On the stability of the Kolmogorov spectra of weak turbulence

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(Submitted 17 October 1986)
Zh. Eksp. Teor. Fiz. **93**, 172–177 (July 1987)

The problem of the stability of the Kolmogorov spectra of weak turbulence is analytically solved for the first time. The spectrum of the isotropic perturbations of the steady-state distribution of the capillary waves on the surface of shallow water is found. It is shown that the Kolmogorov solution is stable against excitations of packets localized in **k** space: the packets are carried into the runoff region without increasing in size.

INTRODUCTION

The Kolmogorov spectra of turbulence are universal steady-state distributions specified by one macroscopic parameter: the flux of the corresponding integral of the motion. In the theory of weak turbulence the Kolmogorov spectra can be obtained as exact solutions to the corresponding kinetic equations.[1-3] There, however, still remains open the question of the spectrum of weak perturbations in the background of the steady-state solutions, in particular, the question of the stability of the latter. Unfortunately, in contrast to the case of equilibrium distributions, which are also specified by one macroscopic parameter, in the case of highly nonequilibrium systems we do not (perhaps for the present) have universal criteria for stability (i.e., criteria of the type of the Boltzmann H theorem in conjunction with the condition that the entropy be an extremum at equilibrium). Thus, we must solve the problem of the stability of the Kolmogorov solutions separately in each specific case.

As will be shown in the present paper, there is a particular class of Kolmogorov spectra for which the solution to the stability problem is very simple. To describe this class, we must distinguish between the proper and improper Kolmogorov spectra. We shall, following Zakhavrov,[3] refer to the situation in which the major portion of the turbulence energy is concentrated in a region of size of the order of that of the course as the proper Kolmogorov situation in energy terms. For example, for the cases in which the three-wave interactions are allowed, and the source of the waves occur in a region of small **k**, this means that the energy integral

$$E=\int \varepsilon_k\, d\mathbf{k}=\int \omega_k n_k\, d\mathbf{k} \infty k^{\alpha+d-s_0}$$

diverges as $k\to 0$, i.e, that $\delta=\alpha+d-s_0<0$. Here the wave frequency $\omega_k \propto k^{\alpha}$, the Kolmogorov spectrum $n_k \propto k^{-s_0}$, and d is the dimensionality of the space. Correspondingly, we shall call the situation in which the opposite sign of δ is observed, i.e., in which $\delta>0$, the improper Kolmogorov situation: in this situation the energy (the flux of which realizes the Kolmogorov spectrum) accumulates in the region of large k. The nature of the establishment of the Kolmogorov spectrum should depend essentially on the sign of δ (see Ref. 3). The kinetic equation admits of self-similar solutions $n_k(t)=f(t)\varphi(\xi)$ that depend on the variable[3] $\xi=kt^{-1/\delta}$. For $\delta>0$ this solution describes a Kolmogorov-spectrum establishment wave propagating to the right ($k_{\text{bound}} \propto t^{1/\delta}$). In the proper Kolmogov situation ($\delta<0$) the steady-state solution is established in the region of large k in a non-self-similar fashion (see, for example, the results of the numerical experiment reported in Ref. 4). Apparently, the self-similar solutions in this case describe a wave propagating to the left from the source, or the evolution of the initial distribution without energy pumping.

In the present paper we consider the intermediate situation $\delta=0$ (the symmetric case), to which pertains, as will be shown in §1, the physically interesting case of capillary-wave turbulence on the surface of shallow water. The kinetic equation describing the evolution of the occupation numbers $n_{\mathbf{k}}(t)$ for the waves can be written in two equivalent forms:

$$\frac{\partial n_{\mathbf{k}}}{\partial t}=I_{\mathbf{k}}\{n_{\mathbf{k}'}\},\quad \frac{\partial n_{\mathbf{k}}}{\partial t}\omega_k k^{d-1}=\frac{\partial \varepsilon_k}{\partial t}=-\frac{\partial P_k}{\partial k},$$

where the energy flux P_k in **k** space can be expressed in terms of the collision integral I_k as follows:

$$P_k=-\int^{k} \omega_k I_k\, d\mathbf{k}.$$

The Kolmogorov spectrum n_k^0 is the steady-state solution that realizes a constant energy flux, i.e., the situation in which $\partial P_k/\partial k=0$; therefore, $I_k\{n_{k'}^0\}\propto k^{-\alpha-d}$ (this means that $I_{\lambda k}\{n_{\lambda k'}^0\}=\lambda^{-\alpha-d} I_k\{n_{k'}^0\}$). As to the operator of the linearized—against the background of n_k^0—kinetic equation

