Scalar field diffusion in a stochastic medium

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We suggest a method for calculating the diffusion coefficient $\varkappa_T(\xi)$ for a scalar field (density, temperature, and so on) in a stochastic, uniform, isotropic, incompressible medium. The method enables us to calculate $\varkappa_T(\xi)$ up to the most encountered case $\xi \approx 1$ ($\xi^2 = \langle u^2 \rangle \tau_0^2 / 3R_0^2$, where $\langle u^2 \rangle$ is the mean square average of the velocity, and τ_0 and R_0 are characteristic time and spatial scales of the velocity fluctuations). The relative error in calculation of $\varkappa_T(\xi)$ for $\xi \approx 1$ is 10–20% and decreases when the parameter ξ decreases. The method is useful also for non-Gaussian ensembles of a random velocity field.

I. INTRODUCTION

In many problems in physics and astrophysics one encounters the problem of the transfer of a scalar field in a medium with random velocities or forces. Amongst them, for instance, is the problem of the transport of particles or heat in a medium whose velocity field is random and is determined by the correlation tensors of different ranks. A similar problem is that of finding the distribution function of particles in a random force field. Related to this type is the problem of the magnetic field diffusion. Notwithstanding the fact that the magnetic field is not a scalar, we can apply to it the method proposed below.

It is well known that one obtains for averaged scalar quantities (temperature, density, distribution function, and so on) diffusion equations with a diffusion coefficient which depends on the characteristics of the medium. Many papers have been devoted to evaluating these diffusion coefficients. For instance, the diffusion coefficient for the average magnetic field in a medium with velocities assumed to be δ-correlated in time was found in Ref. 1. The analogous problem for a scalar field was solved in Refs. 2–6 without assuming a δ-correlation. In all papers it was assumed that the ensemble of random velocities or forces was Gaussian. The important role of the higher correlators of the random field was shown in Refs. 7 and 8. A theoretical study of the temperature of the ocean surface when the inflow of heat from the atmosphere has a stochastic nature was given in Ref. 9, where also the intermittency effect was predicted, i.e., the presence of sporadically appearing rare and strong temperature peaks.

We choose in what follows, for the sake of argument, the temperature as the scalar field. The random medium is in our problem determined by two parameters. The first of them is $\eta = R_0(4\varkappa_m \tau_0)^{-1/2}$, where R_0 and τ_0 are characteristic space and time scales of the correlation of the random velocity field $\mathbf{u}(\mathbf{r},t)$, and $\varkappa_m$ is the molecular diffusion coefficient. The parameter η is equal to the ratio of the correlation scale R_0 to the length for equalizing the temperature over a correlation time τ_0 through the molecular heat conduction mechanism. In the majority of actual cases $\eta \gg 1$. The other parameter is $\xi = u_0\tau_0/3R_0$, where $u_0 = (\langle u^2 \rangle)^{1/2}$ is a characteristic rate of the velocity fluctuations of the medium. Most papers are restricted to the case $\xi \ll 1$, which corresponds to weak random velocity fluctuations, and also to a medium with δ-correlation in time. However, in many actual situations the characteristic dimensions of the correlation region are of the order $u_0\tau_0$, i.e., $\xi \approx 1$. The study in Ref. 3 comes closest to this case; in it the turbulent diffusion coefficient $\varkappa_T$ is calculated through the use of the Green function of the usual heat-conduction equation, where instead of the molecular coefficient $\varkappa_m$ the required quantity $\varkappa_T$ itself is substituted. This gives integral equations from which $\varkappa_T$ is determined. As a result one obtains for $\varkappa_T$ a series whose convergence is not proven. For $\xi \gg 1$ this series gives $\varkappa_T \sim \xi^{-1}$ which contradicts the result obtained from an analysis of the initial equations (see below).

In contrast to the above mentioned-papers we have calculated the diffusion coefficient in a stochastic medium without restricting ourselves to the δ-approximation or to the approximation $\xi \ll 1$. The proposed calculation method is applicable also for non-Gaussian random velocity ensembles.

2. STATEMENT OF THE PROBLEM

The transfer of heat in an incompressible medium with a given turbulent velocity field $\mathbf{u}(\mathbf{r},t)$ is described by the equation

$$(\partial/\partial t - \varkappa_m \nabla^2) T(\mathbf{r}, t) = -\mathbf{u}(\mathbf{r}, t) \nabla T(\mathbf{r}, t). \tag{1}$$

We write the temperature field $T(\mathbf{r},t)$ as the sum of an averaged and a fluctuating component $T = \langle T \rangle + T_1$, $\langle T_1 \rangle = 0$. Averaging Eq. (1) we get

$$(\partial/\partial t - \varkappa_m \nabla^2) \langle T \rangle = f_0(r, t), \quad f_0(\mathbf{r}, t) = -\langle \mathbf{u}(\mathbf{r}, t) \nabla T_1(\mathbf{r}, t) \rangle. \tag{2}$$

The angle brackets $\langle \cdot \cdot \cdot \rangle$ indicate averaging over an ensemble of stochastic fields $\mathbf{u}(\mathbf{r},t)$ where we assume that the medium as a whole is at rest, i.e., $\langle \mathbf{u} \rangle = 0$. We assume the velocity field to be statistically isotropic, uniform, and stationary. Using the Green function of Eq. (1) for an infinite medium

$$G(R, \tau) = \theta(\tau)(4\pi\varkappa_m \tau)^{-3/2} \exp(-R^2/4\varkappa_m \tau), \tag{3}$$

where $\theta(\tau) = 1$ for $\tau > 0$ and $\theta(\tau) = 0$ for $\tau < 0$, we get an integral equation for $T(\mathbf{r},t)$:

$$T(\mathbf{r}, t) = \int d\mathbf{r}' \, G(R, t) T(\mathbf{r}', 0)$$

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$$+\int d\mathbf{r}' \int_0 dt'\, G(R,\tau)\,(-\mathbf{u}(\mathbf{r}',t')\nabla')\,T(\mathbf{r}',t'). \tag{4}$$

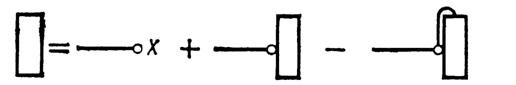
Here $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, $\tau=t-t'$ and the temperature at time $t=0$ is given by the field $T(\mathbf{r},0)$.

