Spectral representation of the polarizability tensor for macroscopic objects

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The polarizability of a macroscopic object of arbitrary shape is examined by expanding the potential in eigenfunctions of local (surface) modes. Some of the properties of these functions and of the corresponding eigenvalues are discussed. A general formula is derived for the spectral representation of the polarizability tensor. The connection between the proposal approach and standard perturbation theory is elucidated. The splitting of local levels due to interactions between two objects is considered.

1. INTRODUCTION

The calculation of the polarizability tensor $\hat{\alpha}$ for homogeneous macroscopic objects is a classic problem (with a multitude of applications) in the electrodynamics of continuous media.[1-3] It also becomes necessary to determine $\hat{\alpha}$ in the theory of inhomogeneous media, for example in calculating the electrical conductivity (as well as the permittivity, thermal conductivity, and so forth) of systems with a low concentration of one component (see Refs. 4–6). However, the exact solution for the polarizability is available only in a few cases: for ellipsoidal objects,[1-3] pairs of circular cylinders (see Ref. 6, for example), and certain others. It is of interest, therefore, both to elucidate the general properties (particularly the analytic properties) of the polarizability tensor for arbitrarily shaped inclusions, and to devise a regular method of calculating $\hat{\alpha}$ in such cases.

Note that the natural variable for $\hat{\alpha}$ is not the frequency ω, but the quantity $z = z(\omega)$, where in the electrical conductivity problem, for example, z is the ratio of the conductivity of the inclusion to that of the medium. The analytic properties of $\hat{\alpha}$ are correspondingly found to be much simpler in the complex z-plane than in the complex ω-plane (see Ref. 1). It can be shown (compare with Ref. 7) that the polarizability is analytic over the entire z-plane, except for the negative real axis, where there are singularities. It was suggested in Refs. 6 and 7 that these singularities are associated with the eigenfrequencies of local modes of the so-called LC model,[6,7] for which $z(\omega) = -\omega^2/\Omega^2$, where Ω is the Thomson frequency.[1] This suggestion was confirmed in Refs. 6 and 7 through some exactly solved examples. In these same examples, it can be shown (see section 6 of the present paper) that the electric potential in the polarizability problem is an expansion in the eigenfunctions of the appropriate local modes. It seems logical to use such an expansion for arbitrarily shaped objects as well, including those that are multiply connected.

In the present paper, several general properties of the local modes associated with an arbitrarily shaped object are discussed. It is shown that local modes exist only for real, negative $z = -\varepsilon_\nu$, where the $\varepsilon_\nu \geqslant 0$ are the eigenvalues (the spectrum) for this problem, and depend only on the shape of the object. The eigenfunctions $\psi_\nu(\mathbf{r})$ corresponding to the ε_ν possess a number of specific features. Thus, not only are the $\psi_\nu(\mathbf{r})$ orthonormal, but so are their gradients $\nabla\psi_\nu(\mathbf{r})$. It turns out that neither the set $\{\psi_\nu(\mathbf{r})\}$ nor the set $\{\nabla\psi_\nu(\mathbf{r})\}$ is complete in the usual sense (note that such completeness is not required in the polarizability problem). It has not been possible to find a relation to replace the usual completeness condition in the present case. In this paper, therefore, we establish only a sufficient condition for solubility of the polarizability problem by expansion in the set of functions $\{\psi_\nu(\mathbf{r})\}$.

The proposition that the potential can be expanded in the set $\{\psi_\nu(\mathbf{r})\}$ makes a formal solution of the problem possible for an arbitrarily shaped object in a uniform external electric field. This solution satisfies the original equation whenever a certain condition (the expandability condition) holds [see Eq. (18)]. It is highly significant that the expandability condition (which has a simple meaning—see section 2) is a much weaker constraint on $\{\psi_\nu(\mathbf{r})\}$ than the usual completeness condition. We will show in this paper that the validity of Eq. (18) is sufficient for a systematic and consistent solution of the polarizability problem. In the exactly soluble cases examined (see section 6), Eq. (18) holds. For arbitrarily shaped objects, the validity of the expandability condition is an assumption which must be mathematically justified; its validity is assumed in the present paper.

The spectral representation of $\hat{\alpha}$ derived here (see Eqs. (26) and (28)) possesses the required analytic properties in the complex z-plane. The polarizability is given by a sum of terms of the form $a_\nu/(z+\varepsilon_\nu)$, where a_ν is the square of the "dipole moment" of the νth local mode, and ε_ν is its corresponding eigenvalue. The dimensionless quantities ε_ν and a_ν are determined solely by the geometry (shape) of the inclusion, and can be found by solving for the eigenvalues, for example numerically. A knowledge of the sets of numbers $\{\varepsilon_\nu\}$ and $\{a_\nu\}$ for an arbitrarily shaped object enables one to find its polarizability for an arbitrary, possibly complex, value of z. For the static polarizability, the characteristics of the local modes then play the formal role of auxiliary quantities.

For the dynamic polarizability ($z = z(\omega)$), the local modes have a real physical meaning, since they can appear as resonances in $\hat{\alpha}$ at the frequencies $\omega = \omega_\nu$ where the ω_ν are determined by the equation $z(\omega_\nu) = -\varepsilon_\nu$. (Here the quantities a_ν are directly related to the corresponding oscillator strengths.) Such modes have long been known for simple objects, such as spheres and cylinders: these are surface (neglecting delay) phonons, plasmons, and so on (see Refs. 8 and 9 for example). The local modes in the LC model considered in Refs. 6 and 7 also fall into this category of excitations. From a formal standpoint, the only thing that distinguishes one of these modes from another is the specific way in which z depends on ω. The approach taken here therefore enables one to examine such surface modes with a common outlook, for arbitrarily shaped objects.

Note that local modes can be treated as a set of quasi-

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particles. The polarizability problem then takes the guise of a problem dealing with the response of an ideal (or, with nonlinear effects taken into account, nonideal) gas of quasiparticles to an external influence. Apart from the felicity of casting the polarizability problem in the familiar language of quasiparticles, this is also convenient in that it enables one to make use of various methods, including approximate methods, analogous to those used in quantum mechanics. As an example, we consider in section 5, the splitting of local levels due to the interaction of a pair of inclusions.

2. LOCAL MODES

For definiteness, we shall consider local modes within the confines of the conductivity problem.

In a medium of conductivity σ_1, let there be an arbitrarily shaped (possibly multiply connected) inclusion of finite size, with conductivity σ_2. If such a system is immersed in a uniform (quasistationary) alternating electric field of frequency ω, then σ_1 and σ_2 will depend on ω [$\sigma_1 = \sigma_1(\omega)$, $\sigma_2 = \sigma_2(\omega)$], and will in general be complex quantities.

In an inhomogeneous medium, the electric field potential $\psi(\mathbf{r})$ obeys the relation

$$\nabla[\sigma(\mathbf{r})\nabla\varphi(\mathbf{r})]=0 \tag{1}$$

(the coordinate origin is assumed to be at the center of the inclusion). We may write $\sigma(\mathbf{r})$ in the form

$$\sigma(\mathbf{r})=\sigma_1[1-(1-z)\theta(\mathbf{r})], \qquad z=z(\omega)=\sigma_2(\omega)/\sigma_1(\omega), \tag{2}$$

where $\theta(\mathbf{r}) = 1$ inside the object and $\theta(\mathbf{r}) = 0$ outside. Making use of (2), the equation for the potential takes the form

$$\nabla^2\varphi(\mathbf{r})-(1-z)\nabla[\theta(\mathbf{r})\nabla\varphi(\mathbf{r})]=0. \tag{3}$$

The usual boundary conditions apply to $\varphi(\mathbf{r})$: continuity of both the potential and the normal component of the current density $\mathbf{J} = -\sigma(\mathbf{r})\nabla\varphi$ at the surface of the inclusion.