$$\frac{\partial}{\partial t}\delta n_{\mathbf{k}}=\hat{L}\delta n_{\mathbf{k}}, \tag{1}$$

as can easily be seen, it has a homogeneity index equal to δ: $\hat{L}\propto k^{s_0-\alpha-d}$. In the $\delta=0$ case the eignefuctions of the linearized kinetic equation are obvious: they are the power functions

$$\delta n_k^{(s)}=k^{-s}e^{\Gamma(s)t}. \tag{2}$$

Thus, the stability problem reduces to the problem of finding the function $\Gamma(s)$ [by direct evaluation of the integrals in (1)] and the study of the evolution of localized perturbations that are superpositions of the eigenfunctions (2):

$$\delta n_k(t)=\int_{\gamma} a(s)k^{-s}e^{\Gamma(s)t}\, ds, \tag{3}$$

where $a(s)$ and the contour γ are such that $\delta n_k\to 0$ as $k\to 0$, ∞.

The paper is organized as follows. In §1 we find the Kolmogorov solution for weak capillary-wave turbulence on the surface of shallow water and its perturbation spectrum in the case when the perturbations are weak and isotropic (i.e., depend only on the modulus k), and have the form (2); §2 is

 0038-5646/87/070097-04$04.00

devoted to the physical interpretation of the solutions obtained, i.e., to the study of the evolution of perturbations localized in k space.

§1. THE STEADY-STATE SOLUTION AND THE SPECTRUM OF WEAK PERTURBATIONS

Let us consider the waves on the surface of a liquid of small depth. In the long wave (i.e., $k \to 0$) limit their dispersion law is close to the dispersion law for sound:

$$\omega_k = (gh)^{1/2} k\left[1+\left(\frac{\beta}{\rho g}-\frac{h^2}{3}\right)k^2\right] \equiv ck(1+a^2k^2), \qquad (4)$$

where h and ρ are the depth and density of the liquid, β is the coefficient of surface tension, and g is the acceleration due to gravity. We shall limit ourselves to the case when $3\beta > \rho g h^2$ and Eq. (4) represents a decay dispersion law, the three-wave processes are allowed, and the evolution of the occupation numbers n_k for the waves is governed by the kinetic equation

$$\frac{\partial n_\mathbf{k}}{\partial t} = \int |V_{k12}|^2 \delta(\omega_k-\omega_1-\omega_2)\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$$

$$\times [n_1n_2 - n_\mathbf{k}(n_1+n_2)]\,d\mathbf{k}_1\,d\mathbf{k}_2$$

$$-2\int |V_{1k2}|^2 \delta(\omega_1-\omega_k-\omega_2)\,\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)$$

$$\times [n_\mathbf{k}n_2 - n_1(n_\mathbf{k}+n_2)]\,d\mathbf{k}_1\,d\mathbf{k}_2. \qquad (5)$$

Here the matrix element of the three-wave interaction in the long-wave limit has the squared modulus $|V_{kk_1k_2}|^2 = bkk_1k_2$ (see Ref. 2), and is scale-invariant, with a homogeneity index $m = 3/2$. As to the dispersion law (4), it is approximately scale-invariant when $ak \ll 1$: in this limit (5) possesses a steady-state power-function Kolmogorov solution. Assuming that $ak \ll 1$, and that the turbulence is isotropic (i.e., that $n_\mathbf{k}$ depends only on the modulus of $\mathbf{k}$), we can perform in (5) three of the four integrations, after which we obtain

$$\frac{\partial n_\mathbf{k}}{\partial t} = \frac{b}{a}\left\{\int_0^k k_1(k-k_1)[n_1n_2-n_k(n_1+n_2)]\,dk_1\right.$$

$$\left.-2\int_k^\infty k_1(k_1-k)[n_2n_k-n_1(n_2+n_k)]\,dk_1\right\}. \qquad (6)$$

Here $n_1 = n_{k_1}$ and $n_2 = n_{k-k_2}$. As noted in Ref. 2, the presence of a in the denominator of the collision term makes impossible the passage in the kinetic equation from the weak-turbulence limit (where $n_k \ll a^3/b$) to the strong-turbulence limit (where $a = 0$).

