Nonlinear wave solutions of electron MHD in a uniform plasma

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Two- and three-dimensional electron vortices are studied, in which the curl of the canonical electron momentum is frozen in an electron fluid moving through a stationary uniform ion background. Conditions are found under which such traveling vortices can exist, and families of analytical solutions for traveling helicon vortices are constructed in two and three dimensions. The existence of an infinite-dimensional set of stable generalized 2D vortices is proved for the case in which the magnetic field has three nonzero components but depends on just two Cartesian coordinates. Plane helicon waves of finite amplitude are shown to be exact solutions of the electron MHD equations. A purely electron instability of such waves is shown to exist for arbitrary amplitudes.

INTRODUCTION

According to current ideas, strong plasma turbulence is characterized by the appearance of weakly interacting stable structures—solitons, discontinuities (in the case of collapse), and vortices. The difficulties in the theory of vortex turbulence are associated with the essential multidimensionality of vortex motion and the variability of vortex structure.

In order to distinguish the qualitatively different types of behavior in two- and three-dimensional objects, we will use the term "vortices" for localized traveling solutions which transport trapped particles over long distances without permitting their "own" fluid to mix with the surrounding flow. The other type of localized propagating solutions we will call solitons. The latter can occur only in systems which are nonlinear and dispersive. For vortices dispersion does not play an essential role. This classification underscores the importance of vortices in plasma transport processes. For stationary localized solutions the difference between vortices and solitons is less important, although there is still a difference in the way material moves.

In the present paper we will discuss vortex motion in a cold electron fluid (thus neglecting finite-Larmor-radius effects), assuming that the background ions are uniform and stationary. Such effects are important in systems in which there is substantial electron drift motion across the magnetic field (Hall effect), e.g., z-pinches,[1] plasma erosion switches,[2] and in some circumstances laser-produced coronas.

It is well known that if we neglect thermal corrections the magnetohydrodynamic equations for each species of the plasma reduce to the statement that the curl of the canonical momentum $\mathbf{p}_\alpha = m_\alpha \mathbf{v}_\alpha + (e_\alpha/c)\mathbf{A}$ of that component is "frozen in" to the corresponding flow velocity $\mathbf{v}_\alpha$, i.e.,

$$\partial \operatorname{curl} \mathbf{p}_\alpha/\partial t = \operatorname{rot}[\mathbf{v}_\alpha \operatorname{curl} \mathbf{p}_\alpha]. \tag{1}$$

(Cross products and dot products are indicated in displayed equations by enclosing the two vectors being multiplied in square brackets and parentheses, respectively.) In electron magnetohydrodynamics (EMHD) we have $\mathbf{v}_e = -(c/4\pi ne)\operatorname{curl}\mathbf{H}$, where $n = \text{const}$ is the plasma density, and Eq. (1) can be written in the form

$$\partial\boldsymbol{\Omega}/\partial t = \operatorname{curl}[\boldsymbol{\Omega} \operatorname{curl} \mathbf{H}], \quad \boldsymbol{\Omega} \equiv \mathbf{H} - \Delta\mathbf{H}, \tag{2}$$

where the spatial coordinates have been nondimensionalized using c/ω_{pe} as a scale length (here $\omega_{pe} = (4\pi ne^2/m)^{1/2}$ is the plasma frequency], the magnetic field $\mathbf{H}$ has been scaled with some arbitrary H_0, and the time t with the the inverse gyrofrequency mc/eH_0.

Formally speaking, the present paper is devoted entirely to an investigation of the properties of the solutions of the EMHD equation (2). In Sec. 1 we study the general features of propagating solutions of Eq. (2). Our interest in such objects arises mainly from their importance in particle and heat transport processes, especially in the case of vortices. We show that in the absence of an external magnetic field only vortex solutions are possible, for which the frozen-in quantity $\boldsymbol{\Omega} = \mathbf{H} - \Delta\mathbf{H}$ vanishes identically in the region of free flow outside the separatrix, indicating that such vortices are localized exponentially within a region of order unity (of order c/ω_{pe} in dimensional units). In Sec. 2 we demonstrate the existence of such vortices by explicitly constructing a three-parameter family of localized solutions with spherical separatrices. These solutions are related to the spherical Hill's vortex[3] in ideal fluids, though they are considerably more complicated.

We obtain necessary conditions for the existence of solutions in the presence of a uniform external magnetic field. In particular, it is found that three-dimensional solitons are possible only for propagation at an angle with respect to the external magnetic field.

Since the opportunities for doing anything analytically are very limited in the three-dimensional case, particularly in regard to the question of stability, the theory of electron vortices up until now has treated two-dimensional flows in the (x, y) plane with the the magnetic field directed parallel to the z axis and no quantities depending on z (Refs. 4, 5). This geometry is degenerate for two reasons: it is two-dimensional ($\partial/\partial z = 0$), and the helicon frequency of the perturbations vanishes, since there is no variation parallel to the magnetic field.

A possible next step in the direction of a three-dimensional theory, including stability, is the study of two-dimensional systems generalized in such a way that the magnetic field and the current density have all three components but depend on only two spatial coordinates. In Sec. 3 the equa-

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tions of generalized two-dimensional EMHD are derived and traveling vortex solutions are obtained from them. It is natural to call these helicon vortices, just as in the three-dimensional case.