For certain constraints (see below) on $z = z(\omega)$, Eq. (3) will have nontrivial, nonsingular solutions $\psi_\nu(\mathbf{r})$ at certain frequencies $\omega = \omega_\nu$ (where $\psi_\nu(\mathbf{r}) \to 0$ as $r \to \infty$), even in the absence of an alternating external field. We shall refer to the functions $\psi_\nu(r)$ as the eigenfunctions (the subscript ν enumerates the various solutions), and, to the quantities $\varepsilon_\nu = -z(\omega_n)$ at the corresponding frequencies ω_ν as the eigenvalues of the local-mode problem. Thus, the equation determining $\psi_\nu(\mathbf{r})$ and ε_ν is of the form

$$\nabla^2\psi_\nu(\mathbf{r})-(1+\varepsilon_\nu)\nabla[\theta(\mathbf{r})\nabla\psi_\nu(\mathbf{r})]=0, \qquad \varepsilon_\nu=-z(\omega_\nu). \tag{4}$$

At the boundary of the inclusion, both $\psi_\nu(\mathbf{r})$ and the normal component of the "current density" are continuous:

$$\mathbf{n}\nabla\psi_\nu^{(1)}=-\varepsilon_\nu\mathbf{n}\nabla\psi_\nu^{(2)}, \tag{5}$$

where $\mathbf{n}$ is the unit vector normal to the interface, and $\psi_\nu^{(1)}$ and $\psi_\nu^{(2)}$ respectively refer to the medium ("conductivity" 1) and the inclusion ("conductivity" $z(\omega_\nu) = -\varepsilon_\nu$). Furthermore, the functions $\psi_\nu(\mathbf{r})$ must satisfy the condition at infinity: $\psi_\nu(\mathbf{r}) \to 0$ as $r \to \infty$. Note that in general, the functions $\psi_\nu(\mathbf{r})$ are complex, while the quantities ε_ν are real and positive, as will be shown below.

Multiplying Eq. (4) by $\psi_\nu^*(\mathbf{r})$, where the asterisk denotes complex conjugation, and integrating over all space, we obtain (after integrating by parts)

$$\left.\begin{aligned}&(1+\varepsilon_\nu)\int|\mathbf{e}_\nu(\mathbf{r})|^2\theta(\mathbf{r})\,d\mathbf{r}=\int|\mathbf{e}_\nu(\mathbf{r})|^2\,d\mathbf{r},\\ &\varepsilon_\nu=\left\{\int|\mathbf{e}_\nu(\mathbf{r})|^2[1-\theta(\mathbf{r})]\,d\mathbf{r}\right\}\left[\int|\mathbf{e}_\nu(\mathbf{r})|^2\theta(\mathbf{r})\,d\mathbf{r}\right]^{-1}.\end{aligned}\right\} \tag{6}$$

Here

$$\mathbf{e}_\nu(\mathbf{r})=\nabla\psi_\nu(\mathbf{r}). \tag{7}$$

For brevity, we shall call the $\mathbf{e}_\nu(\mathbf{r})$ the eigenvectors.

According to (6), the eigenvalues ε_ν are real and positive ($\varepsilon_\nu \geqslant 0$), so that local modes arise only for real, negative values of the parameter $z(\omega)$ in the appropriate range of frequencies (for $\omega = \omega_\nu$, where the ω_ν, are given by $z(\omega_\nu) = -\varepsilon_\nu$). This condition is satisfied, for example, by systems in which there are surface phonons, plasmons, and the like,[8,9] as well as the LC model.[6,7] In general, local modes must exist in all systems of the type considered (inclusion in a medium) with a real, negative ratio of conductivities, permittivities, etc. Note also that the ε_ν are dimensionless, and depend solely on the shape of the inclusion, not on its volume. The set of numbers $\{\varepsilon_\nu\}$ comprises the spectrum of an arbitrarily shaped object.

We now elucidate the orthogonality conditions imposed on the set of eigenfunctions $\{\psi_\nu(\mathbf{r})\}$ by the form of Eq. (4). Divide (4) by $1 + \varepsilon_\nu$, multiply by $\psi_\mu^*(\mathbf{r})$, and integrate over all space. Integrating by parts, we then obtain

$$\frac{1}{1+\varepsilon_\nu}\int\mathbf{e}_\nu(\mathbf{r})\mathbf{e}_\mu^*(\mathbf{r})\,d\mathbf{r}=\int\mathbf{e}_\nu(\mathbf{r})\mathbf{e}_\mu^*(\mathbf{r})\theta(\mathbf{r})\,d\mathbf{r}, \tag{8}$$

where $\mathbf{e}_\nu(\mathbf{r})$ is the same as in (7). Next, writing an equation like (4) for $\psi_\mu^*(\mathbf{r})$, multiplying it by $\psi_\nu(\mathbf{r})$, and integrating, we arrive in similar fashion at Eq. (8′), which differs from (8) by the replacement $\varepsilon_\nu \to \varepsilon_\mu$. Subtracting (8′) from (8), we obtain

$$\left(\frac{1}{1+\varepsilon_\nu}-\frac{1}{1+\varepsilon_\mu}\right)\int\mathbf{e}_\nu(\mathbf{r})\mathbf{e}_\mu^*(\mathbf{r})\,d\mathbf{r}=0,$$

i.e. the vectors $\mathbf{e}_\nu(\mathbf{r})$ and $\mathbf{e}_\mu(\mathbf{r})$ are orthogonal when $\varepsilon_\nu \neq \varepsilon_\mu$. We assume that the $\mathbf{e}_\nu(\mathbf{r})$ are normalized:

$$\int|\mathbf{e}_\nu(\mathbf{r})|^2\,d\mathbf{r}=1. \tag{9}$$

Then if the eigenvectors are nondegenerate, the vectors $\mathbf{e}_\nu(\mathbf{r})$ form an orthonormal set,

$$\int\mathbf{e}_\nu(\mathbf{r})\mathbf{e}_\mu^*(\mathbf{r})\,d\mathbf{r}=\delta_{\mu\nu}. \tag{10}$$

We shall assume hereafter that when there is degeneracy, the set $\{\mathbf{e}_\nu(\mathbf{r})\}$ can similarly be orthogonalized, so that Eq. (10) also holds for that case. Note that when (10) is taken into account, the following local orthogonality conditions follow from (8):

$$\left.\begin{aligned}&\int\mathbf{e}_\nu(\mathbf{r})\mathbf{e}_\mu^*(\mathbf{r})\theta(\mathbf{r})\,d\mathbf{r}=\frac{1}{1+\varepsilon_\nu}\delta_{\mu\nu},\\ &\int\mathbf{e}_\nu(\mathbf{r})\mathbf{e}_\mu^*(\mathbf{r})[1-\theta(\mathbf{r})]\,d\mathbf{r}=\frac{\varepsilon_\nu}{1+\varepsilon_\nu}\delta_{\mu\nu}.\end{aligned}\right. \tag{10′}$$

In the first of these equations, the integration is over the inclusion volume (or area in the two-dimensional case), while in the second, it is over the region outside the object.