Subtracting from Eq. (4) its averaged part we get an integral equation for the fluctuations $T_1(\mathbf{r},t)$:

$$T_1(\mathbf{r},t)=\int_0^\infty dt' \int d\mathbf{r}'\, G(R,\tau)\,[-\mathbf{u}(\mathbf{r}',t')\nabla'\langle T(\mathbf{r}',t')\rangle$$
$$+\langle \mathbf{u}(\mathbf{r}',t')\nabla' T_1(\mathbf{r}',t')\rangle-\mathbf{u}(\mathbf{r}',t')\nabla' T_1(\mathbf{r}',t')]. \tag{5}$$

It is clear from this expression that $T_1(\mathbf{r},t)$ depends on the gradient of the averaged temperature.

It is convenient to use in what follows a diagram notation. Let a line connecting $\mathbf{r},t$ and $\mathbf{r}',t'$ correspond to the Green function $G(\mathbf{R},\tau)$, a small circle corresponds to the operator $(-\mathbf{u}\nabla)$, and we denote the average temperature $\langle T\rangle$ by a cross and the fluctuating temperature T_1 by a rectangle. Equation (5) takes the following graphical form:



(6)

The arched line indicates statistical averaging over the velocity field for all quantities which stand under it. One integrates over the coordinates $\mathbf{r}',t'$ corresponding to the right-hand ends of the straight lines.

We write $T_1(\mathbf{r},t)$ as the sum of two quantities: $T_1=T_s+T_a$, where T_s (T_a) is a quantity which is even (odd) in the number of velocity components. The even part of $T_1(\mathbf{r},t)$ can according to Eq. (6) be expressed in terms of the odd one:

$$T_s(\mathbf{r},t)=\int d\mathbf{r}' \int_0^\infty dt'\, G(R,\tau)\,[\langle \mathbf{u}(\mathbf{r}',t')\nabla' T_a(\mathbf{r}',t')\rangle$$
$$-\mathbf{u}(\mathbf{r}',t')\nabla' T_a(\mathbf{r}',t')]. \tag{7}$$

Using Eq. (7) we get from (6) an equation for solely the odd part of $T_1(\mathbf{r},t)$:

$$T_a(\mathbf{r},t)=\int d\mathbf{r}'\int_0^\infty dt'\, G(\mathbf{r}-\mathbf{r}',t-t')\Big\{-\mathbf{u}(\mathbf{r}',t')\nabla'\langle T(\mathbf{r}',t')\rangle$$
$$+\int d\mathbf{r}''\int_0^\infty dt''[\mathbf{u}(\mathbf{r}',t')\nabla' G(\mathbf{r}'-\mathbf{r}'',t'-t'')\mathbf{u}(\mathbf{r}'',t'')\nabla'' T_a(\mathbf{r}'',t'')$$
$$-\mathbf{u}(\mathbf{r}',t')\nabla' G(\mathbf{r}'-\mathbf{r}'',t'-t'')\langle \mathbf{u}(\mathbf{r}'',t'')\nabla'' T_a(\mathbf{r}'',t'')\rangle$$
$$-\langle \mathbf{u}(\mathbf{r}',t')\nabla' G(\mathbf{r}'-\mathbf{r}'',t'-t'')\mathbf{u}(\mathbf{r}'',t'')\nabla'' T_a(\mathbf{r}'',t'')\rangle]\Big\}. \tag{8}$$

It is convenient to regard the terms with averaging as a part of the free term of Eq. (8). Introducing the notation

$$F_+(1)=\langle T(1)\rangle+\int d2\, G(1,2)\langle \mathbf{u}(2)\nabla_2 T_a(2)\rangle,$$
$$F_-(1)=\int d2\langle \mathbf{u}(1)\nabla_1 G(1,2)\mathbf{u}(2)\nabla_2 T_a(2)\rangle, \tag{9}$$

we seek the solution of (8) in the form

$$T_a(1)=-\int d2\, M(1,2)\,[\mathbf{u}(2)\nabla_2 F_+(2)+F_-(2)], \tag{10}$$

where we have introduced the two-point resolvent function $M(1,2)=M(\mathbf{r}_1,t_1,\mathbf{r}_2,t_2)$ which we shall denote in diagrams by an oval and which satisfies the equation

$$M(1,2)=G(1,2)+\int d3\int d4\, G(1,3)\mathbf{u}(3)\nabla_3 G(3,4)$$
$$\times \mathbf{u}(4)\nabla_4 M(4,2). \tag{11}$$

If we know the function M we get, by substituting (10) into (9), a set of ordinary, rather than stochastic, integral equations for the functions F_+ and F_-:

$$F_+(1)=\langle T(1)\rangle-\int d2\int d3\, G(1,2)\,[\langle \mathbf{u}(2)\nabla_2 M(2,3)\mathbf{u}(3)\rangle$$
$$\nabla_3 F_+(3)+\langle \mathbf{u}(2)\nabla_2 M(2,3)\rangle F_-(3)], \tag{12}$$
$$F_-(1)=-\int d2\int d3[\langle \mathbf{u}(1)\nabla_1 G(1,2)\mathbf{u}(2)\nabla_2 M(2,3)\mathbf{u}(3)\rangle$$
$$\nabla_3 F_+(3)+\langle \mathbf{u}(1)\nabla_1 G(1,2)\mathbf{u}(2)\nabla_2 M(2,3)\rangle F_-(3)].$$

Since the velocity field $\mathbf{u}(\mathbf{r},t)$ is uniform and stationary, the kernels in (12) are functions of the coordinate and time differences, i.e., they can easily be solved explicitly if we use a Fourier transform with respect to the coordinates and a Laplace transform with respect to the time. Finding the fluctuating part of the temperature has thus been reduced to determination of the auxiliary function $M(1,2)$ which has, however, a fundamental value. The function $M(1,2)$ has the meaning of a Green function and describes the propagation of temperature fluctuations in the medium. We note that owing to the presence of the $\theta(\tau)$ function in (3) the function M contains $\theta(t_1-t_2)$ and we can integrate in Eq. (11) from $t'=-\infty$ to $t'=+\infty$.