Section 4 is devoted to the question of the stability of vortices and electron motions described by Eq. (2) and its implications for flows with length scales $l \gg 1$ (in dimensional variables, $l \gg c/\omega_{pe}$). In this limit the electron inertia can be neglected and the frozen-in quantity is the magnetic field itself:

$$\partial \mathbf{H}/\partial t = \text{curl}\,[\mathbf{H}\ \text{curl}\ \mathbf{H}]. \tag{3}$$

We look at stability by analyzing the integrals of motion, which for Eq. (2) are the energy,

$$W = \frac{1}{2}\int (\mathbf{H}^2 + (\text{curl}\ \mathbf{H})^2)\, d^3\mathbf{r} = \frac{1}{2}\int \mathbf{H}\mathbf{\Omega}\, d^3\mathbf{r}, \tag{4}$$

the canonical momentum,

$$\mathbf{P} = -\int (\mathbf{A} + \text{curl}\ \mathbf{H})\, d^3\mathbf{r} = \frac{1}{2}\int [\mathbf{r}\mathbf{\Omega}]\, d^3\mathbf{r}, \tag{5}$$

the canonical angular momentum,

$$\mathbf{M} = -\int [\mathbf{r}, \mathbf{A} + \text{curl}\ \mathbf{H}]\, d^3\mathbf{r} = \frac{1}{3}\int [\mathbf{r}, [\mathbf{r}\mathbf{\Omega}]]\, d^3\mathbf{r}, \tag{6}$$

and the conserved frozen-in integrals, which are the fluxes of the frozen-in quantity $\mathbf{\Omega}$ within the tubes generated by displacing any closed contour in the streaming direction with the electron velocity. In the limit where the electron inertia vanishes, $\mathbf{\Omega}$ is replaced by $\mathbf{H}$ in Eqs. (4)–(6).

The best way to find stable vortices is by using the method of Arnol'd,[6] first employed in ideal hydrodynamics, but carried over to EMHD practically without modification. (Ideal hydrodynamics is a special case of EMHD; in the limit $l \ll 1$ Eq. (2) goes over to the equation for the conservation of vorticity.) The method is essentially a study of the energy (4) at a conditional extremum for fixed values of the frozen-in conserved quantities. Examination of a narrow class of variations

$$\delta\mathbf{\Omega} = \text{curl}\ [\boldsymbol{\xi}\mathbf{\Omega}], \quad \text{div}\,\boldsymbol{\xi} = 0, \tag{7}$$

which conserve an infinite number of the frozen-in integrals, increases the chances of obtaining a conditional extremum W, giving rise to a large (infinite-dimensional) set of stable vortices.[5,6]

We have been able to prove three-dimensional stability in EMHD only with zero electron inertia (Eq. (3)] and an arbitrary constant field $\mathbf{H} = (0, 0, H_z(x, y))$, which, however, is marginally stable against perturbations with $\partial/\partial z = 0$. Arnol'd's method works better in applications to the generalized two-dimensional geometry, where we have found a broad class of stable circular flows resembling a pinch, as well as monopoles and traveling dipoles.

In Sec. 5 we discuss several aspects of degeneracy in EMHD. In contrast with ideal fluids, Eq. (2) admits nontrivial linear waves superposed on a uniform external field $\mathbf{H}_0$. Linearizing Eq. (2) about H_0 yields the dispersion relation for helicons,

$$\omega(\mathbf{k}) = \pm k(\mathbf{k}\mathbf{H}_0)/(1+k^2). \tag{8}$$

Since the nonlinearity in the frozen-in equation has a rotational character, it drops out in the one-dimensional case and the exact solution is the same as the linearized one. This means that plane helicon waves of finite amplitude satisfying the dispersion relation (8) are exact solutions of Eq. (2). This was pointed out previously[7] for the special case $\mathbf{k}\|\mathbf{H}_0$. It is found that these waves are unstable against two- and three-dimensional perturbations (for infinitesimal perturbations this follows from the form of the dispersion relation (8), which allows three-wave decay processes). This means that in the nonlinear stage of the instability the solution can evolve into a two- or three-dimensional object.

1. GENERAL PROPERTIES OF LOCALIZED PROPAGATING SOLUTIONS

We consider three-dimensional solutions of Eq. (2) propagating with some velocity $\mathbf{u}$, i.e., $\mathbf{H} = \mathbf{H}(\mathbf{r} - \mathbf{u}t)$, $\partial/\partial t = -\mathbf{u}\cdot\nabla$. For such solutions Eq. (2) takes the form $\text{curl}\,[\mathbf{\Omega} \times (\mathbf{u} + \text{curl}\ \mathbf{H})] = 0$, whence we obtain

$$[\mathbf{\Omega}, \mathbf{u} + \text{curl}\ \mathbf{H}] = \nabla\varphi. \tag{9}$$