We now determine the asymptotic behavior (for $r\to\infty$) of $\psi_\nu(\mathbf{r})$. In order to do so, we define the Green's function of the Laplacian operator

$$\nabla_r^2 g(\mathbf{r}-\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'), \tag{11}$$

which is well known (see Ref. 2, for example) to be of the form

$$g(\mathbf{r}-\mathbf{r}')=-\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \quad (D=3),$$

$$g(\mathbf{r}-\mathbf{r}')=\frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'| \quad (D=2), \tag{12}$$

where D is the dimensionality of the space. Making use of (11) and integrating by parts, Eq. (4) can be written in integral form:

$$\psi_\nu(\mathbf{r})=-(1+\varepsilon_\nu)\int d\mathbf{r}'\theta(\mathbf{r}')\mathbf{e}_\nu(\mathbf{r}')\nabla_{r'}g(\mathbf{r}-\mathbf{r}'). \tag{13}$$

In the three-dimensional case, as $r\to\infty$, the function $g(\mathbf{r}-\mathbf{r}')$ takes the form

$$g(\mathbf{r}-\mathbf{r}')=-\frac{1}{4\pi}\left\{\frac{1}{r}+\frac{\mathbf{r}\mathbf{r}'}{r^3}+\ldots\right\}. \tag{14}$$

so that we obtain from (13) the following asymptotic expression for $\psi_\nu(\mathbf{r})$:

$$r\to\infty: \quad \psi_\nu(\mathbf{r})\approx\frac{\mathbf{d}_\nu\mathbf{r}}{r^3}, \quad \mathbf{d}_\nu=\frac{1+\varepsilon_\nu}{4\pi}\mathbf{u}_\nu \quad (D=3), \tag{15}$$

$$\mathbf{u}_\nu=\int\mathbf{e}_\nu(\mathbf{r})\theta(\mathbf{r})\,d\mathbf{r}. \tag{16}$$

Similarly, in the two-dimensional case, we obtain

$$r\to\infty: \quad \psi_\nu(\mathbf{r})\approx 2\frac{\mathbf{d}_\nu\mathbf{r}}{r^2}, \quad \mathbf{d}_\nu=\frac{1+\varepsilon_\nu}{4\pi}\mathbf{u}_\nu \quad (D=2). \tag{17}$$

According to (15) and (17), when $\mathbf{u}_\nu\neq 0(\mathbf{d}_\nu\neq 0)$ the function $\psi_\nu(\mathbf{r})$ becomes dipolar as $r\to\infty$ (note that such a function is not normalizable in the two-dimensional case). If in the state ν we have $\mathbf{u}_\nu=0(\mathbf{d}_\nu=0)$, then the trailing terms in the expansion (14) must be taken into account, so that as $r\to\infty$, $\psi_\nu(\mathbf{r})$ will be a quadrupole (or octupole, etc.). Note that such states (with $\mathbf{u}_\nu=\mathbf{d}_\nu=0$) do not contribute to the polarizability tensor of an inclusion in a uniform external field (see (26)).

As we have already remarked in the Introduction, the set of functions $\{\psi_\nu(\mathbf{r})\}$, and the set $\{\mathbf{e}_\nu(\mathbf{r})\}$ as well, is not complete in the usual sense. We can be assured of this by considering some exactly soluble cases. Because of the unique nature of the local modes, it is difficult to formulate relations which are the inverse of the orthogonality condition. We therefore limit ourselves to deriving a sufficient condition (the expandability condition) for solving the polarizability problem by expansion in the set $\{\psi_\nu(\mathbf{r})\}$. This is most simply accomplished as follows.

Using an expansion in the set $\{\psi_\nu(\mathbf{r})\}$, we find in section 3 an expression for the electric field strength $\mathbf{E}(\mathbf{r})$ (see (24)). Inside a perfectly conducting inclusion ($z\to\infty$), $\mathbf{E}(\mathbf{r})$ must vanish. Taking the limit as $z\to\infty$ in (24), we find from this requirement that (from here on, we assume that the function $\psi_\nu(\mathbf{r})$ is real)

$$\theta(\mathbf{r})\sum_\nu(1+\varepsilon_\nu)e_{\nu\alpha}(\mathbf{r})u_{\nu\beta}=\theta(\mathbf{r})\delta_{\alpha\beta}, \tag{18}$$

where $\mathbf{e}_\nu(\mathbf{r})$ and $\mathbf{u}_\nu$ are the same as in (7) and (16). (Note that Eq. (18) fails to converge to an identity only for points $\mathbf{r}$ inside the object). It is straightforward to show that when (18) is satisfied, the expression (23) derived in section 3 for the potential satisfies the original equation (3). Thus, (18) is the desired expandability condition, which is consistent with the requirement of physical sensibility. We may rewrite (18) in the form

$$\theta(\mathbf{r})\frac{\partial}{\partial x_\alpha}\left\{\sum_\nu(1+\varepsilon_\nu)\psi_\nu(\mathbf{r})u_{\nu\beta}-x_\beta\right\}=0. \tag{18'}$$

Up to a certain constant factor, the expression in the braces is the expansion (inside the object) of the vector $\mathbf{r}$ in the set of functions $\{\psi_\nu(\mathbf{r})\}$. It is therefore clear that the condition (18) is a rather weak constraint on $\{\psi_\nu(\mathbf{r})\}$.

Consideration of a number of specific shapes (cylinder sphere pair of cylinders—see section 6) indicates that (18) is valid for these cases. Equation (18) obviously also holds for an ellipsoid, since the field inside is uniform.[1] As we shall show later (see sections 3 and 4) when (18) is satisfied, one can consistently solve the polarizability problem in the general case as well. We therefore proceed from the assumption that the expandability condition (18) is valid for an arbitrarily shaped inclusion.

Note that integration of (18) with respect to $\mathbf{r}$ gives

$$\sum_\nu(1+\varepsilon_\nu)u_{\nu\alpha}u_{\nu\beta}=v\delta_{\alpha\beta}. \tag{19}$$

where v is the volume (area, for $D=2$) of the inclusion.

3. GENERAL EXPRESSION FOR THE POLARIZABILITY TENSOR

We now consider the polarizability problem for an inclusion in a uniform external electric field $\mathbf{E}_0$. We write the potential $\varphi(\mathbf{r})$ in the form

$$\varphi(\mathbf{r})=-\mathbf{E}_0\mathbf{r}+\psi(\mathbf{r}), \tag{20}$$

where the first term on the right-hand side of (20) corresponds to the field $\mathbf{E}_0$ and the second describes the distortion of the potential due to the inclusion. Substituting (20) into (3) gives an equation for $\psi(\mathbf{r})$:

$$\nabla^2\psi(\mathbf{r})-(1-z)\nabla[\theta(\mathbf{r})\nabla\psi(\mathbf{r})]=-(1-z)\nabla[\theta(\mathbf{r})\mathbf{E}_0]. \tag{21}$$

The function $\psi(\mathbf{r})$ is localized to the region surrounding the inclusion ($\psi(\mathbf{r})\to 0$ as $r\to\infty$), so it is natural to seek $\psi(\mathbf{r})$ as an expansion in the set $\{\psi_\nu(\mathbf{r})\}$:

$$\psi(\mathbf{r})=\sum_\nu A_\nu\psi_\nu(\mathbf{r}). \tag{22}$$

Here the $\psi_\nu(\mathbf{r})$ are the real eigenfunctions of the local-mode problem, and are associated with an object having the same geometry as the inclusion under consideration. Substituting (22) into (21) and making use of (4) and (10), we can determine the coefficients A_ν. The result is

$$\varphi(\mathbf{r})=-\mathbf{E}_0\mathbf{r}-E_{0\beta}(1-z)\sum_\nu(1+\varepsilon_\nu)\frac{\psi_\nu(\mathbf{r})u_{\nu\beta}}{z+\varepsilon_\nu}, \tag{23}$$

where the $\mathbf{u}_\nu$ are the same as in (16). Equation (23) satisfies Eq. (3) if (18) holds.

