wall with two degrees of freedom. By choosing an appropriate value for the parameter $\varepsilon$ we can bring all the energies of the domain walls into coincidence with $C_2 = 1/3$.

Near orientational phase transition points (i.e., those with $C_2 = C_{\rm ph}$ and $C_2 = 1$), the second derivatives of the energies of domain walls with a single angular degree of freedom with respect to the parameter $C_2$ have a singularity which is characteristic of the phenomenological theory of phase transitions:

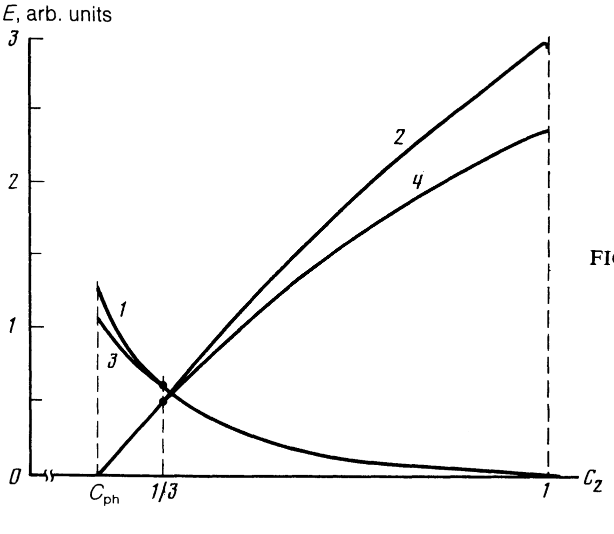


FIG. 1.

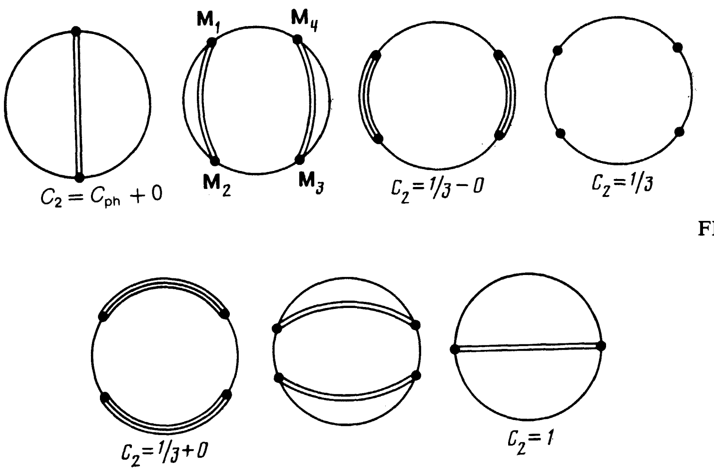


FIG. 2.

$$\frac{d^2}{dC_2^2} E[\mathbf{M}_1 \leftrightarrow \mathbf{M}_2, \varphi = \text{const}] \sim \frac{\text{const}}{(C_2 - C_{ph})^{1/2}}, C_2 \geqslant C_{ph}, \tag{2.7}$$

$$\frac{d^2}{dC_2^2} E[\mathbf{M}_1 \leftrightarrow \mathbf{M}_4, \varphi = \text{const}] \sim \frac{\text{const}}{(1-C_2)^{1/2}}, \quad C_2 \leqslant 1.$$

The corresponding derivatives of the energies of domain walls with two degrees of freedom, on the other hand, have no singularities.

While the bifurcations of singular points and of the separatrix contour which arise as the structural parameter $C_2$ crosses the values $C_2 = C_{ph}$ and $C_2 = 1$ correspond to the phenomenological model of orientational phase transitions, reflecting both the creation or annihilation of angular phases (on the one hand) and the classification of domain walls between angular phases (on the other), the bifurcation which arises as the value $C_2 = 1/3$ is crossed is more peculiar.

Specifically, as the parameter $C_2$ crosses this value no change occurs in the number or type of singular points. At the bifurcation point itself, the separatrix contour consists of separatrices characterized by a single angular degree of freedom.

The sequence of bifurcations is shown schematically in Fig. 2. The circle represents the great-circle intersection of the sphere with a plane passing through the poles; the points on the circle are singular points (uniform phases). Single lines between singular points correspond to separatrices which in turn give rise to domain walls with a single angular degree of freedom. The double lines in the cases $C_2 \neq C_{ph}$, 1, 1/3 correspond to pairs of separatrices which give rise to domain walls with two degrees of freedom. Four lines must converge at each singular point, provided that the condition $C_2 \neq 1/3$ holds. Comparing the separatrix contours for $C_2 = 1/3 \pm 0$, we reach the conclusion that a bifurcation at the point $C_2 = 1/3$ can be determined as an interchange bifurcation of a separatrix contour, since the bifurcation diagrams for $C_2 = 1/3 \pm 0$ correspond to the interchange of single and triple lines between singular points on the circle.

If we consider a potential

$$U(\mathbf{m}, \mathbf{C}) = {}^1/_2 C_1 (u_1 + u_2) + {}^1/_2 C_2 (u_1^2 + u_1 u_2 + u_2^2), \tag{2.8}$$

instead of the integrable potential (1.4), we can scale the energy of the domain walls in accordance with

$$E[C_1, C_2] = C_1^{1/2} E[1, C_2/C_1], \quad C_1 > 0, \tag{2.9}$$

where $E[1, C_2/C_1]$ is given by (2.2)–(2.5) when the substitution $C_2 \to C_2/C_1$ is made. The scaling laws (2.8) and (2.9) can be used to study a model of orientational phase transitions in any of the parameters ($C_1$ or $C_2$).

The anisotropy energy (2.8) can be written in the form

$$U = {}^1/_2 \alpha (\mathbf{m}, A_\varepsilon \mathbf{m})^2 + {}^1/_2 \beta (\mathbf{m}, A_\varepsilon \mathbf{m}) - {}^1/_2 \alpha (1+\varepsilon) m_z^2, \tag{2.10}$$

where $\mathbf{m} = (m_x, m_y, m_z)$, the diagonal matrix $A_\varepsilon$ is

$$A_\varepsilon = -\operatorname{diag}(1+\varepsilon, 1, 0),$$

and the new arbitrary constants $\alpha$ and $\beta$ are defined by

$$\alpha = C_2, \quad \beta = -C_1 + 2C_2(2+\varepsilon).$$

The case considered above corresponds to $C_1 > 0$ or

$$\beta < 2\alpha(2+\varepsilon).$$

The direct analogy between problems of the integrability of Landau-Lifshitz equations (1.1) and the equations of motion of a solid around a fixed point[4] suggests that it might be possible to find a complete description of the series of bifurcations of the analogous periodic motions of a solid.

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