We assume that the magnetic field is localized as a result of the convergence of the energy integral (4). This means that $\mathbf{H}$ has to fall off at infinity faster than $r^{-3/2}$. It follows from (9) that $\varphi(\mathbf{r}) \to 0$ as $r \to \infty$. The geometrical meaning of Eq. (9) is that the streamlines of the divergenceless fields $\mathbf{\Omega}$ and $\mathbf{u} + \text{curl}\mathbf{H}$ lie on a level surface of $\varphi(\mathbf{r})$, which is constant along these lines. Since $\text{curl}\ \mathbf{H} \to 0$ as $\mathbf{r} \to \infty$, from any point on this surface outside some closed set bounded by a separatrix it is possible to get to infinity along a line of the field $\mathbf{u} + \text{curl}\ \mathbf{H}$. Consequently $\varphi = 0$ at every such point (outside the separatrix). Note that to within a sign the field $\mathbf{u} + \text{curl}\ \mathbf{H}$ is the velocity of electrons coming in from infinity in the frame of reference moving with the localized solution we are studying. The existence of the separatrix implies the existence of some "forbidden" region, where the current lines of external electrons cannot penetrate and which contains the current lines of trapped electrons translating together with the solution. In this case the solution describes a vortex. Note the necessary condition for the existence of a vortex: $|\text{curl}\ \mathbf{H} > \mathbf{u}$ at least at one point.

Thus outside the separatrix (for vortices) or throughout all of space (for solitons) we have $\varphi = 0$, whence we obtain $\mathbf{\Omega} = \mathbf{H} - \Delta\mathbf{H} = 0$. This means that there are no localized three-dimensional helicon solitons (in the absence of a magnetic field), and vortices are exponentially localized in a region which extends over a distance of order unity (c/ω_{pe} in dimensional variables).

It is also noteworthy that the frozen-in quantity vanishes identically outside the separatrix of the magnetic field. This implies that the magnetic field has a singularity near the separatrix, where the higher derivatives must experience a discontinuity. The same conclusion applies to the vortices moving in the external uniform magnetic field (see below). It follows that even though the solutions obtained by explicit matching (cf. Ref. 8, etc.) are not analytic, this is not a shortcoming of the method but a distinctive property of vortices.

We now treat the case of localized solutions propagating in a nonuniform external magnetic field $\mathbf{H}_0$. In place of (9) we obtain the analogous expression

$$[\mathbf{H}_0 + \mathbf{\Omega}, \mathbf{u} + \text{curl}\ \mathbf{H}] = \nabla\varphi. \tag{10}$$

We first assume $\mathbf{u}\|\mathbf{H}_0$. In that case also φ vanishes identically, outside a possible separatrix, whence

$$\boldsymbol{\Omega}=\lambda\,\text{curl}\,\mathbf{H},\quad \lambda=H_0/u=\text{const}. \tag{11}$$

Equation (11) can be rewritten in the form

$$(\text{curl curl}-\lambda\,\text{curl}+1)\mathbf{H}\equiv(\mu_1\,\text{curl}-1)(\mu_2\,\text{curl}-1)\mathbf{H}=0,$$
$$\mu_{1,2}=(\lambda\pm(\lambda^2-4)^{1/2})/2. \tag{12}$$

The general solution of Eq. (11) is the sum of two force-free fields:

$$\mathbf{H}=\mathbf{H}_1+\mathbf{H}_2;\quad \mathbf{H}_1=\mu_1\,\text{curl}\,\mathbf{H}_1,\ \mathbf{H}_2=\mu_2\,\text{curl}\,\mathbf{H}_2.$$

In the soliton case, Eq. (12) is satisfied throughout all of space; it follows that there can be no solitons that propagate parallel to an uniform external magnetic field (in this limit the equations are linear). We can show that the solutions of (12) are nonlinear (these are discussed in Sec. 5) by multiplying (12) by $\exp(i\mathbf{k}\cdot\mathbf{r})$ and integrating over all of space, which yields $|\lambda| = k + k^{-1}\geqslant 2$; consequently, μ_1 and μ_2 are real. The force-free fields $\mathbf{H}_1$ and $\mathbf{H}_2$ themselves have either infinite or vanishing energy.[4]

In the vortex case the Fourier method fails and the condition $|\lambda|\geqslant 2$ need not hold. For $|\lambda|<2$ the complex force-free fields may still be localized ($\int \mathbf{H}_{1,2}^2 d^3\mathbf{r}=0$, $0<\int(\text{Re}\mathbf{H}_{1,2})^2 d^2\mathbf{r}<\infty$). In this case, since μ_1 and μ_2 are complex conjugate, the sum $\mathbf{H}=\mathbf{H}_1+\mathbf{H}_2$ may still be chosen real. Thus the necessary condition for the existence of vortices propagating parallel to the external magnetic field is $u>H_0/2$ (in dimensional variables, $u>H_0/(16\pi mn)^{1/2}$). In Sec. 2 we show that this condition is also sufficient.

For solutions propagating at an angle to the external magnetic field ($\mathbf{u}\times\mathbf{H}_0\neq 0$), the function φ in Eq. (10) is not localized and there are no topological limitations on the existence of helicon vortices and solitons.

2. THREE-DIMENSIONAL TRAVELING VORTICES

In this section we derive a solution describing a toroidal vortex propagating parallel to a constant magnetic field. In this way we will prove the existence of three-dimensional traveling vortices in EMHD and verify the conclusions of the preceding section "experimentally," i.e., for a specific example.