Equation (23) gives for the electric field $\mathbf{E}=-\nabla\varphi$

$$E_\alpha(\mathbf{r}) = E_{0\beta}\left\{\delta_{\alpha\beta} + (1-z)\sum_\nu (1+\varepsilon_\nu)\frac{e_{\nu\alpha}(\mathbf{r})u_{\nu\beta}}{z+\varepsilon_\nu}\right\}. \quad (24)$$

For a perfectly conducting ($z \to \infty$) inclusion with (18) taken into account, $\mathbf{E} = 0$ inside the object. The exterior surface ($S^{(1)}$) boundary condition for such an inclusion is then automatically satisfied: the tangential component of the electric field vanishes. If the inclusion is in fact a dielectric ($\sigma_2 = 0$, i.e., $z = 0$), then the normal component of the current density at $S^{(1)}$, $j_n = \sigma_1 E_n$ must vanish:

$$n_\alpha E_{0\beta}\left[\delta_{\alpha\beta} + \sum_\nu \frac{1+\varepsilon_\nu}{\varepsilon_\nu} e_{\nu\alpha}^{(1)}(\mathbf{r})u_{\nu\beta}\right]_{S^{(1)}} = 0.$$

Hence, making use of (5), we can transform to a condition at the interior surface $S^{(2)}$ which is satisfied identically by virtue of (18).

Far from the object ($r \to \infty$), the potential must be that of a dipole:

$$\varphi(\mathbf{r}) = -\mathbf{E}_0\mathbf{r} + \mathbf{pr}/r^3 + \ldots, \quad p_\alpha = \Lambda_{\alpha\beta}E_{0\beta}, \quad \hat{\Lambda} = v\hat{\alpha}. \quad (25)$$

Using (15), we determine from (23) the asymptotic form of the potential φ, whereupon by equating this to (25), we obtain an expression for the tensor $\hat{\Lambda}$:

$$\Lambda_{\alpha\beta}(z) = -\frac{1-z}{4\pi}\sum_\nu (1+\varepsilon_\nu)^2 \frac{u_{\nu\alpha}u_{\nu\beta}}{z+\varepsilon_\nu}. \quad (26)$$

When $r \to \infty$ in the two-dimensional case, we have

$$\varphi(\mathbf{r}) = -\mathbf{E}_0\mathbf{r} + 2\mathbf{pr}/r^2 + \ldots, \quad p_\alpha = \Lambda_{\alpha\beta}E_{0\beta}, \quad \hat{\Lambda} = s\hat{\alpha}, \quad (27)$$

where s is the area of the inclusion. Note that the dipole moment $\mathbf{p}$ determined via (27) is half that in Refs. 5 and 6. Using the same approach in this case ($D = 2$), we also obtain Eq. (26) for the tensor Λ.

Equation (26) provides a spectral representation of the polarizability tensor of an arbitrarily shaped macroscopic object (inclusion). The dependence of Λ on the argument σ_2/σ_1 is set out explicitly in (26). All of the information about the geometry of the inclusion is in fact contained in the quantities ε_ν and $\mathbf{u}_\nu$, which can be determined from the local-model problem, and are associated with the given shape. Equation (26) takes a more lucid form if we introduce the dipole moments of the local modes $\mathbf{d}_\nu$, as defined in (15) and (17):

$$\Lambda_{\alpha\beta}(z) = -4\pi(1-z)\sum_\nu \frac{d_{\nu\alpha}d_{\nu\beta}}{z+\varepsilon_\nu}. \quad (28)$$

It is straightforward to show that the tensor $\hat{\alpha} = v^{-1}\hat{\Lambda}$ is dimensionless, and therefore depends only on the shape of the object.

According to Refs. 5 and 6, the effective conductivity σ_e of a system with a low density of inclusions can be expressed in terms of their polarizability. The analytic properties of $\hat{\Lambda}$ are therefore the same in the complex z-plane as those of the conductivity (see Ref. 7), except at $z = \infty$, where in contrast to σ_e, Λ does not have a pole. In fact, in accord with Ref. 7, Eq. (26) is analytic over the entire z-plane except for the negative real axis. Along this semi-axis, the polarizability has simple poles (the spectrum ε_ν being discrete), which coalesce into a cut if the spectrum ε_ν is continuous. Note that like the conductivity, Eq. (26) satisfies a dispersion relation, which in the present case takes the form (compare with Ref. 7)

$$\hat{\Lambda}(z) = \hat{\Lambda}(\infty) - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\hat{\Lambda}(-t+i0)}{t+z}\,dt. \quad (29)$$

We remark that it is possible to arrive at a general expression for the polarizability tensor in a number of other ways. The dipole moment of an inclusion in a medium can be expressed as

$$\mathbf{p} = \frac{1}{4\pi\sigma_1}\int_v (\mathbf{j} - \sigma_1\mathbf{E})\,d\mathbf{r} = -\frac{1-z}{4\pi}\int_v \mathbf{E}\,d\mathbf{r}, \quad z = \frac{\sigma_2}{\sigma_1}. \quad (30)$$

The integration here is over the volume of the object. Substitution into (39) of the electric field $\mathbf{E}$ from (24) leads to the following expression for the tensor $\hat{\Lambda}$:

$$\Lambda_{\alpha\beta}(z) = -\frac{1-z}{4\pi}\left\{v\delta_{\alpha\beta} + (1-z)\sum_\nu (1+\varepsilon_\nu)\frac{u_{\nu\alpha}u_{\nu\beta}}{z+\varepsilon_\nu}\right\}. \quad (31)$$

It is not hard to see that Eqs. (26) and (31) become identical if (19) is satisfied. On the other hand, Eq. (19) follows from (31) as the condition for the absence of a pole of $\hat{\Lambda}$ at $z = \infty$.

According to §92 of Ref. 1, the scattering of electromagnetic waves from a small (relative to a wavelength) dielectric particle is completely determined by the polarizability of the particle. Consequently, dipolar local modes, particularly optical surface phonons, can be detected as resonances in the scattering cross section of long-wavelength electromagnetic radiation. For a particle in vacuum, $z(\omega)$ is the permittivity of its constituent material. In the frequency range where $z(\omega)$ is real and negative, there can be surface modes with frequencies ω_ν determined by the equation $z(\omega_\nu) = -\varepsilon_\nu$, where ε_ν is the spectrum of local modes associated with an object of the specified shape. If the spectrum ε_ν is discrete, then when $\omega \to \omega_\nu$,

$$\hat{\Lambda} \approx \hat{F}_\nu/(\omega - \omega_\nu),$$

where $\hat{F}_\nu$ is the corresponding oscillator strength. Consider two (geometrically) identical particles with different permittivities $z^{(1)}(\omega)$ and $z^{(2)}(\omega)$. The frequencies $\omega_\nu^{(1)}$ and $\omega_\nu^{(2)}$ of the surface modes of these particles, corresponding to a single value of ε_ν, are given by the equations $z^{(1)}(\omega_\nu^{(1)}) = -\varepsilon_\nu$ and $z^{(2)}(\omega_\nu^{(2)}) = -\varepsilon_\nu$. Thus $\omega_\nu^{(1)}$ and $\omega_\nu^{(2)}$ are related by

$$z^{(1)}(\omega_\nu^{(1)}) = z^{(2)}(\omega_\nu^{(2)}).$$

The spectral expansion (28) of $\hat{\Lambda}$ also gives a relation between the corresponding oscillator strengths:

$$\hat{F}_\nu^{(1)}[dz^{(1)}(\omega)/d\omega]_{\omega=\omega_\nu^{(1)}} = \hat{F}_\nu^{(2)}[dz^{(2)}(\omega)/d\omega]_{\omega=\omega_\nu^{(2)}}.$$

These two relations, which are easily established for objects having a simple geometry (sphere, cylinder), should be valid (although with different $z(\omega)$) for particles of any shape.