Knowing the function $M(1,2)$ makes it possible to write down the term $f_0(\mathbf{r},t)=-\langle \mathbf{u}(\mathbf{r},t)\nabla T(\mathbf{r},t)\rangle$ [which occurs on the right-hand side of Eq. (2)] for the average temperature:

$$f_0(\mathbf{r},t)=F_-(\mathbf{r},t)+\int d\mathbf{r}'\int_0 dt'[\langle \mathbf{u}(\mathbf{r},t)\nabla M(\mathbf{r},t,\mathbf{r}',t')$$
$$\times \mathbf{u}(\mathbf{r}',t')\nabla'\rangle F_+(\mathbf{r}',t')$$
$$+\langle \mathbf{u}(\mathbf{r},t)\nabla M(\mathbf{r},t,\mathbf{r}',t')\rangle F_-(\mathbf{r}',t')]$$
$$\equiv \nabla_i\nabla_j\int_0^t d\tau\int d\mathbf{R}\,\Phi_{ij}(\mathbf{R},\tau)\langle T(\mathbf{r}-\mathbf{R},t-\tau)\rangle. \tag{13}$$

One easily finds the explicit form of $\Phi_{ij}(\mathbf{R},\tau)$ if one writes down the graphical expression in analytical form. It is important to note that the tensor $\Phi_{ij}(\mathbf{R},\tau)$ is obtained after averaging expressions such as $\langle u_i(1)M(1,2)u_j(2)\rangle$ and differs significantly from zero inside a spatial scale $\sim R_0$ and a time scale $\sim\tau_0$, as does the correlator $\langle u_i(1)u_j(2)\rangle$. In contrast, the average temperature $\langle T(\mathbf{r},t)\rangle$, which really means the temperature averaged over scales $L\gg R_0$ and times $\gg\tau_0$, is clearly a smooth function over scales R_0 and τ_0. To a good approximation we can thus put in (13) $\langle T(\mathbf{r}-\mathbf{R},t-\tau)\rangle\approx\langle T(\mathbf{r},t)\rangle$ and we get

$$f_0(\mathbf{r},t)\approx\varkappa_T(t)\nabla^2\langle T(\mathbf{r},t)\rangle, \tag{14}$$

$$\varkappa_T(t)=\frac{1}{3}\sum_i\int_0^t d\tau\int d\mathbf{R}\,\Phi_{ii}(\mathbf{R},\tau). \tag{15}$$

In this approximation it is sufficient to take

$$F_+(1)\approx\langle T(1)\rangle,$$
$$F_-(1)\approx-\int d2\int d3\langle \mathbf{u}(1)\nabla_1 G(1,2)\mathbf{u}(2)\nabla_2 M(2,3)\mathbf{u}(3)\rangle$$
$$\times\nabla_3\langle T(3)\rangle,$$

i.e., $f_0(\mathbf{r},t)$ takes the form

$$f_0(\mathbf{r},t)\approx\int d\mathbf{r}'\int_0^\infty dt'\langle\mathbf{u}(\mathbf{r},t)\nabla M(\mathbf{r},t,\mathbf{r}',t')\mathbf{u}(\mathbf{r}',t')\rangle \times\nabla\langle T(\mathbf{r}',t')\rangle$$
$$-\int d\mathbf{r}'\int d\mathbf{r}''\int_0^\infty dt'\int_0^\infty dt''\langle\mathbf{u}(\mathbf{r},t)\nabla G(\mathbf{r}-\mathbf{r}',t-t')\mathbf{u}(\mathbf{r}',t') \times\nabla' M(\mathbf{r}',t',\mathbf{r}'',t'')\mathbf{u}(\mathbf{r}'',t'')\rangle\nabla''\langle T(\mathbf{r}'',t'')\rangle. \quad (16)$$

The tensor $\Phi_{ij}(\mathbf{R},\tau)$ is in this case approximately equal to

$$\Phi_{ij}(\mathbf{R},\tau)\approx\langle u_i(1)M(1,2)u_j(2)\rangle + \int_0^\infty dt_3\int d\mathbf{r}_3\langle u_i(1)G(1,2)\mathbf{u}(3)\nabla_3 M(3,2)u_j(2)\rangle. \quad (17)$$

For brevity we have used here the notation $u_i(\mathbf{r}_1,t_1)\equiv u_i(1)$, $G(\mathbf{r}_1-\mathbf{r}_2,t_1-t_2)\equiv G(1,2)$, and so on. The second term in (17) is odd in the number of u_i components and vanishes for mirror-symmetric turbulence, i.e., for a medium with zero helicity. Since the function $M(1,2)$ is even in the number of u_i components, the second term in (17) also vanishes for Gaussian ensembles of the velocity field.

Expression (15) for $\varkappa_T(t)$ is practically independent of the time for $t>\tau_0$ since the tensor $\Phi_{ij}(\mathbf{R},\tau)$ is rapidly damped at $\tau>\tau_0$. The turbulent diffusion coefficient (thermal diffusivity) is given in that case by the expression

$$\varkappa_T(\xi,\eta)={}^1/_3\sum_i\int_0^\infty d\tau\int d\mathbf{R}\,\Phi_{ii}(\mathbf{R},\tau). \quad (18)$$

We can integrate in Eq. (18) over τ from $-\infty$ to $+\infty$, since the tensor $\Phi_{ij}(\mathbf{R},\tau)$ contains $\theta(\tau)$. We actually evaluate it for a Gaussian ensemble $\mathbf{u}(\mathbf{r},t)$ when all odd correlators of the velocity field vanish while the even ones are expressed as a sum of all possible pair correlators. For this case we have

$$\varkappa_T(\xi,\eta)={}^1/_3\sum_i\int d\mathbf{R}\int_{-\infty}^{\infty} d\tau\langle u_i(\mathbf{r},t)M(\mathbf{r},t,\mathbf{r}',t')u_i(\mathbf{r}',t')\rangle. \quad (19)$$

Substituting (14) into Eq. (2) gives a diffusion equation for $\langle T(\mathbf{r},t)\rangle$ with a renormalized diffusion coefficient:

$$\frac{\partial}{\partial t}\langle T\rangle-[\varkappa_m+\varkappa_T(t)]\nabla^2\langle T\rangle=0. \quad (20)$$

Practically always we have $\varkappa_T\gg\varkappa_m$, i.e., $\eta\to\infty$ and we restrict ouselves in what follows to just that case. As $\eta\to\infty$ the coefficient $\varkappa_T$ is a function solely of the parameter ξ.