The solution is axisymmetric, so the magnetic field can be represented in the form

$$\mathbf{H}=r^{-1}[\nabla\psi(r,z,t),\mathbf{e}_\varphi]+r^{-1}f(r,z,t)\mathbf{e}_\varphi,$$

where r, φ, z are cylindrical coordinates and $\mathbf{e}_\varphi$ is the unit vector in the azimuthal direction. Substituting this expression in (2) we obtain two scalar equations:

$$d(\psi-\tilde\Delta\psi)/dt=0,$$
$$dr^{-2}(f-\tilde\Delta f)/dt=r^{-1}[\nabla(r^{-2}\tilde\Delta\psi),\ \nabla(\psi-\tilde\Delta\psi)]_\varphi,$$
$$d/dt=\partial/\partial t+r^{-1}[\nabla f,\nabla]_\varphi,\quad \tilde\Delta=r(\partial/\partial r)r^{-1}(\partial/\partial r)+\partial^2/\partial z^2. \tag{13}$$

For a vortex propagating parallel to the z axis we have

$$\partial/\partial t=-u\partial/\partial z,\quad d/dt=r^{-1}[\nabla\tilde f,\nabla]_\varphi,\quad \tilde f=f+ur^2/2.$$

It follows from Eqs. (13) that

$$\tilde\Delta\psi-\psi=F(\tilde f),\quad \tilde\Delta f-f+F'(\tilde f)\tilde\Delta\psi=r^2P(\tilde f), \tag{14}$$

where F and P are arbitrary functions. To construct solutions we employ a method widely used recently for finding vortex solutions.[8] We choose functions F and P in such a way that Eqs. (14) are linear inside and outside the sphere $R=R_0$. We find solutions inside and outside the sphere and then join them, imposing the condition that the functions ψ and f be continuously differentiable and that $\tilde f$ be constant on $R=R_0$ (the functions F and P can have jumps only on surfaces of constant $\tilde f$). Let

$$F=\begin{cases} c\tilde f \\ b\tilde f \end{cases},\quad P=\begin{cases} d, & R<R_0 \\ 0, & R>R_0 \end{cases}.$$

Then for $R>R_0$

$$\tilde\Delta\psi-\psi=b(f+ur^2/2),\quad \tilde\Delta f-f+b\tilde\Delta\psi=0.$$

These equations have the solution

$$f=[A\chi(k_1R)+A^*\chi(k_2R)]r^2,$$
$$\psi=\left[\frac{A\chi(k_1R)}{k_1^2-1}+\frac{A^*\chi(k_2R)}{k_2^2-1}\right]br^2-\frac{bur^2}{2},$$

$$\chi(\zeta)=K_{3/2}(\zeta)\zeta^{-3/2},\quad k_{1,2}=1-b^2/2\pm(b^4/4-b^2)^{1/2}, \tag{15}$$

where k_1 and k_2 are the corresponding roots with positive real part, and $(\pi/2\zeta)^{1/2}K_{3/2}(\zeta)$ is the modified spherical Bessel function of the second kind. For $b^2\leqslant 4$, the roots k_1 and k_2 would be purely imaginary, so we must have $b^2<4$. Since $k_1=k_2^*$ and $\chi(\zeta^*)=\chi^*(\zeta)$ hold, f and ψ must be real functions. The function X vanishes exponentially at infinity, so we have $f\to 0$ and $\psi\to -bur^2/2$. For $\mathbf{r}\to\infty$ the magnetic field satisfies $H_z=\partial\psi/r\partial r=H_0$, so we must have $H_0=-bu$.

For $R<R_0$ we have

$$\tilde\Delta\psi-\psi=c(f+ur^2/2),\quad \tilde\Delta f-f+c\tilde\Delta\psi=r^2d,$$
$$f=(D_1j_1(\varkappa_1R)+D_2j_1(\varkappa_2R)-Rd)r^2/R,$$
$$\psi=-c\left(\frac{D_1}{\varkappa_1^2+1}\frac{j_1(\varkappa_1R)}{R}+\frac{D_2}{\varkappa_2^2+1}\frac{j_1(\varkappa_2R)}{R}+\frac{u}{2}-d\right)r^2,$$
$$\varkappa_{1,2}^2=-1+c^2/2\pm(c^4/4-c^2)^{1/2}, \tag{16}$$

where $j_1(\zeta)=(\pi/2\zeta)^{1/2}J_{3/2}(\zeta)$ is the spherical Bessel function of the first kind.

It is possible to join the solutions (15) and (16) for given R_0, b, and c by setting $u=-H_0/b$ and choosing the constants Re A, Im A, D_1, D_2, and d so that all the matching conditions are satisfied. We will not write down the resulting solutions here because they are so messy. It should be noted that $|b|=|H_0/u|<2$, in complete agreement with the conclusions of Sec. 1, and that b can be chosen arbitrarily subject to this restriction. For the special case $b=0$ the solution describes vortices with $H_0=0$. In this case outside $R=R_0$ we have $\tilde\Delta\psi=\psi$ and $\tilde\Delta f=f$, i.e., $\mathbf{H}=\Delta\mathbf{H}$.

Since $\tilde f=\text{const}=0$ in these solutions for $R=R_0$, the functions $F(\tilde f)$ and consequently $\tilde\Delta\psi$ are continuous. Thus the magnetic field and current density associated with the vortices are continuous, but there is a jump in the second derivatives of the magnetic field.

Vortices propagating obliquely to the magnetic field are not axisymmetric, which renders the task of finding analytical solutions practically hopeless.