4. PERTURBATION THEORY

If the conductivity σ_2 of an inclusion is almost the same as the conductivity σ_1 of the medium ($|(\sigma_1 - \sigma_2)/\sigma_1| \ll 1$, i.e., $|1 - z| \ll 1$), then we can use standard perturbation theory, expanding in the small parameter $1 - z$, to calculate the potential $\varphi(\mathbf{r})$ and the polarizability tensor $\hat{\Lambda}$ (see Ref. 6, for example). This does not mean, however, that an analogous perturbation theory can also be developed for the quantities ε_ν and $\psi_\nu(\mathbf{r})$, since the appropriate parameter in those cases is not small: $|1 - z(\omega_\nu)| = 1 + \varepsilon_\nu \geqslant 1$. It can thus be shown

that when $|1-z| \ll 1$, the method of the previous section is no longer valid, since it requires, at first glance, a preliminary accurate determination of ε_ν and $\psi_\nu(\mathbf{r})$.

It turns out, however, that the appropriate expansions for $\varphi(\mathbf{r})$ and $\hat{\Lambda}$ in powers of $1-z$ can also be derived directly from the general expressions (23) and (26), without calculating ε_ν and $\psi_\nu(\mathbf{r})$. In deriving these expansions, we make use of certain sum rules stemming from (18). The fact that the series for $\varphi(\mathbf{r})$ and $\hat{\Lambda}$ derived in this way are the same as the results obtained from standard perturbation theory is indicative of the consistency of the method employed in the present paper, and of the assumption made in deriving it that Eq. (18) is valid.

To obtain the perturbation theory series in $1-z$, we write Eq. (3) in integral form:

$$\varphi(\mathbf{r}) = -\mathbf{E}_0\mathbf{r} - (1-z)\int d\mathbf{r}'\theta(\mathbf{r}')\frac{\partial\varphi(\mathbf{r}')}{\partial x_\alpha'}\frac{\partial g(\mathbf{r}-\mathbf{r}')}{\partial x_\alpha'}. \quad (32)$$

The function $g(\mathbf{r}-\mathbf{r}')$ here is defined by (11) and (12). Carrying out an iterative solution of (32),

$$\varphi(\mathbf{r}) = \varphi^{(0)}(\mathbf{r}) + \varphi^{(1)}(\mathbf{r}) + \varphi^{(2)}(\mathbf{r}) + \ldots, \quad (33)$$

where $\varphi^{(n)} \propto (1-z)^n$, we obtain

$$\varphi^{(0)}(\mathbf{r}) = -\mathbf{E}_0\mathbf{r}, \quad (34)$$

$$\varphi^{(n)}(\mathbf{r}) = -(1-z)\int d\mathbf{r}'\theta(\mathbf{r}')\frac{\partial\varphi^{(n-1)}(\mathbf{r}')}{\partial x_\alpha'}\frac{\partial g(\mathbf{r}-\mathbf{r}')}{\partial x_\alpha'}, \quad n \geqslant 1. \quad (35)$$

Together with (34), the recursion relation (35) enables one to find any term of the perturbation series for the potential $\varphi(\mathbf{r})$. From (32)–(35), it is also easy to obtain the corresponding expansion for the tensor $\hat{\Lambda}$ (see Ref. 6, for example).

Let us now find the expansion for a potential of the form (33) from the general equation (23). The latter equation implies that $\varphi^{(0)}(\mathbf{r})$ is given by (34), and when $n \geqslant 1$, we obtain

$$\varphi^{(n)}(\mathbf{r}) = -(1-z)^n E_{0\beta} B_n^{\beta}(\mathbf{r}), \quad n \geqslant 1, \quad (36)$$

$$B_n^{\beta}(\mathbf{r}) = \sum_\nu \frac{\psi_\nu(\mathbf{r})u_{\nu\beta}}{(1+\varepsilon_\nu)^{n-1}}. \quad (37)$$

In order to determine the quantities $B_n^{\beta}(\mathbf{r})$ (i.e., the aforementioned sum rules), we multiply Eq. (13) by $u_{\nu\beta}(1+\varepsilon_\nu)^{1-n}$ and sum over ν. The result is

$$B_n^{\beta}(\mathbf{r}) = -\int d\mathbf{r}'\theta(\mathbf{r}')\frac{\partial B_{n-1}^{\beta}(\mathbf{r}')}{\partial x_\alpha'}\frac{\partial g(\mathbf{r}-\mathbf{r}')}{\partial x_\alpha'}, \quad n \geqslant 1, \quad (38)$$

where, according to (18),

$$\theta(\mathbf{r})\partial B_0^{\beta}(\mathbf{r})/\partial x_\alpha = \theta(\mathbf{r})\delta_{\alpha\beta}. \quad (39)$$

Equations (38) and (39) make it possible to find the desired values of $B_n^{\beta}(\mathbf{r})$. It follows from (36)–(38) that the $\varphi^{(n)}(\mathbf{r})$ in (36) satisfy the recursion relation (35). Thus, an expansion of (23) in powers of $1-z$ gives the same results as standard perturbation theory.

Similarly, we can find an expansion for the polarizability tensor as well. Accurate to $(1-z)^2$, we find from (26) that

$$\Lambda_{\alpha\beta} = -\frac{1}{4\pi}(1-z)\sum_\nu (1+\varepsilon_\nu)u_{\nu\alpha}u_{\nu\beta} - \frac{1}{4\pi}(1-z)^2\sum_\nu u_{\nu\alpha}u_{\nu\beta} + \ldots. \quad (40)$$

To calculate the term of order $(1-z)$, we make use of Eq. (19). For the term in $(1-z)^2$, we note that from (38) and (39), with $n=1$,

$$\sum_\nu \psi_\nu(\mathbf{r})u_{\nu\beta} = -\int d\mathbf{r}'\theta(\mathbf{r}')\frac{\partial g(\mathbf{r}-\mathbf{r}')}{\partial x_\beta'}. \quad (41)$$

In obvious fashion, we then obtain from (41) a string of equations:

$$\sum_\nu e_{\nu\alpha}(\mathbf{r})u_{\nu\beta} = \int d\mathbf{r}'\theta(\mathbf{r}')\frac{\partial^2 g(\mathbf{r}-\mathbf{r}')}{\partial x_\alpha \partial x_\beta}, \quad (42)$$

$$\sum_\nu \mathbf{e}_\nu(\mathbf{r})\mathbf{u}_\nu = \theta(\mathbf{r}), \quad (43)$$

$$\sum_\nu u_{\nu\alpha}u_{\nu\beta} = \int d\mathbf{r}\,\theta(\mathbf{r})\int d\mathbf{r}'\theta(\mathbf{r}')\frac{\partial^2 g(\mathbf{r}-\mathbf{r}')}{\partial x_\alpha \partial x_\beta}, \quad (44)$$

$$\sum_\nu \mathbf{u}_\nu^2 = v. \quad (45)$$

Substituting (19) and (44) into (40) gives the same results as in Ref. 6. In particular,

$$\mathrm{Sp}\,\hat{\Lambda} = -\frac{v}{4\pi}\{D(1-z) + (1-z)^2 + \ldots\}, \quad (46)$$

which agrees with Ref. 6.