It is interesting to establish the ξ-dependence of $\varkappa_T(\xi)$ for $\xi\gg1$. We can do this directly from Eqs. (11) and (19). We introduce dimensionless quantities $\overline{M}=MR_0^3$, $\overline{G}=GR_0^3$, $\overline{K}_1=R_0^5u_0^{-2}K_1$, $d\overline{\tau}=\tau_0^{-1}d\tau$, $d\overline{\mathbf{r}}=R_0^{-3}d\mathbf{r}$, denoting them with a superior bar. Here $K_1(3,4)\equiv\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4$ is the kernel of Eq. (11). In dimensionless form, Eq. (11) becomes

$$\overline{M}(1,2)=\overline{G}(1,2)+\xi^2\int d\overline{\mathbf{r}}_3\int d\overline{\tau}_3\int d\overline{\mathbf{r}}_4\int d\overline{\tau}_4\overline{G}(1,3) \times\overline{K}_1(3,4)\overline{M}(4,2). \quad (11')$$

Hence it is clear that $\overline{M}(1,2)\propto\xi^{-2}$ as $\xi\to\infty$, and thus $\varkappa_T(\xi)\sim u_0^2\tau_0/\xi^2$ if the medium has no helicity and if the velocity ensemble is Gaussian. When there is helicity and if the ensemble is non-Gaussian, $\varkappa_T(\xi)\sim u_0^2\tau_0/\xi$ as $\xi\to\infty$.

3. CALCULATION OF THE TURBULENT-DIFFUSION COEFFICIENT

For the evaluation of $\varkappa_T(\xi)$ we need explicit expressions for the velocity pair correlator

$$\langle u_i(\mathbf{r},t)u_j(\mathbf{r}',t')\rangle\equiv B_{ij}(\mathbf{R},\tau)=B_{ij}^{(s)}(\mathbf{R},\tau)+e_{ijm}R_mC(R,\tau),$$
$$B_{ij}^{(s)}(\mathbf{R},\tau)=\delta_{ij}B_\perp(R,\tau)+(B_\parallel(R,\tau)-B_\perp(R,\tau))R_iR_j/R^2, \quad (21)$$

where $\mathbf{R}=\mathbf{r}-\mathbf{r}'$, $\tau=t-t'$, and e_{nmr} is the antisymmetric unit pseudotensor. The quantities $B_\parallel$ and $B_\perp$ determine the velocity correlations along and at right angles to $\mathbf{R}$, respectively. The incompressibility condition div $\mathbf{u}=0$ gives the relation $B_\perp=B_\parallel+(R/2)\partial B_\parallel/\partial R$. The term with $C(R,\tau)$ describes the turbulent helicity $(2RC(R,\tau)=\langle\mathbf{u}(1)\times\mathbf{u}(2)\rangle)$. It is convenient to use a Fourier transform

$$B_{nm}(\mathbf{R},\tau)=(2\pi)^{-3}\int d\mathbf{p}\,\widetilde{B}_{nm}(\mathbf{p},\tau)e^{i\mathbf{pR}}. \quad (22)$$

where the tensor $\widetilde{B}_{nm}(\mathbf{p},\tau)$ has the form

$$\widetilde{B}_{nm}(\mathbf{p},\tau)=\Pi_{nm}(\mathbf{p})f(p,\tau)+ie_{nmr}p_rD(p,\tau),$$
$$\Pi_{nm}(\mathbf{p})=\delta_{nm}p^2-p_np_m, \quad (23)$$
$$f(p,\tau)=-\frac{1}{2p}\frac{\partial}{\partial p}\widetilde{B}_\parallel(p,\tau),\qquad D(p,\tau)=\frac{1}{p}\frac{\partial}{\partial p}\widetilde{C}(p,\tau).$$

For the actual calculations we take

$$B_\parallel(R,\tau)=B_0\exp(-R^2/R_0^2-|\tau|/\tau_0),\qquad B_0\equiv{}^1/_3\langle u^2\rangle={}^1/_3u_0^2,$$
$$f(p,\tau)={}^1/_4\pi^{3/2}B_0R_0^5\exp[-(pR_0)^2/4-|\tau|/\tau_0]. \quad (24)$$

The correlators

$$f(p,\tau)=3\pi^2B_0p_0^{-4}\delta(p-p_0)\exp(-\tau^2/\tau_0^2), \quad (25)$$
$$f(p,\tau)=3\pi^2B_0p_0^{-4}\delta(p-p_0)\exp(-|\tau|/\tau_0), \quad (26)$$

were also used in Refs. 3, 5, 6. The correlator (24) decreases monotonically with increasing R while the correlators (25) and (26) oscillate and decrease more slowly than (24). For this reason the spatial scale $\sim p_0^{-1}$ is less rigorously defined in (25) and (26) than for the correlator (24).

We shall perform all concrete calculations for Gaussian velocity ensemble $\mathbf{u}(\mathbf{r},t)$ and for the case $\varkappa_m\to0\,(\eta\to\infty)$. If we iterate Eq. (11):

$$M(1,2)=G(1,2)$$
$$+\int d3\,d4\,G(1,3)\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4G(4,2)$$
$$+\int d3\,d4\,d5\,d6\,G(1,3)\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4G(4,5)\cdot \times\mathbf{u}(5)\nabla_5G(5,6)\mathbf{u}(6)\nabla_6G(6,2)+\ldots \quad (27)$$

and substitute the series (27) in Eq. (19) for $\varkappa_T(\xi)$, we obtain a series of approximations for $\varkappa_T(\xi)$. We get (if there is no helicity) for the correlators (24)–(26)

$$\varkappa_T(\xi)=B_0\tau_0(1-5\xi^2+125\xi^4/2-\ldots),\quad \xi^2=B_0\tau_0^2/R_0^2, \quad (28)$$

$$\varkappa_T(\xi)=\tfrac{1}{2}\pi^{1/2}B_0\tau_0(1-0.207\xi_1^2+\ldots),\quad \xi_1^2=B_0\tau_0^2p_0^2,\tag{29}$$

$$\varkappa_T(\xi)=B_0\tau_0(1-\tfrac{1}{2}\xi_2^2+\ldots),\quad \xi_2^2=B_0\tau_0^2p_0^2.\tag{30}$$

The terms with ξ^2 results from allowance for fourth-order velocity correlators, terms $\propto\xi^4$ from sixth order correlators, and so on. Expressions (28)–(30) are asymptotic (divergent) series and can be used for the calculation of $\varkappa_T(\xi)$ only for $\xi\ll 1$. The accuracy of the calculation for $\varkappa_T(\xi)$ is determined by the first discarded term. For instance, taking the first two terms in (28) into account gives $\varkappa_T(\xi)$ for $\xi^2=0.05$ with an error of 20%. For realistic situations $\xi\approx 1$ and the series (28)–(30) are completely useless for calculating $\varkappa_T(\xi)$. Comparison of the terms $\propto\xi_1^2$ and $\propto\xi_2^2$ with the term $5\xi^2$ shows the same level of turbulence development corresponds to $\xi_1\approx 5$ for the correlator (25) and to $\xi_2\approx 3.16$ for the correlator (26) when $\xi=1$.