3. TWO-DIMENSIONAL HELICON VORTICES

The case of magnetic field configurations with all three components, depending on just two of the three spatial coordinates, is intermediate between three-dimensional and conventional two-dimensional geometries. This geometry, which we refer to as "generalized two-dimensional," also admits stable vortex solutions, the class of which is larger than in the ordinary two-dimensional case.[5] In this section we derive a set of generalized two-dimensional EMHD equations from (2) under the assumption that $\partial/\partial z = 0$ and find the corresponding traveling helicon waves.

In this limit the magnetic field can be written in the form

$$\mathbf{H}=[\nabla a(x, y, t), \mathbf{e}_z]+h(x, y, t)\mathbf{e}_z, \tag{17}$$

where a and h are the z-components of the vector potential and the magnetic field strength. Substituting (17) in (2), we obtain the system of equations

$$\partial\theta/\partial t+(h, \theta)=0, \quad \theta\equiv a-\Delta a,$$

$$\partial\omega/\partial t+(h, \omega)=(a, \theta), \quad \omega\equiv h-\Delta h, \tag{18}$$

where Δ is the Laplacian operator and where for conciseness we have written $(A, B)\equiv \mathbf{e}_z\cdot\nabla A\times\nabla B$. With the idea in mind of studying two-dimensional stability (Sec. 4), we write the integrals of motion for the system (18):

$$W_2=(^1/_2)\int(h\omega-a\theta)\,d^2\mathbf{r},$$

$$\mathbf{P}_2=\int\omega\mathbf{r}\,d^2\mathbf{r}, \quad M_2=\int\omega\mathbf{r}^2\,d^2\mathbf{r}, \tag{19}$$

$$I_F=\int F(\theta)\,d^2\mathbf{r}, \quad J_F=\int\omega F(\theta)\,d^2\mathbf{r}, \tag{20}$$

where F is an arbitrary function.

We can investigate the general properties of two-dimensional traveling solutions just as in the three-dimensional case. Avoiding detailed discussions, we merely note that the situation in two dimensions is analogous to that in three, with the exception that when an external magnetic field is present, localized states can propagate only parallel to it. It follows that two-dimensional helicon solitons can not exist. Below we prove the existence of vortices by constructing a specific family of solutions.

It follows from (18) that a vortex propagating in the x-direction with velocity u satisfies

$$\Delta a-a=F(h+uy), \quad \Delta h-h+F'(h+uy)a=P(h+uy). \tag{21}$$

We can obtain a solution to these equations by the same method as in the previous section. Letting F and P be different linear functions inside and outside the circle $r=r_0$, when there is no external magnetic field we find a dipole solution in the form

$$a=\begin{cases}(c_1r^{-1}J_1(k_1r)+c_2r^{-1}J_1(k_2r)+c_3)y, & r<r_0\\ c_4r^{-1}K_1(r)y, & r>r_0\end{cases},$$

$$h=\begin{cases}(d_1r^{-1}J_1(k_1r)+d_2r^{-1}J_1(k_2r)+d_3)y, & r<r_0\\ d_4r^{-1}K_1(r)y, & r>r_0\end{cases}. \tag{22}$$

It is necessary to supplement this solution with a system of algebraic equations for the parameters c_i, d_i, k_i found by matching the functions a and h and their derivatives at $r=r_0$. The result is a three-parameter family of two-dimensional traveling helicon vortices. As in the three-dimensional case, outside the separatrix $r=r_0$ the "generalized vorticities" θ and ω vanish identically.

Similarly, we can also construct a vortex solution in the presence of a uniform external magnetic field for $u>H_0/2$. As in the three-dimensional case, the magnetic field and current density are continuous in traveling dipole vortices.

4. STABILITY OF EMHD VORTICES AND FLOWS

The linear stability of current configurations against helicon perturbations was first treated in Ref. 9. The most suitable technique for proving stability is by studying their integrals of motion. According to a theorem of Lyapunov, a sufficient condition for stability of a system is the existence of a conditional absolute extremum of any integral of the motion when an arbitrary number of other integrals of the motion are maintained constant. In the method of Arnol'd[6] all the frozen-in integrals are fixed. The method is convenient because the most general variation of the field can be written explicitly in the form of a series $\delta\mathbf{\Omega}+\delta^2\mathbf{\Omega}+\ldots$, where the first variation is given by Eq. (7) and the others are defined by

$$\delta^N\Omega=\text{curl}\,[\xi, \delta^{N-1}\Omega]/N.$$

To investigate the extremum it is enough in practice to calculate the first two terms of the series.

In analogy to the ideal fluid case,[6] the first variation of the energy integral (4) vanishes for steady flows satisfying $\text{curl}\,(\mathbf{\Omega}\times\text{curl}\,\mathbf{H})=0$, as may easily be seen through integration by parts. Thus for the Hamiltonian to be extremal it suffices that the second variation δ^2W be positive or negative definite. We treat the stability of Eqs. (2) and (3) separately.