5. INTERACTION OF INCLUSIONS

In considering an ensemble of particles (inclusions), it is sometimes necessary to take account of mutual effects ("interaction") which lead to changes in the spectrum of local modes. Thus, when two identical particles interact there is a splitting of local levels that correspond to an isolated inclusion. Averaging over various pairs of particles gives rise to a peaked broadening of these levels. This broadening must be taken into consideration when one investigates the analytic properties of the electrical conductivity of two-component media.[6,7] The same effect must also be taken into account in estimating resonance widths in cross sections for long-wavelength electromagnetic scattering from a system of particles.

We consider a pair of identical inclusions, and find the splitting of some local level associated with an isolated inclusion, due to the interaction between the defects. As for the local level, we assume that it is nondegenerate and corresponds to a dipole state. We also assume that the distance ρ between inclusions is large compared with their characteristic size R, so that we can make use of perturbation theory (in the parameter $R/\rho \ll 1$), as in quantum mechanics,[10] to find the splitting.

The equation for the eigenfunctions $\psi_\nu(\mathbf{r})$ and eigenvalues ε_ν of a system consisting of a pair of inclusions is of the form (4), where

$$\theta(\mathbf{r}) = \theta^{(1)}(\mathbf{r}-\mathbf{r}_1) + \theta^{(2)}(\mathbf{r}-\mathbf{r}_2). \quad (47)$$

Here $\theta^{(i)}(\mathbf{r}-\mathbf{r}_i)$ refers to the ith inclusion ($i = 1,2$) and $\mathbf{r}_i$ is the location of its center. For an individual inclusion, we have

$$\nabla^2\psi_\lambda^{(i)}(\mathbf{r}-\mathbf{r}_i)-(1+\varepsilon_\lambda)\nabla[\theta^{(i)}(\mathbf{r}-\mathbf{r}_i)\nabla\psi_\lambda^{(i)}(\mathbf{r}-\mathbf{r}_i)]=0, \tag{48}$$

where ε_λ is the unperturbed level. We seek a solution of (4), with $\theta(\mathbf{r})$ taken from (47), in the form (omitting the subscript ν)

$$\psi(\mathbf{r})=A\psi_\lambda^{(1)}(\mathbf{r}-\mathbf{r}_1)+B\psi_\lambda^{(2)}(\mathbf{r}-\mathbf{r}_2). \tag{49}$$

Substituting (49) into (4) (taking (47) into account), and after some manipulations involving Eq. (48), we obtain

$$A\frac{\varepsilon_\lambda-\varepsilon}{1+\varepsilon_\lambda}\nabla^2\psi_\lambda^{(1)}(\mathbf{r}-\mathbf{r}_1)+B\frac{\varepsilon_\lambda-\varepsilon}{1+\varepsilon_\lambda}\nabla^2\psi_\lambda^{(2)}(\mathbf{r}-\mathbf{r}_2)$$
$$-A(1+\varepsilon)\nabla[\theta^{(2)}(\mathbf{r}-\mathbf{r}_2)\nabla\psi_\lambda^{(1)}(\mathbf{r}-\mathbf{r}_1)]$$
$$-B(1+\varepsilon)\nabla[\theta^{(1)}(\mathbf{r}-\mathbf{r}_1)\nabla\psi_\lambda^{(2)}(\mathbf{r}-\mathbf{r}_2)]=0. \tag{50}$$

Multiply (50) by $\psi_\lambda^{(1)}(\mathbf{r}-\mathbf{r}_1)$ and integrate over all $\mathbf{r}$; then multiply by $\psi_\lambda^{(2)}(\mathbf{r}-\mathbf{r}_2)$ and likewise integrate over all space. As a result, using the normalization condition (9) and omitting overlap integrals of order $1/\rho^{D+1}$ and higher, we obtain

$$A\frac{\varepsilon-\varepsilon_\lambda}{1+\varepsilon_\lambda}+B(1+\varepsilon)J_1=0,\quad B\frac{\varepsilon-\varepsilon_\lambda}{1+\varepsilon_\lambda}+A(1+\varepsilon)J_2=0, \tag{51}$$

$$J_i=\int d\mathbf{r}\,\theta^{(i)}(\mathbf{r}-\mathbf{r}_i)\nabla\psi_\lambda^{(1)}(\mathbf{r}-\mathbf{r}_1)\nabla\psi_\lambda^{(2)}(\mathbf{r}-\mathbf{r}_2). \tag{52}$$

To calculate the overlap integrals (52) when $\rho\gg R$ ($\rho=\mathbf{r}_1-\mathbf{r}_2$, R is the characteristic size of an inclusion), we make use of the asymptotic expressions (15) and (17). We find then that $J_1=J_2=J$, where

$$J(\rho)=\frac{\partial\psi_\lambda^{(2)}(\rho)}{\partial\rho}\mathbf{u}_\lambda^{(1)}=\frac{\partial\psi_\lambda^{(1)}(\rho)}{\partial\rho}\mathbf{u}_\lambda^{(2)},\qquad \rho=\mathbf{r}_1-\mathbf{r}_2, \tag{53}$$

$$\mathbf{u}_\lambda^{(i)}=\int\nabla\psi_\lambda^{(i)}(\mathbf{r})\theta^{(i)}(\mathbf{r})\,d\mathbf{r}. \tag{54}$$

For $\psi_\lambda^{(i)}$ in (53), we must take Eqs. (15) and (17). Thus, the in three-dimensional case, we find from (53) and (15) that

$$J(\rho)=\frac{1+\varepsilon_\lambda}{4\pi}\{\rho^2\mathbf{u}_\lambda^{(1)}\mathbf{u}_\lambda^{(2)}-3(\rho\mathbf{u}_\lambda^{(1)})(\rho\mathbf{u}_\lambda^{(2)})\}\rho^{-5}, \tag{55}$$

from which it follows that $\mathbf{J}(\rho)$ is of order $(R/\rho)^3$.

The condition for the set of equations (51) to be solvable, bearing in mind that $\mathbf{J}_1=\mathbf{J}_2$, is $A^2=B^2$, and hence $B=\pm A$. Thus, as in quantum mechanics,[10] the eigenfunctions of the two-center problem are given by symmetric and antisymmetric combinations of functions belonging to an isolated inclusion. The levels for these states are

$$\varepsilon^{\pm}(\rho)=\varepsilon_\lambda\mp(1+\varepsilon_\lambda)^2J(\rho) \tag{56}$$

with $\mathbf{J}(\rho)$ from (53).