In (6) the kernel of the collision term is already scale-invariant. The exponent of the steady-state Kolmogorov solution is most easily obtained by writing (6) in the form $\partial\varepsilon_k/\partial t + \partial P_k/\partial k = 0$, where $\varepsilon_k = k\omega_k n_k = ck^2 n_k$ is the density, and

$$P_k = -c\int^k k'^2 I_{k'}\,dk'$$

is the energy flux in k space. To the steady-state solution corresponds the constancy of the flux, i.e., the situation in which $\partial P_k/\partial k = 0$, and since for $n_k = k^{-s_0}$ we have $P_k \propto k^{6-2s_0}$, $s_0 = 3$. The deviation of the Kolmogorov exponent from the exponent obtained from the standard formula $s_0 = m + d = \frac{3}{2} + 2 = \frac{7}{2}$ is due to the presence in (6) of the factor a, which has the dimension of length.[2,5]

Since $\alpha = 1$, $d = 2$, and $s_0 = 3$, $\delta = \alpha + d - s_0 = 0$—capillary wave turbulence on the surface of shallow water—pertains to the situation of interest to us here, namely, a situation intermediate between the proper and improper Kolmogorov situations.

Let us linearize (6) against the background of the Kolmogorov solution $n_k^0 = Nk^{-3}$ by setting $n_k = n_k^0 + \delta n_k$ ($\delta n_k \ll n_k^0$):

$$\frac{a}{2bN}\frac{\partial \delta n_k}{\partial t} = -\delta n_k \int_0^\infty kk_1^{-2}\,dk_1$$

$$+\int_0^\infty \delta n_{k_1}[(k_1-k)^{-2}+(k_1+k)^{-2}-2k^{-2}]\,dk_1. \qquad (7)$$

Equation (7) possesses proper solutions of the form (2). The eigenvalues $\Gamma(s)$ are given by the integral

$$\Gamma(s) = \frac{2bN}{a}\int_0^\infty [x^{1-s}(x-1)^{-2}+x^{1-s}(x+1)^{-2}-2x^{1-s}-2x^{-2}]\,dx$$

$$= \frac{2bN}{a}(1-s)\,\mathrm{ctg}\,\frac{\pi s}{2}, \qquad (8a)$$

which converges in the band $2 < \mathrm{Re}\, s < 4$. This means that, among the perturbations having the asymptotic forms $\delta n_k \propto k^{-s}$, only those with $2 < \mathrm{Re}\, s < 4$ are localized: for them the interaction of the waves with wave vectors of the same order of magnitude is the main thing. But in the case of nonlocalized perturbations we should explicitly specify the scales k_0 and k_m of the source and sink, respectively, and take account of the finiteness of the inertial interval (k_0, k_m). We shall limit ourselves to the study of the evolution of localized perturbations.

As can be seen from (8a), $\Gamma(3) = 0$, which corresponds to neutral stability of the Kolmogorov solution against changes in the magnitude of the energy flux. Notice that $\Gamma(s)$ in the interval $2 < \mathrm{Re}\, s < 4$ is, generally speaking, a complex quantity. This is due to the fact that, in contrast to the weakly nonequilibrium situation, the operator of the linear part of the kinetic equation, linearized against the background of the Kolmogorov spectrum is in general non-Hermitian (see, for example, Ref. 6). This is the cause of the difference in the behavior of small deviations: the perturbations of the equilibrium spectrum attenuate monotonically, while in the Kolmogorov case the occupation numbers can oscillate about the steady-state values (see §2).

Among the proper solutions (2) are those for which $\mathrm{Re}\,\Gamma(s) > 0$ [see (8a)]. Naturally, this does not yet imply the instability of the Kolmogorov solution, since the eigenfunctions k^{-s} do not satisfy the boundary conditions. It is more convenient to formulate the physical conditions on the perturbations in terms of $\Psi(x) = \delta n_k/n_k^0$, where $x = \ln(k/k_0)$ (when expressed in terms of the variable x, (3) goes over into the Fourier transform). The requirement that the perturbation energy be finite, i.e., that

$$\delta E = \int_0^\infty ck^2 \delta n_k\,dk \le M < \infty,$$

leads to the requirement that $\Psi(x)\to 0$ as $x\to\pm\infty$, with $\Psi\propto e^{-\varepsilon_1 x}$ for $x\to+\infty$ and $\Psi\propto e^{\varepsilon_2 x}$ for $x\to-\infty$, where $\varepsilon_1,\varepsilon_2>0$. It is also convenient to shift the region of analyticity of $\Gamma(s)$ by going over from s to $z=(s-3)/2$:

$$\frac{a}{2bN}\Gamma(z)=2(1+z)\operatorname{tg}\pi z. \tag{8b}$$

§2. EVOLUTION OF LOCALIZED PERTURBATIONS IN THE BACKGROUND OF THE KOLMOGOROV SPECTRUM

As we shall now show, the function $\Gamma(z)$, (8b), has been constructed in such a way that any localized perturbation is carried into the region of large x. Let us first consider smooth perturbations in the form of a "hump" locally given by the formula $\Psi(x,t)=A\exp[-2zx+\Gamma(z)t]$, [where $z(x)$ is a slowly varying real function of x ($|\partial z/\partial x|\ll|z/x|$)], which is such that, to the right of the peak of the function $\Psi(x)$, $0\leqslant z<\frac{1}{2}$, while to the left $-\frac{1}{2}<z\leqslant 0$. Since we are considering localized perturbations ($|z|<\frac{1}{2}$), the evolution of each section is determined only by wave vectors of the same order of magnitude, i.e., by the quantity $\Gamma(z(x))$. It can easily be seen from (8b) that $\Gamma(z)$ is real on the real axis, with sign $\Gamma(z)=\text{sign } z$ and $\Gamma(0)=0$ in the region $-\frac{1}{2}<z<\frac{1}{2}$. This means that a perturbation of the above-described shape evolves in the following fashion: the right slope increases, while the left slope decreases, the height of the peak remaining unchanged. As is easy to understand, this corresponds to the motion of the packet to the right. The apex moves according to the law

$$k(t)=k_0\exp[\Gamma'(0)t]=k_0\exp(2bNt/a).$$

The exponential character of the motion is easy to understand if account is taken of the fact that the condition $\delta=0$ implies independence of the characteristic nonlinear-interaction time of k. Therefore, during the motion along the spectrum any wave, for example, doubles its wave vector over a time period that is independent of k.

An arbitrary localized solution to Eq. (7) can be written in the form

$$\Psi(x,t)=\int_\gamma \Psi(z)\exp[-2zx+\Gamma(z)t]dz, \tag{9}$$

where the contour γ traverses across the analyticity band of $\Gamma(z)$ ($|\text{Re } z|<\frac{1}{2}$) from $-i\infty$ to $+\infty$ and $\Psi(z)$ is the Fourier transform of the initial perturbation $\Psi(x,0)$. Making the contour γ coincident with the axis $\text{Re } z=0$, $z=i\sigma$, we see that, since $a\,\text{Re}\,\Gamma(i\sigma)=-2bN\sigma\,\text{th}\,\sigma<0$, any localized perturbation attenuates. Let us discuss this in a somewhat greater detail. Let initially $\Psi(z)=\exp[(z-z_0)^2/\Delta^2]$ be a narrow Guassian packet ($\Delta\ll|z_0|=|\varkappa_0+i\sigma_0|=(\varkappa_0^2+\sigma_0^2)^{1/2}$ Let us evaluate the integral (9) by the method of steepest descent. The saddle point z^* is given by the equation

$$2\frac{z^*-z_0}{\Delta^2}-2x+t\frac{\partial\Gamma(z^*)}{\partial z}=0, \tag{10}$$

i.e., depends on x and t. Let us elucidate the behavior of the maximum of the envelope $\Psi(x,t)$. Let us denote the coordinate of the peak by $x_0(t)$. As is easy to obtain from (9) and (10), $x_0(0)=-\varkappa_0/\Delta^2$, the saddle point for $x=x_0$, occurs on the imaginary axis

$$z^*(x_0,0)=i\sigma_0,\qquad a\Gamma(i\sigma_0)=4bN(i\,\text{th}\,\sigma_0-\sigma_0\,\text{th}\,\sigma_0).$$

For $t\ll\sigma_0\cosh^2\sigma_0$ (see below)

$$x_0(t)=-\frac{\varkappa_0}{\Delta^2}+t\,\text{Re}\frac{\partial\Gamma(z_0^*)}{\partial z},$$

$$z_0^*=z_0^*(x_0,t)=i\sigma_0-it\Delta^2\left(\text{th}\,\sigma_0+\frac{\sigma_0}{\text{ch}^2\sigma_0}\right),$$

the saddle point of $z_0^*(t)$, moves along the imaginary axis to the zero point. Since the velocity $\text{Re}(\partial\Gamma(i\sigma)/\partial z)=4bN/a\cosh^2\sigma>0$ on the imaginary axis, the narrow Gaussian packet moves in the direction of large x, the height of the envelope peak decreasing in the process ($\text{Re}\,\Gamma(i\sigma)<0$). Since $\text{Im}\,\Gamma(i\sigma)\neq 0$, the attenuation of the packet is accompanied by oscillations. For $\sigma_0\ll 1$ the period of the oscillations is much shorter than the damping time.