Stability of EMHD flows without the effect of electron inertia

We consider a perturbation of the Hamiltonian $w=\frac{1}{2}\int H^2d^3\mathbf{r}$ together with an arbitrary three-dimensional field variation of the form

$$\delta\mathbf{H}=\text{curl}\,[\xi\mathbf{H}], \quad \delta^2\mathbf{H}=^1/_2\,\text{curl}\,[\xi\delta\mathbf{H}], \quad \text{div}\,\xi=0; \tag{23}$$

$$\delta^2w=(^1/_2)\int(2\mathbf{H}\,\delta^2\mathbf{H}+(\delta\mathbf{H})^2)\,d^3\mathbf{r}$$

$$=(^1/_2)\int\text{curl}\,[\xi\mathbf{H}]\,(\text{curl}\,[\xi\mathbf{H}]-[\xi\,\text{curl}\,\mathbf{H}])\,d^3\mathbf{r}. \tag{24}$$

In the general case expression (24) is nondefinite, but for an arbitrary time-independent field

$$\mathbf{H}=(0,\ 0,\ H_z(x,\ y)) \tag{25}$$

we have

$$\delta^2w=(^1/_2)\int[(\mathbf{H}\nabla)\xi]^2\,d^3\mathbf{r}\geqslant 0, \tag{26}$$

where equality is attained only in the case $\partial\xi/\partial z\equiv 0$, i.e., for two-dimensional perturbations. Equilibrium is irrelevant for these, since any field of the form (25) is an equilibrium field if one neglects electron inertia. Taking into account the z-dependence implies that helicons transporting a positive value of the energy are excited. Thus electron flows with straight parallel magnetic field lines are stable in three dimensions if they give rise to a minimum of the energy for fixed values of the frozen-in integrals. We emphasize that this result applies to electron flows for which the scale lengths in both the initial and perturbed flows are large.

We now consider the case of generalized two-dimensional geometry under exactly the same assumptions, i.e., the system of equations arising from (18) for $l \ll 1$:

$$\partial a/\partial t+(h,\ a)=0,\qquad \partial h/\partial t+(a,\ \Delta a)=0. \tag{27}$$

The following quantities are conserved by Eqs. (27):

$$w_2=(1/2)\int(h^2+(\nabla a)^2)\,d^2\mathbf{r}, \tag{28}$$

$$\mathbf{p}_2=\int h\mathbf{r}\,d^2\mathbf{r},\qquad m_2=\int h\mathbf{r}^2\,d^2\mathbf{r}, \tag{29}$$

$$i_F=\int F(a)\,d^2\mathbf{r},\qquad j_F=\int hF(a)\,d^2\mathbf{r}. \tag{30}$$

It is not difficult to find the general form of the variation that conserves all the frozen-in integrals (30). This is accomplished most simply starting from the three-dimensional variation (23), if we choose an infinitesimal displacement ξ consistent with $\partial\xi/\partial z=0$, analogous to (17):

$$\xi=[\nabla\alpha(x,\ y),\ \mathbf{e}_z]+\beta(x,\ y)\mathbf{e}_z.$$

Then

$$\delta a=(\alpha,\ a),\qquad \delta h=(\alpha,\ h)+(\beta,\ a),$$

$$\delta^2 a=(\alpha,\ \delta a)/2,\qquad \delta^2 h=[(\alpha,\ \delta h)+(\beta,\ \delta a)]/2. \tag{31}$$

Using (31) in analogy with (24) we calculate the second variation[1)] of the energy (28) for a steady flow satisfying $(h,a)=(a,\Delta a)=0$:

$$\delta^2 w_2=(1/2)\int[(\nabla(\alpha,a))^2+(\beta,a)^2+(\alpha,a)(\alpha,\Delta a)]\,d^2\mathbf{r}. \tag{32}$$

It is noteworthy that the quantity h drops out of (32), i.e., the extremal properties of the Hamiltonian ω_2 over the class of frozen-in flows depend only on $a(x,y)$.

We mention several consequences of (32):

1. An obvious sufficient criterion for two-dimensional stability is that the gradients of a and Δa point in the same direction (it follows from time independence that they must be parallel). This is not a necessary condition.

2. We consider the case of circular stationary vortices, $a=a(r)$ and $h=h(r)$. For such flows (32) goes over to

$$\delta^2 w_2=\frac{1}{2}\int\Big\{(\beta,a)^2+\Big(a'(r)\frac{\partial}{\partial r}\frac{\partial\alpha}{r\,\partial\varphi}\Big)^2+\Big[\frac{a'(r)}{r^2}\Big]^2\Big[\Big(\frac{\partial^2\alpha}{\partial\varphi^2}\Big)^2-\Big(\frac{\partial\alpha}{\partial\varphi}\Big)^2\Big]\Big\}\,r\,dr\,d\varphi. \tag{33}$$

Fourier-expanding the periodic (in φ) function $\alpha(r,\varphi)$, we have

$$\int_0^{2\pi}\Big[\Big(\frac{\partial^2\alpha}{\partial\varphi^2}\Big)^2-\Big(\frac{\partial\alpha}{\partial\varphi}\Big)^2\Big]d\varphi=2\pi\sum_{N=-\infty}^{\infty}N^2(N^2-1)\,|\alpha_N(r)|^2,$$

which shows that (33) is positive definite. Consequently any stationary circular vortex solution $a=a(r)$, $h=h(r)$ of Eqs. (27) is stable, since it minimizes the energy at fixed values of the frozen-in integrals.

3. The one-dimensional case $a=a(r)$, $h=h(r)$ describes a set of plane sheared magnetic surfaces perpendicular to the x axis: $\mathbf{H}=(0,\ -a'(x),\ h(x))$. Since the two-dimensional [i.e., (x,y)] stability does not depend on h, it is identical with that of the field $(0,\ -a'(x),\ 0)$, which by (26) is three-dimensionally stable. By the same reasoning, this configuration is (x,z)-stable. This evidently constitutes an example of three-dimensional stability of a field with straight field lines (including those in which shear is present) derived from Eq. (3).