As noted in Ref. 7, the imaginary part of the function $f=\sigma_e/\sigma_1$ on the branch cut (for $z=-t+i0$, $t>0$) plays an important role in the theory of conductivity of two-component media. Following the development in Ref. 6, let us investigation the form of Im f in the neighborhood of the isolated local level ε_λ when there is a low number density of inclusions ($c\ll1$). According to Ref. 6, to terms of the order of the square of the number density, we have ($t\neq\varepsilon_\lambda$)

$$\mathrm{Im}\,f\sim N^2\,\mathrm{Im}\int\mathrm{Sp}\,\hat{\Lambda}^{(2)}(\rho)\,d\rho, \tag{57}$$

where $\hat{\Lambda}^{(2)}$ is the polarizability tensor of a pair of inclusions, and N is their number density. Making use of (26) and the results of the present section, we find from (57) (assuming that $\varepsilon_\lambda\sim1$) that

$$\mathrm{Im}\,f\sim c^2/|t-\varepsilon_\lambda|^2,\quad c\ll|t-\varepsilon_\lambda|\ll1. \tag{58}$$

Here c is the dimensionless number density of inclusions (the fractional volume that they occupy), and $c\ll1$.

Equation (58) holds for both two-dimensional (compare with Ref. 6) and three-dimensional systems. The constraints on $|t-\varepsilon_\lambda|$ indicated in (58) have the following origin. Distances $\rho\gg R$ are responsible for the condition $|t-\varepsilon_\lambda|\ll1$, which was put to important use above. (The range $|t-\varepsilon_\lambda|\sim1$ corresponds to distances $\rho\sim R$, so that here Im f does not take on a universal form.) On the other hand, the interaction radius of inclusions ($\sim R/|t-\varepsilon_\lambda|^{1/D}\gg R$) should be small compared with the mean distance between inclusions ($\sim N^{-1/D}$), which is the condition for applicability of the c^2 approximation.[6] This gives the second constraint on $|t-\varepsilon_\lambda|$ indicated in (58).

According to (58), at the boundary of the region of applicability ($|t-\varepsilon_\lambda|\sim c$), the quantity Im f is of order unity. To study the region $|t-\varepsilon_\lambda|<c$, one must go beyond the scope of the c^2 approximation. If we digress from consideration of this narrow range of values of t, the imaginary part of f at $t\sim\varepsilon_\lambda$ will have a sharp peak of height ~1 and width $\sim c$. The combined width of the resonance lines in the long-wavelength electromagnetic scattering cross section from an ensemble of particles is of the same order of magnitude ($\sim c\ll1$).

6. EXACTLY SOLUBLE CASES

The exact eigenvalues ε_ν and eigenfunctions $\psi_\nu(\mathbf{r})$ of the local modes can be obtained for certain simply-shaped objects. As our first example, we examine a spherical inclusion. In the polarizability problem, it is most convenient to employ real eigenfunctions. We therefore introduce two types of spherical harmonics $\mathbf{Y}_{lm}^{(\lambda)}(\theta,\varphi)$ ($\lambda=1,2$), which are the real and imaginary parts of the usual spherical harmonics $Y_{lm}(\theta,\varphi)$ (using standard notation—see Ref. 10):

$$Y_{lm}^{(1)}(\theta,\varphi)=a_{lm}P_l^m(\cos\theta)\cos m\varphi,\qquad 0\leqslant m\leqslant l,$$
$$Y_{lm}^{(2)}(\theta,\varphi)=a_{lm}P_l^m(\cos\theta)\sin m\varphi,\qquad 1\leqslant m\leqslant l,$$
$$a_{lm}=\left[\frac{2l+1}{2\pi(1+\delta_{m0})}\frac{(l-m)!}{(l+m)!}\right]^{1/2},$$
$$\int Y_{lm}^{(\lambda)}(\theta,\varphi)\,Y_{l'm'}^{(\lambda')}(\theta,\varphi)\,d\sigma=\delta_{\lambda\lambda'}\delta_{ll'}\delta_{mm'}. \tag{59}$$

According to (9), inside (i) and outside (e) a sphere of radius R, the normalized eigenfunctions in the local-mode problem for a sphere take the form

$$\psi_{\lambda lm}^{(i)}(r,\theta,\varphi)=[(2l+1)R]^{-1/2}(r/R)^lY_{lm}^{(\lambda)}(\theta,\varphi)\quad r<R,$$
$$\psi_{\lambda lm}^{(e)}(r,\theta,\varphi)=[(2l+1)R]^{-1/2}(R/r)^{l+1}Y_{lm}^{(\lambda)}(\theta,\varphi)\quad r>R. \tag{60}$$

Here $0\leqslant m\leqslant l$ for $\lambda=1$, and $1\leqslant m\leqslant l$ for $\lambda=2$ and $l=1$. The functions (60) correspond to the eigenvalues

$$\varepsilon_{\lambda lm}=\varepsilon_l=(l+1)/l,\ l=1,\ 2,\ \ldots\ , \tag{61}$$

which are degenerate in λ and m. It is straightforward to demonstrate that the eigenfunctions (60) satisfy both the full and "local" orthogonality conditions (10) and (10′) with $\nu=\{\lambda lm\}$.

Only the three functions with $l=1$ (ψ_{111}, ψ_{211}, and ψ_{110}) show dipolar behavior outside the sphere, with corresponding vectors $\mathbf{d}_\nu$ from (15):

$$\mathbf{d}_{111}=\frac{R^{3/2}}{(4\pi)^{1/2}}\{1,0,0\},\quad \mathbf{d}_{211}=\frac{R^{3/2}}{(4\pi)^{1/2}}\{0,1,0\},$$

$$\mathbf{d}_{110}=\frac{R^{3/2}}{(4\pi)^{1/2}}\{0,0,1\}. \tag{62}$$

For a sphere with isotropic polarizability, (28) gives the well known relation[1]

$$\Lambda=-R^3\frac{1-z}{2+z}=-R^3\frac{\sigma_1-\sigma_2}{2\sigma_1+\sigma_2}. \tag{63}$$

In the present case, the sums in (18) and (19) contain only the first term, so these relations converge to trivial identities.

As we have noted, the fairly simple form of Eq. (63) is related to the fact that for the sphere, only three eigenfunctions out of the whole set $\{\psi_\nu(\mathbf{r})\}$ are dipolar: the local modes associated with an ellipsoid also possess this property. These examples do not enable us to convincingly demonstrate the effectiveness of the spectral method in calculating the polarizability. Notice that according to the results of the previous section, taking the interaction (for $\rho\gg R$) of inclusions into account leads to a splitting of the local levels and an increase in the number of eigenfunctions with dipolar behavior. A more complicated picture is to be expected for distances ρ comparable to the size of the inclusions. It would be more interesting and convincing to test the spectral approach in just such a nontrivial situation. Unfortunately, the polarizability problem for two spheres separated by an arbitrary distance does not have a closed-form analytic solution: however, the corresponding two-dimensional problem for a pair of circular cylinders can be solved exactly (see Ref. 6, for example). As our second example, we therefore examine the spectral properties of this two-dimensional system.