But any perturbation is ultimately carried into the region of large x, leaving behind a damped trail. Indeed, let us consider $t\to\infty$ and $x/t=V=\text{const}$. For $x\to+\infty$, $a(z)$ is analytic in the region $\text{Re } z\geqslant 0$. Shifting the contour of integration to the saddle point, which, for $V\geqslant\Gamma'(0)$, lies on the real axis, and is given by the equation $\Gamma'(\varkappa)=V$, we obtain

$$\Psi(x,t)\propto\exp t[\Gamma(\varkappa)-\varkappa\Gamma'(\varkappa)].$$

For the function $\Gamma=(1+\varkappa)\tan\varkappa$, it is easy to show that, in the region $\varkappa>0$, $\Gamma(\varkappa)-\varkappa\Gamma'(\varkappa)<0$ (which corresponds to the absence of convective instability). As to the trail left behind, for $t\to\infty$ and finite x ($ax\ll 2bNt$) the saddle point is given by the condition $\Gamma'(z^*)=0$ or $\sin 2\pi z^*=-2\pi(1+z^*)$.

This equation posseses the approximate solution $z^*\approx-\frac{1}{4}\pm 2i/5$, and since $\text{Re}\,\Gamma(z^*)<0$, for $ax\ll 2bNt$,

$$\Psi(x,t)\propto\exp[x/2+\Gamma(z^*)t]. \tag{11}$$

Equation (11) demonstrates the stability of the Kolmogorov spectrum against isotropic perturbations: any localized perturbation is carried into the region of attenuation (i.e., of large k), leaving behind in the inertial region a damped trail.

Apparently, this character of the evolution of localized perturbations (the drift into the region of attenuation) is fairly general. Let us consider the model four-wave problem

$$\omega^{d/\alpha-1}\frac{\partial n_\omega}{\partial t}=\int_0^\infty\!\!\int d\omega_1\,d\omega_2\int_{\omega-\omega_2}^\infty d\omega_3\,U_{\omega\omega_1\omega_2\omega_3}\delta(\omega+\omega_1-\omega_2-\omega_3)[n_1n_2n_3-nn_2n_3-nn_1n_2-nn_1n_3],$$

$$U=(\omega\omega_1\omega_2\omega_3)^{\mu/4},\qquad n=n_\omega=n(\omega),\qquad n_i=n_{\omega_i},\qquad i=1,2,3.$$

Here the Kolmogorov solution, which is determined by the constant particle number flux, is equal to $n_\omega^0=\omega^{-(\mu+3)/3}=\omega^{-}$, which corresponds to a source in the region of high ω and to damping in the low ω region. Let us consider the situation intermediate between the proper and improper Kolmogorov particle-number spectra: $\mu-2\nu+d/\alpha+3=0$. Under this condition the eigenfunctions of the linearized kinetic equation have the form $\delta n_\omega=\omega^{-s}e^{\Gamma(s)t}$, and the eigenvalues of $\Gamma(s)$ can also be computed directly. Let us, without giving the unwieldy expression for the function $\Gamma(s)$, indicate the following properties of the function:

1) $\Gamma(s)$ is defined in the interval

$$\frac{3\nu+1}{4} < \mathrm{Re}\, S < \frac{5\nu-1}{4}.$$

Note that the Kolmogorov exponent always lies in the middle of the "band of localizability" of isotropic perturbations [see also (8a)].

2) On the real axis ($\mathrm{Im}\, s = 0$) we have sign $\Gamma(s) = -$ sign$(s - s_0)$.

3) On the imaginary axis we have $\mathrm{Re}(\partial\Gamma/\partial s) < 0$.

Thus, in the case of the described spectrum with a particle flux the localized perturbations are carried to the left.

Here it is pertinent to note that the motion of a narrow Gaussian packet ($z_0 = 0, \Delta \ll 1$) is, as we have seen, determined by the group velocity $\partial\Gamma(s_0)/\partial s$. But according to Ref. 7, this same quantity $\Gamma'(s_0)$ specifies the direction of the corresponding flux of the integral of the motion.

Summarizing, we can assert that, in the symmetric case, isotropic perturbations in the background of the Kolmogorov spectrum should be carried into the runoff region without increasing in magnitude. If the perturbations have an oscillating structure in k space [i.e., if ($\sigma_0 \neq 0$, $\Psi \sim \exp(i\sigma_0 \ln k)$], then the drift will undergo damping and oscillation in time.

The author is grateful to V. E. Zakharov for stimulating conversations, and to V. S. L'vov, S. L. Musher, and M. D. Spektor for useful discussions.

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Translated by A. K. Agyei