Thus generalized two-dimensional EMHD admits an infinite-dimensional class of stable stationary vortices. The circular stable vortices are "parametrized" by the two functions $a(r)$ and $h(r)$.

The existence of stable stationary monopole vortices implies that stable traveling dipole vortices also exist. This can be shown by taking account of momentum conservation, as was done by Filippov and Yan'kov.[5]

Stability of EMHD flows including the effect of electron inertia

This problem is substantially harder. The difficulty is associated at least with the fact that even in the ideal-fluid limit of Eq. (2) not one stable localized three-dimensional flow is known. We discuss the two-dimensional stability of Eqs. (18). Similarly to (31), the field variations that conserve the frozen-in integrals (2) take the form

$$\delta\theta=(\alpha,\ \theta),\qquad \delta\omega=(\alpha,\ \omega)+(\beta,\ \theta).$$

In fact the stability theorem can be proved in this case without recourse to the variational method. Expressing the field a as a superposition of line vortices by transforming with the modified Bessel function of the second kind,

$$a(\mathbf{r})=(2\pi)^{-1}\int\theta(\mathbf{r}')K_0(|\mathbf{r}-\mathbf{r}'|)\,d^2\mathbf{r}',$$

we bring the energy integral (19) into the form

$$W_2=\frac{1}{2}\int(h^2+(\nabla h)^2)\,d^2\mathbf{r}-\frac{1}{4\pi}\int\theta(\mathbf{r})\theta(\mathbf{r}')K_0(|\mathbf{r}-\mathbf{r}'|)\,d^2\mathbf{r}\,d^2\mathbf{r}'. \tag{34}$$

Let us consider z-pinch-like flows with $h=0$, perturbed in such a way that none of the frozen-in integrals change. If all the I_F integrals defined in (20) are invariant, then the allowed changes in θ reduce to incompressible deformations of the regions corresponding to different values of θ. For such variations the first term in (34), being zero, can only grow, while the second can be interpreted as the energy of interaction of θ "charges" with a monotonic interaction potential $K0(r)$. From this we conclude, in analogy to the result of Filippov and Yan'kov,[5] that circular flows with $a=a(r)$ and $h=0$ are two-dimensionally stable if the quantity

$$|\theta(r)|=|a(r)-a''(r)-a'(r)/r|$$

decreases monotonically. Note that, in contrast to ordinary two-dimensional EMHD with $a\equiv 0$ (Ref. 5), these stable flows are obtained for fixed values of the frozen-in integrals along with an energy minimum, not a maximum.

The topic of two-dimensional helicon instability may be of independent practical interest, since numerical solution of the linearized problem[10] shows that three-dimensional ($k_z \neq 0$) stability follows from two-dimensional ($k_z = 0$) stability for current configurations of the z-pinch type.

Returning to the three-dimensional stability problem, we remark that when electron inertia is included in the treatment not all fields of the form $\mathbf{H} = (0, 0, h(x,y))$ are stationary, but only those for which $(h, \Delta h) = 0$. Among these are the circular vortices. If the quantity $|\omega(r)|$ decreases monotonically they are (x, y)-stable, maximizing the energy.[5] If the diameter of the vortex satisfies $l \gg 1$, then, as we have seen, it minimizes the energy with respect to three-dimensional perturbations when we neglect inertial corrections. Hence inclusion of even a small nonvanishing electron inertia changes the relative extremum given by (26) into a saddle point, and these two-dimensional perturbations can become unstable against three-dimensional perturbations inclined at a small angle ($\sim l^{-1}$ for $l \gg 1$) with respect to the magnetic field.

5. FINITE-AMPLITUDE HELICON WAVES

In this section we discuss the behavior of plane helicon waves propagating in a uniform magnetic field $\mathbf{H}_0$. Such waves are exact solutions of Eq. (2) and may therefore have any amplitude. We discuss several degenerate cases for which the waves do not interact, and consider their stability.

If we include the uniform magnetic field $\mathbf{H}_0$, Eq. (2) may be rewritten as follows:

$$\partial(\mathbf{H}-\Delta\mathbf{H})/\partial t+(\mathbf{H}_0\nabla)\,\mathrm{curl}\,\mathbf{H}=\mathrm{curl}\,[\mathbf{H}-\Delta\mathbf{H},\,\mathrm{curl}\,\mathbf{H}]. \tag{35}$$

The solutions of the linearized form of (35) are plane helicon waves, the dispersion relation of which is given by (8). If the magnetic field is a function of only one spatial variable, then the nonlinearity in (35) vanishes and the exact equation reduces to the linearized one. This shows that linear helicon waves propagating rectilinearly, or any combination of such waves propagating parallel to one another, constitutes an exact solution of Eq. (2).