The reason why the iteration of Eq. (11) is so inefficient for calculating $\varkappa_T(\xi)$ is that the kernel of this equation contains Green functions $G(R,\tau)$ that describe the transfer of heat by the molecular heat-conduction mechanism and not by the convective heat-transfer mechanism. In what follows, we obtain from the initial Eq. (11) a new equation for the random function $M(1,2)$ the kernel of which describes more directly the convective heat transfer mechanism. Iteration of this new equation enables us to obtain for $\varkappa_T$ a new asymptotic series which is suitable to find $\varkappa_T(\xi)$ up to the most interesting case $\xi\approx 1(\xi_1\approx 5,\ \xi_2\approx 3.16)$.

The formal solution of Eq. (11) can be written down if we use the resolvent operator $\hat{R}_0(1,2)$:

$$M(1,2)=G(1,2)+\int d3\,\hat{R}(1,3)G(3,2).\tag{31}$$

We introduce here a new auxiliary equation with a kernel depending on the differences $\mathbf{R}=\mathbf{r}-\mathbf{r}'$ and $\tau=t-t'$, the solution of which can be obtained using Fourier transforms in $\mathbf{R}$ and τ:

$$M_+(1,2)=G(1,2)+\int d3\int d4\,G(1,3)K(3,4)M_+(4,2).\tag{32}$$

We can in principle choose the kernel in (32) arbitrarily (but damped at infinity).

We shall look for the solution of Eq. (11) in the form of a sum $M=M_++M_-$. Subtracting Eq. (32) from (11) [or from (31)] we get an equation with the same kernel as the initial Eq. (11). The solution of this equation can be expressed in terms of the same resolvent:

$$\begin{aligned}M_-(1,2)=&\int d3\int d4\,G(1,3)\,[\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4M_+(4,2)\\&-K(3,4)M_+(4,2)+\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4M_-(4,2)]\\=&\int d3\int d4\,G(1,3)\,[\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4-K(3,4)]M_+(4,2)\\&+\int d3\int d4\int d5\,\hat{R}(1,3)G(3,4)\,[\mathbf{u}(4)\nabla_4G(4,5)\mathbf{u}(5)\nabla_5\\&-K(4,5)]M_+(5,2).\end{aligned}\tag{33}$$

Using the fact that according to (31) $RG=M-G$ we get

$$\begin{aligned}M_-(1,2)=\int d3\int d4\,M(1,3)\,[\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4\\-K(3,4)]M_+(4,2).\end{aligned}\tag{34}$$

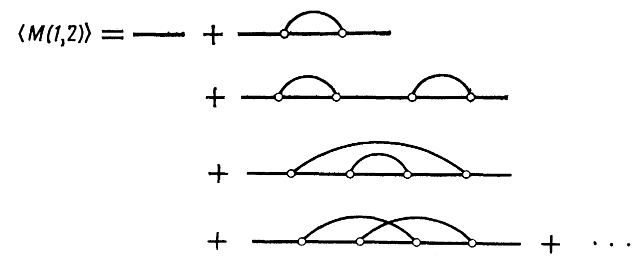
Adding to both sides of (34) the function M_+ we get a new equation for the function M:

$$\begin{aligned}M(1,2)=&M_+(1,2)\\&+\int d3\int d4\,M(1,3)\,[\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\nabla_4\\&-K(3,4)]M_+(4,2).\end{aligned}\tag{35}$$

Instead of the single Eq. (11) we have thus introduced two Eqs. (32) and (35) which are equivalent to the initial Eq. (11). The value of such a replacement consists in that one can solve Eq. (32) exactly and, through a physically based choice of the kernel of the function (32), we can take into account the characteristic features of the convective heat transfer. It is clear that the replacement of one equation by two may be accomplished in an infinite number of ways. We denote by $M_1(\mathbf{R},\tau)$ the solution of the equation

$$\begin{aligned}M_1(1,2)=&G(1,2)\\&+\int d3\int d4\,G(1,3)\langle\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\rangle\nabla_4M_1(4,2).\end{aligned}\tag{36}$$

The function $M_1(\mathbf{R},\tau)$ is a relatively good approximation to the average value $\langle M\rangle$ which one can graphically write down (for a Gaussian velocity ensemble) in the form of a sum of all possible graphs:



(37)

The function $M_1(\mathbf{R},\tau)$ which is the solution of (36) corresponds formally to a summation in (37) of ladder type graphs:

$$\begin{aligned}M_1(1,2)\equiv&M_1(\mathbf{R},\tau)\\=&G(1,2)+\int d3\int d4\,G(1,3)\langle\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\rangle\\&\times\nabla_4G(4,2)+\int d3\int d4\int d5\int d6\,G(1,3)\\&\times\langle\mathbf{u}(3)\nabla_3G(3,4)\mathbf{u}(4)\rangle\\&\times\nabla_4G(4,5)\langle\mathbf{u}(5)\nabla_5G(5,6)\mathbf{u}(6)\rangle\nabla_6G(6,2)+\ldots\end{aligned}\tag{38}$$

We note that the ladder type graphs (38) are much larger than other graphs of the same order in the velocity correlators, since they possess a larger symmetry in the angular variables than the non-ladder graphs. Ww now give the explicit form of $M_1(\mathbf{R},\tau)$. It is convenient to us a Fourier transform

$$M_1(\mathbf{R},\tau)=(2\pi)^{-4}\int d\mathbf{p}\int_{-\infty}^{\infty}d\omega\,\tilde{M}_1(\mathbf{p},\omega)\exp[i(\mathbf{pR}+\omega\tau)].\tag{39}$$

Substituting (39) into Eq. (36) and bearing in mind that $\tilde{G}(p,\omega)=(\varkappa_mp^2+i\omega)^{-1}$ we get

$$\tilde{M}_1(\mathbf{p},\omega)=(\varkappa_mp^2+S(\mathbf{p},\omega)+i\omega)^{-1},\tag{40}$$

$$S(\mathbf{p},\omega)=p_ip_j\int d\mathbf{R}\int_{-\infty}^{\infty} d\tau\, B_{ij}(\mathbf{R},\tau)G(R,\tau)\exp[-i(\mathbf{pR}+\omega t)].\qquad(41)$$