The problem of the spectrum of local modes for a pair of circular cylinders (see Ref. 6) can be solved in bipolar coordinates.[2] The centers of the circles of radius R are placed on the x-axis symmetrically about the point $x=0$ (see Fig. 1 in Ref. 6). We denote the distance between the centers of the inclusions $\rho(\rho\geqslant 2R)$. The bipolar coordinates (ξ,θ) are defined by[2]

$$x=\frac{a\,\mathrm{sh}\,\xi}{\mathrm{ch}\,\xi+\cos\theta}\qquad y=\frac{a\sin\theta}{\mathrm{ch}\,\xi+\cos\theta}. \tag{64}$$

The boundary of the right-hand circle is $\xi=\xi_0$, and its interior is $\xi>\xi_0$. Likewise, for the left-hand circle we have $\xi\leqslant-\xi_0$, and the region outside the inclusions is specified by $-\xi_0\leqslant\xi\leqslant\xi_0$. With this geometry, (64) gives

$$\xi_0=\ln\{[\rho+(\rho^2-4R^2)^{1/2}]/2R\},\quad a=\tfrac{1}{2}(\rho^2-4R^2)^{1/2}. \tag{65}$$

As noted in Refs. 6 and 7, two sets of eigenvalues are associated with a pair of cylinders of the same radius: $\varepsilon_{1n}=\tanh n\xi_0$ and $\varepsilon_{2n}=\coth n\xi_0$ ($n=1,2,\ldots$). Four types of eigenfunctions $\psi_{\lambda n}(\mathbf{r})$ ($\lambda=1,2,3,4$) are found to correspond to these, so that each of the levels ε_{1n} and ε_{2n} is doubly degenerate. The normalized eigenfunctions of the first type (corresponding to ε_{1n}) are of the form

$$\begin{aligned}
&\psi_{1n}^{(e)}=A_n\,\mathrm{ch}\,n\xi\sin n\theta,\qquad |\xi|\leqslant\xi_0,\\
&\psi_{1n}^{(1)}=A_n\,\mathrm{ch}\,n\xi_0e^{-n(\xi-\xi_0)}\sin n\theta,\qquad \xi\geqslant\xi_0.\\
&\psi_{1n}^{(2)}=A_n\,\mathrm{ch}\,n\xi_0e^{n(\xi+\xi_0)}\sin n\theta,\qquad \xi\leqslant-\xi_0,\\
&A_n=[(1-\varepsilon_{1n})/2\pi n]^{1/2},\qquad \varepsilon_{1n}=\mathrm{th}\,n\xi_0.
\end{aligned} \tag{66}$$

Here $\psi^{(e)}$, $\psi^{(1)}$, and $\psi^{(2)}$ denote functions ψ in the exterior region, inside the right-hand inclusion, and inside the left-hand inclusion respectively. Comparing the asymptotic expression for $\psi_{1n}^{(e)}$ with (17) ($\xi\approx 2ax/r^2$ and $\theta\approx\pi-2ayr^{-2}$ as $r\to\infty$), we find the components of the vector $\mathbf{d}_{1n}$:

$$(d_{1n})_x=0,\ (d_{1n})_y=(-1)^{n+1}naA_n. \tag{67}$$

The normalized eigenfunctions of the second type (corresponding to ε_{2n}) take the form

$$\begin{aligned}
&\psi_{2n}^{(e)}=B_n\,\mathrm{sh}\,n\xi\cos n\theta,\qquad |\xi|\leqslant\xi_0,\\
&\psi_{2n}^{(1)}=B_n\,\mathrm{sh}\,n\xi_0e^{-n(\xi-\xi_0)}\cos n\theta,\qquad \xi\geqslant\xi_0,\\
&\psi_{2n}^{(2)}=-B_n\,\mathrm{sh}\,n\xi_0e^{n(\xi+\xi_0)}\cos n\theta,\qquad \xi\leqslant-\xi_0,\\
&B_n=[(\varepsilon_{2n}-1)/2\pi n]^{1/2},\ \varepsilon_{2n}=\mathrm{cth}\,n\xi_0.
\end{aligned} \tag{68}$$

In the same manner, we find the components of the vector $\mathbf{d}_{2n}$:

$$(d_{2n})_x=(-1)^n naB_n,\qquad (d_{2n})_y=0. \tag{69}$$

Functions ψ_{3n} of the third type (with eigenvalues $\varepsilon_{3n}=\varepsilon_{1n}$) are obtained from (66) via the replacement $\sin n\theta\to\cos n\theta$. Finally, functions ψ_{4n} of the fourth type (with levels $\varepsilon_{4n}=\varepsilon_{2n}$) are obtained from (68) by replacing $\cos n\theta$ with $\sin n\theta$. There are no dipolar terms in the asymptotic expansion of ψ_{3n} and ψ_{4n}, so $\mathbf{d}_{3n}=\mathbf{d}_{4n}=0$. It is straightforward to show that the set of eigenfunctions $\{\psi_{\lambda n}\}$ satisfies all of the orthogonality relations; conditions (18) and (19) are satisfied as well. The sum in (18) is then a series expansion of x and y in terms of $\psi_{1n}^{(1)}$, $\psi_{1n}^{(2)}$ and $\psi_{2n}^{(1)}$, $\psi_{2n}^{(2)}$ (compare with Eq. (2.2) in the Appendix of Ref. 6).

From the general equation (28), we obtain

$$\begin{aligned}
&\Lambda_{xx}=-2a^2(1-z)\sum_{n=1}^{\infty}n\frac{\varepsilon_{2n}-1}{z+\varepsilon_{2n}},\\
&\Lambda_{yy}=-2a^2(1-z)\sum_{n=1}^{\infty}n\frac{1-\varepsilon_{1n}}{z+\varepsilon_{1n}}
\end{aligned} \tag{70}$$

for the diagonal terms of the polarizability tensor of a pair of circular cylinders. A spectral expansion for $\hat{\Lambda}$ of the same form as (70) has been obtained in Ref. 6 using the customary method, by solving the electrostatic problem. When comparing, it must be borne in mind that it has been assumed in Eq. (48) of Ref. 6 that $z=-\omega^2/\Omega^2$, $\varepsilon_{1n}=\omega_{1n}^2/\Omega^2$, $\varepsilon_{2n}=\omega_{2n}^2/\Omega^2$. Furthermore, because of a different definition of the dipole moment (compare Eq. (27) of the present paper with Eq. (22) of Ref. 6), the result in Ref. 6 must be reduced by a factor of two for consistency with (70). With

these remarks in mind, Eq. (70) is the same as Eq. (48) of Ref. 6. Also note that with the method of Ref. 6, the electric potential is in fact an expansion in the set of eigenfunctions $\{\psi_{1n}\}$ and $\{\psi_{2n}\}$, so that the two approaches are equivalent even at this level.

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[1]L. D. Landau and E. M. Lifshitz, Electrodinamika sploshnykh sred, Nauka, Moscow (1982) (Electrodynamics of Continuous Media, Pergamon Press, 1984).
[2]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Vol. 2, McGraw-Hill, New York (1953), Ch. 10.
[3]W. R. Smythe, *Static and Dynamic Electricity*, 2nd ed., McGraw-Hill, New York (1950).
[4]V. M. Finkelberg, Dokl. Akad Nauk SSSR **152**, 320 (1963).
[5]B. Ya. Balagurov, Zh. Tekh. Fiz. **52**, 850 (1982) [Sov. Phys. Tech. Phys. **27**, 544 (1982)].
[6]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **89**, 1796 (1985) [Sov. Phys. JETP **62**, 1036 (1985)].
[7]B. Ya. Balagurov, Zh. Eksp. Teor. Fiz. **88**, 1664 (1985) [Sov. Phys. JETP **61**, 991 (1985)].
[8]V. M. Agranovich, Usp. Fiz. Nauk **115**, 199 (1975) [Sov. Phys. Usp. **18**, 99 (1975)].
[9]V. V. Bryksin, D. N. Mirlin, and Yu. A. Firsov, Usp. Fiz. Nauk **113**, 29 (1974).
[10]L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika, Nauka, Moscow (1974) (Quantum Mechanics, Pergamon, 1978), §50, 81.

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