From the linear part of (35) it can be seen that for a helicon with wave vector $\mathbf{k}$ we have

$$\mathrm{curl}\,\mathbf{H} = \frac{\omega(\mathbf{k})\,(1+k^2)}{(\mathbf{k}\mathbf{H}_0)}\,\mathbf{H} = sk\mathbf{H},$$

where s is the sign on the right hand side of (8). By utilizing this property we can identify one more case in which plane waves do not interact with one another even though (35) is nonlinear. If we superpose helicons having identical values of s and of the wavenumber k (the absolute value of the wave vector), then the sum, like the individual waves, satisfies $\mathrm{curl}\,\mathbf{H} = sk\,\mathbf{H}$, and the nonlinearity in (35) vanishes. To this we can moreover add waves with the same value of s, whose wavenumber is equal to $1/k$. If we denote the field of the waves with wavenumber k by $\mathbf{H}_1$ and the field of those with wavenumber $1/k$ by $\mathbf{H}_2$, then we have

$$[\mathbf{H}-\Delta\mathbf{H},\mathrm{curl}\,\mathbf{H}]=\left[(1+k^2)\mathbf{H}_1,\ \frac{s}{k}\mathbf{H}_2\right] +\left[\left(1+\frac{1}{k^2}\right)\mathbf{H}_2,\ sk\mathbf{H}_1\right]=0$$

and it follows that the waves do not interact. This degeneracy turns out to be useful for investigating the stability of such waves.

In investigating the stability of the helicons we start by assuming that the wave amplitude is small in comparison with $\mathbf{H}_0$. According to perturbation theory, we have in this case a three-wave decay instability.[11] Analysis of the dispersion relation (8) reveals that for any helicon $(\omega_0, \mathbf{k}_0)$, we can find two others, $(\omega_1, \mathbf{k}_1)$ and $(\omega_2, \mathbf{k}_2)$, such that the decay conditions

$$\omega_0=\omega_1+\omega_2,\quad \mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2,\quad |\omega_0|>|\omega_1|,\ |\omega_2|,$$

are satisfied, i.e., a small-amplitude helicon is unstable against decay. Note that the matrix element of the interaction,

$$V_{\mathbf{k}_0\mathbf{k}_1\mathbf{k}_2}=\frac{1}{2(2\pi)^{3/2}}\frac{k_0}{1+k_0^2}\left(\frac{\omega(\mathbf{k}_1)\,\omega(\mathbf{k}_2)}{\omega(\mathbf{k}_0)}\right)^{1/2} \times([\mathbf{k}_1,\mathbf{k}_2]\mathbf{e}_z)\left(k_2+\frac{1}{k_2}-k_1-\frac{1}{k_1}\right)$$

(where the z axis is perpendicular to the plane of $\mathbf{k}_0$, $\mathbf{k}_1$, $\mathbf{k}_2$ and we have $s = 1$) vanishes in the degenerate case considered above.

Now suppose that the wave amplitude is not small. Consider the set of waves $\mathbf{k}_1$ and $\mathbf{k}_2$ that are nearly degenerate with respect to the initial wave. In this case the waves interact weakly (not because the amplitudes are small, but because we are close to the degeneracy), and using perturbation theory we can obtain exactly the same results as in the small-amplitude case.[11] In order to convince oneself that close to the degeneracy $\mathbf{k}_1$ and $\mathbf{k}_2$ can be found which satisfy the decay condition, it suffices to consider the case in which $\mathbf{k}_0$, $\mathbf{k}_1$, and $\mathbf{k}_2$ comprise an almost-equilateral triangle whose plane is perpendicular to that of the vectors $\mathbf{H}_0$ and $\mathbf{k}_0$.

Thus helicons of arbitrary amplitude derived from Eq. (2) are unstable. For waves of finite amplitude ($H \sim H_0$) the maximum growth rate is on the order of the frequency.

When these instabilities evolve, objects of higher dimension can develop, e.g., helical solitons. Plane waves cannot decay into vortices because the topology of $\mathbf{\Omega}$ is conserved due to freezing-in; in a plane helicon the lines of the $\mathbf{\Omega}$ field are unentangled spirals, while in a vortex within the separatrix lines of $\mathbf{\Omega}$ are constrained to lie on closed surfaces of constant φ (cf. Eq. (9)], which, in the general case, are toroidal.[12]

CONCLUSION

We have thus considered several particular solutions of the EMHD equation in a uniform plasma, which claim a role in strong helicon turbulence. We have found solutions describing two- and three-dimensional traveling vortices with and without an external magnetic field. The existence of three-dimensional helicon solitons propagating at an angle to the external magnetic field remains an open question.

The qualitative method developed in Sec. 1 for analyzing localized traveling-wave solutions has proven useful in determining the necessary conditions for the existence of such solutions (Secs. 1 and 2 show that these conditions are also close to sufficient) and in classifying them into vortices and solitons. Other systems described by a vector freezing-in equation can be treated similarly.

The results of Sec. 4 show that by virtue of the existence of a finite number of conserved quantities, the EMHD equation admits an infinite-dimensional class of stable stationary vortex solutions. However, it is not possible in practice to prove that traveling vortices (a category with important applications), are stable, especially in three dimensions.

The purely electron instability of finite-amplitude helicon waves found in Sec. 5 may turn out to be important in problems of ponderomotive[13] and ionizational[14] helicon self-focusing.

[1)]In calculating this quantity we used the formula for integration by parts, $\int (a,b)c\, d^2\mathbf{r} = \int a(b,c)\, d^2\mathbf{r}$, and the Jocobi identity $((a,b),c) + ((b,c),a) + ((c,a),b) \equiv 0$. The infinitesimal displacement quantities α and β are assumed to vanish sufficiently rapidly at infinity.

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