We next take $\varkappa_m\to 0$ and use the fact that then $G(R,\tau)\to\theta(\tau)\delta(\mathbf{R})$. In that case

$$S(\mathbf{p},\omega)=p^2\int_0^{\infty} d\tau\, B_{\parallel}(0,\tau)e^{-i\omega\tau}.\qquad(42)$$

For actual calculations we take the correlators (24) and (26), i.e., $B_{\parallel}(0,\tau)=B_0\exp(-|\tau|/\tau_0)$:

$$S(\mathbf{p},\ \omega)=B_0\tau_0p^2(1+i\omega\tau_0)^{-1}.\qquad(43)$$

It is convenient in the calculations to take the inverse Fourier transform with respect to the frequency:

$$\tilde{M}_1(p,\tau)=\frac{\theta(\tau)\exp(-\tau/2\tau_0)}{(4B_0\tau_0^2p^2-1)^{1/2}}\left[\sin\frac{(4B_0\tau_0^2p^2-1)^{1/2}\tau}{2\tau_0}+(4B_0\tau_0^2p^2-1)^{1/2}\cos\frac{(4B_0\tau_0^2p^2-1)^{1/2}\tau}{2\tau_0}\right].\qquad(44)$$

Taking the inverse Fourier transform in the variable $\mathbf{p}$ we get

$$M_1(R,\tau)=\frac{\theta(\tau)\exp(-\tau/2\tau_0)}{8\pi R\tau_0B_0^{1/2}}\left[\delta(R-\tau B_0^{1/2})+2\tau_0\frac{\partial}{\partial\tau}\delta(R-\tau B_0^{1/2})+\psi(R,\tau)+2\tau_0\frac{\partial}{\partial\tau}\psi(R,\tau)\right].\qquad(45)$$

The function $\psi(R,\tau)=0$ for $0<\tau<R/B_0^{1/2}$, while for $\tau>R/B_0^{1/2}$ we have

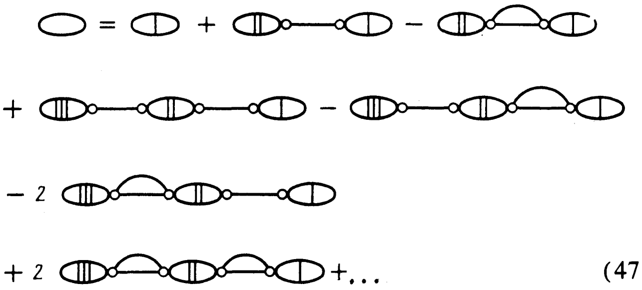
$$\psi(R,\ \tau)=RI_1[(B_0\tau^2-R^2)^{1/2}/2\tau_0B_0^{1/2}]/2\tau_0[B_0(B_0\tau^2-R^2)^{1/2}],\qquad(46)$$

where $I_1(x)$ is a modified Bessel function. The term with the derivative of the δ-function arises when $\tau\ll\tau_0$ for any τ-dependence of $B_{\parallel}(0,\tau)$. We get from (45) the normalization condition

$$\int d\mathbf{R}\,M_1(R,\tau)=\tilde{M}_1(0,\tau)=\theta(\tau)\quad\text{and}\quad M_1(R,0)=\delta(\mathbf{R}).$$

The term with the δ-function and its derivative describe the average convective heat transfer at a rate of $\sim B_0^{1/2}$. The appearance of these terms is completely natural if $\varkappa_m=0$, for in that case Eq. (1) become a wave-type equation. It is interesting that $M_1(R,\tau)$ is independent of the scale R_0. This means that the function $M_1(R,\tau)$ describes convection as a superposition of jet flows damped in time, in which the length transversed by the jet is determined by the average velocity and the average damping time, i.e., $R_0\sim u_0\tau_0$. In contrast to $G(R,\tau)$ the function $M_1(R,\tau)$ thus describes the convective heat transfer directly, and iterations of the new Eq. (35) for $M_1(1,2)$ thus enable us to calculate $\varkappa_T(\xi)$ for an appreciably larger range of values of the parameter ξ up to $\xi=0.5$. To extend the calculation of $\varkappa_T(\xi)$ even further, up to $\xi=1$, we can use the fact that the choice of the kernel in (32) is not unique. In particular, we propose to use the kernel $K_1=\langle u(3)\nabla_3G(3,4)u(4)\rangle\nabla_4$, $K_2=2K_1$, $K_3=3K_1$, and so on, designating the corresponding solutions of (32) by M_1, M_2, M_3, and so on. We shall designate these functions graphically by ovals with one, two, and so on, vertical lines. The function $M_n(R,\tau)$ is obtained from $M_1(R,\tau)$ through the simple substitution $B_0\to nB_0$.

Finally, the series of iterations for $M(1,2)$ is constructed as follows: in Eq. (35) with $M_1(R,\tau)$, we substitute in the integral term $M(1,3)$ from the same Eq. (35) but with the function $M_2(R,\tau)$, in the resulting integral equation, we substitute (35) with $M_3(R,\tau)$, and so on. As a result we get a series of iterations for $M(1,2)$ of the form


(47)

Substituting this expression into (19) we get an asymptotic series for $\varkappa_T(\xi)$: $\varkappa_T(\xi)=\varkappa_T^{(0)}+\varkappa_T^{(1)}+\varkappa_T^{(2)}+\dots$. The fact that the choice of kernels in (32) is not unique enables us in principle to look for another series for $\varkappa_T(\xi)$ which, possibly, is more suitable than the one proposed in the present paper.

We now give the results of the numerical calculations, assuming a Gaussian character of the random-velocity ensemble. The zeroth-order approximation is described by the formula

$$\varkappa_T^{(0)}(\xi)={}^1/_3\sum_i\int d\mathbf{R}\int_{-\infty}^{\infty}d\tau\, M_1(R,\tau)B_{ii}(R,\tau).\qquad(48)$$

For the correlator (24) it gives ($\xi^2=B_0\tau_0^2/R_0^2$)

$$\varkappa_T^{(0)}(\xi)=\frac{B_0\tau_0}{3\xi^2}\left\{1-\frac{1}{\xi^2}+\left(\frac{\pi}{2}\right)^{1/2}\frac{\exp(1/2\xi^2)}{\xi^3}\left[1-\Phi\left(\frac{1}{2^{1/2}\xi}\right)\right]\right\},\qquad(49)$$

where

$$\Phi(x)=\frac{2}{\pi^{1/2}}\int_0^x dt\, e^{-t^2}$$

is the error function. As $\xi\to\infty$ this formula changes into (28) and as $\xi\to\infty$ it gives $\varkappa_T^{(0)}\to B_0\tau_0/3\xi^2$, i.e., for large ξ, $\varkappa_T^{(0)}(\xi)\propto\xi^{-2}$ as we found earlier directly from (11) and (19).

For the correlator (26), Eq. (48) gives the simpler expression

$$\varkappa_T^{(0)}(\xi)=2B_0\tau_0(2+\xi^2)^{-1},\quad \xi^2=B_0\tau_0^2p_0^2.\qquad(50)$$

As $\xi\to 0$ it changes into (30).

The second and third terms in (47) contribute for a Gaussian ensemble only when there is helicity ($h=\mathbf{u}$ curl $\mathbf{u}\neq 0$):

$$\varkappa_T^{(1)}(\xi)=\varkappa_T^{(helicity)}(\xi)=\frac{1}{18\pi^4}\int_0^\infty dp\,p^4\int_0^\infty dq\,q^4\int_0^\infty d\tau\int_0^\infty d\tau'\int_0^\infty d\tau''\,M_1(p,\tau')\times M_2(q,\tau'')D(p,\tau+\tau')D(q,\tau+\tau'').\qquad(51)$$

In the case of maximum helicity $\mathbf{u}\parallel$ curl $\mathbf{u}$, i.e.,

$D(p,\tau) = -\gamma p f(p,\tau)$, where γ characterizes the degree of helicity. For the correlator (24) we have

$$\varkappa_T^{(1)}(\xi) = B_0\tau_0 \frac{16}{9\pi}\gamma^2\xi^2\psi_0(\xi^2)\psi_0(2\xi^2), \qquad \psi_0(\xi^2) = \int_0^\infty \frac{dx\, x^2 e^{-x}}{2\xi^2 x+1}. \tag{52}$$

Expression (52) reaches a maximum $\approx 0.08\gamma^2 B_0\tau_0$ for $\xi \approx 0.4$. The relative contribution $\varkappa_T^{(1)}$ as compared to $\varkappa_T^{(0)}$ increases with increasing ξ and as $\xi \to \infty$ reaches the value $\varkappa_T^{(1)}/\varkappa_T^{(0)} \to 0.21\gamma^2$. For the correlator (26) Eq. (51) gives

$$\varkappa_T^{(1)}(\xi) = {}^1/_2 B_0\tau_0\gamma^2\xi^2(1+\xi^2)^{-1}(2+\xi^2)^{-1}, \quad \xi^2 = B_0\tau_0^2 p_0^2. \tag{53}$$

In that case the maximum of $\varkappa_T^{(1)}$ equals $0.086\gamma^2 B_0\tau_0$ for $\xi^2 = 2^{1/2}$. The ratio $\varkappa_T^{(1)}/\varkappa_T^{(0)} \to 0.25\gamma^2$ as $\xi \to \infty$.

We have calculated from term $\varkappa_T^{(2)}(\xi)$ only for the case of no helicity $(C(R,\tau) = 0)$. For the correlator (24) it is equal to

$$\varkappa_T^{(2)}(\xi) = \frac{4B_0\tau_0\xi^4}{3\pi^{9/2}} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \frac{\exp(-p_1^2-p_2^2-p_3^2)}{(2|\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3|^2\xi^2+3)(2\xi^2 p_1^2+1)} \times\left(\frac{2p_1^2 Q_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}{6\xi^2 p_1^2+1} + \frac{Q_2(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}{6\xi^2 p_2^2+1}\right), \tag{54}$$

$$\begin{aligned} Q_1 &= (\mathbf{p}_1\Pi_2\mathbf{p}_1)((\mathbf{p}_1+\mathbf{p}_2)\Pi_3(\mathbf{p}_1+\mathbf{p}_2)) + (\mathbf{p}_1\Pi_2(\mathbf{p}_1+\mathbf{p}_3))(\mathbf{p}_1\Pi_3(\mathbf{p}_1+\mathbf{p}_2)), \\ Q_2 &= (\mathbf{p}_1\Pi_2\Pi_1(\mathbf{p}_2+\mathbf{p}_3))(\mathbf{p}_2\Pi_3(\mathbf{p}_1+\mathbf{p}_2)) + (\mathbf{p}_1\Pi_3\mathbf{p}_2)((\mathbf{p}_1+\mathbf{p}_3)\Pi_2\Pi_1(\mathbf{p}_2+\mathbf{p}_3)) \\ &\quad + (\mathbf{p}_1\Pi_3(\mathbf{p}_1+\mathbf{p}_2))(\mathbf{p}_2\Pi_1\Pi_2(\mathbf{p}_1+\mathbf{p}_3)) + (\mathbf{p}_1\Pi_2\Pi_1\mathbf{p}_2)((\mathbf{p}_1+\mathbf{p}_2)\Pi_3(\mathbf{p}_1+\mathbf{p}_2)). \end{aligned} \tag{55}$$

For simplicity we write $\mathbf{p}\Pi_2\mathbf{q} \equiv p_i\Pi_{ij}(\mathbf{p}_2)q_j$, and so on. The ninefold integration in (54) can be reduced to a threefold one, if we use formulae such as

$$(2\xi^2p^2+1)^{-1} = \int_0^\infty dt\, \exp(-t(2\xi^2p^2+1)).$$

For the correlator (26) we get instead of (54) $(\xi^2 = B_0\tau_0^2p_0^2)$

$$\varkappa_T^{(2)}(\xi) = \frac{B_0\tau_0\xi^4}{256\pi^3(\xi^2+2)(3\xi^2+2)} \int d\mathbf{n}_1 \int d\mathbf{n}_2 \int d\mathbf{n}_3 \times\frac{2Q_1(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)+Q_2(\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3)}{(|\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3|^2\xi^2+6)}. \tag{56}$$

One can find the integrals in (56) explicitly but the expression obtained is too cumbersome. The terms $\varkappa_T^{(2)}$ take into account the contribution from the sixth-order correlators. One must, however, bear in mind that using the functions M_1, M_2, $\cdots$ means taking into account correlators of all order, but of the ladder type.

An estimate of the next term $\varkappa_T^{(3)}(\xi)$ gives $\varkappa_T^{(3)} \approx 0.03B_0\tau_0$ for $\xi = 1$. The proposed procedure for calculating $\varkappa_T(\xi)$ thus gives an error of 10–20% for $\xi = 1$. The error decreases as the parameter ξ decreases.

According to the self-consistent theory of Ref. 3, the main contribution to $\varkappa_T(\xi)$ is given by Eq. (19), where we have substituted instead of the exact Green function $M(1,2)$ the Green function (3) with the replacement $\varkappa_m \to \varkappa_T$:

$$G_T(R,\tau) = \theta(\tau)[4\pi\varkappa_T(\xi)\tau]^{-3/2}\exp[-R^2/4\varkappa_T(\xi)\tau],$$

$$\varkappa_T(\xi) = {}^1/_3\sum_i \int d\mathbf{R} \int_0^\infty d\tau\, G_T(R,\tau)B_{ii}(R,\tau) = \frac{1}{3\pi^2}\int_0^\infty dp\, p^4 \int_0^\infty d\tau\, \exp[-p^2\varkappa_T(\xi)\tau] f(p,\tau). \tag{57}$$

From (57) it follows that $\varkappa_T(\xi) \propto \xi^{-1}$ as $\xi \to \infty$, which contradicts the correct asymptotic behavior $\propto \xi^{-2}$. For small ξ, Eq. (57) give smaller values than the correct asymptotic relations (28)–(30).

The results of calculating $\varkappa_T^{(0)}(\xi)$ and $\varkappa_T^{(2)}(\xi)$ for the correlator (24) are shown in the figure. It is interesting to note that $\varkappa_T(\xi)$ for the correlator (26) is almost the same as $\varkappa_T(\xi)$ for the correlator (24) if instead of $\xi^2 = B_0\tau_0^2p_0^2$ we take $\xi^2_{\text{eff}} = B_0\tau_0^2p_0^2/10$ [this choice of ξ_{eff} guarantees that the asymptotic behaviors of (28) and (30) are the same].

4. QUALITATIVE DISCUSSION OF THE RESULTS

It is clear from the figure that the value of $\varkappa_T^{(0)}(\xi)$ calculated from (48) is a good approximation for $\varkappa_T(\xi)$. To understand this we qualitatively consider the turbulent heat transfer process. As we already mentioned in the Introduction, this process is characterized by the parameters η and ξ. If $\eta \ll 1$, the temperature is practically constant in the region where the turbulent jets develop and are damped, so that in that case no turbulent heat transfer takes place. The velocity correlators with scales much smaller than $(4\varkappa_m\tau_0)^{1/2}$—the length over which the temperature equalizes through the molecular heat conductivity mechanism—are clearly unimportant for the description of the turbulent heat transfer. In

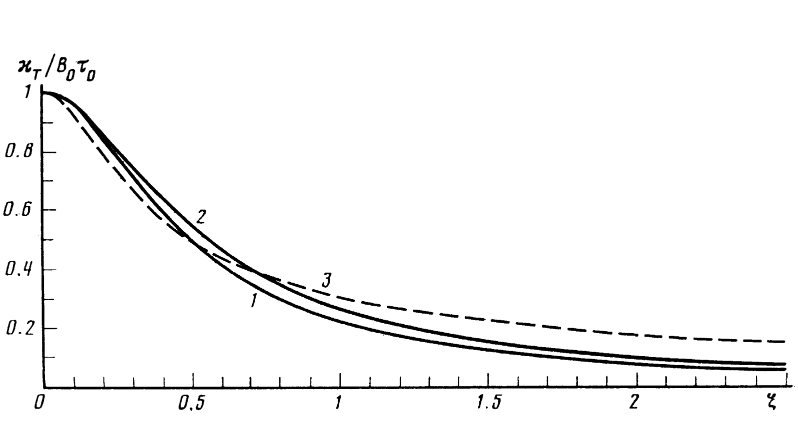


FIG. 1. The turbulent diffusion coefficient $\varkappa_T(\xi)$ for the correlator $B_{\parallel}(R,\tau) = B_0\exp(-R_2/R_0^2 - |\tau|/\tau_0)$ with $\xi^2 = B_0\tau_0^2/R_0^2$. The curves 1, 2, 3 correspond to $\varkappa_T^{(0)}(\xi)$, $\varkappa_T^{(0)}(\xi) + \varkappa_T^{(2)}(\xi)$, and the solution (57).

the case which is most often met with, $\varkappa_m \ll \varkappa_T(\eta \to \infty)$, it is apparently necessary to take into account correlators of all orders, but this is not always the case due to the presence of another parameter ξ which characterizes the role of the small-scale fluctuations inside the main correlation range $\sim R_0$. The case $\xi \ll 1$ corresponds to two situations: a) turbulent jets are extensive and are damped by viscosity simulataneously in the whole volume. Smaller scale jets can arise in that case at the boundary of the main jet and divert a relatively small amount of the transferred heat; b) the lifetime of the turbulent jets is so short that they do not manage to break up into smaller jets. The condition $\xi \ll 1$ means that the motion inside the main correlation region $\sim R_0$ is regular and can be described by the second-order velocity correlator. The series (27) leads in the limit as $\eta \to \infty$ and $\xi \to 0$ to the following expression for $\varkappa_T$:

$$\varkappa_T^{(max)} = \int_0^\infty B_\parallel(0,\tau)\,d\tau. \tag{58}$$

This value is an upper limit for $\varkappa_T$ since: a) as $\varkappa_m \to 0(\eta \to \infty)$ the turbulent jet does not manage during the time of its existence to change its temperature through molecular heat conduction, and b) as $\xi \to 0$ the jet does not manage to break up into smaller jets. The breakup of the main jet into smaller ones can, clearly, only change the amount of heat transferred from one point of space to another. The very largest turbulent jet which are well described by the two-point velocity correlator $B_{ij}(\mathbf{R},\tau)$ are the most important ones for the convective heat transfer. Equation (48) takes into account the second-order correlators and determines the main part of the turbulent diffusion. According to (20) the transfer of the averaged temperature is described by the Green function $G_T(R,\tau)$. It is natural to assume that the propagation of large-scale temperature fluctuations is also approximately described by the same function. Therefore, Eq. (57) for $\varkappa_T$ gives rather good agreement with the exact value of $\varkappa_T(\xi)$. However, when $\xi \approx 1$ it leads to significant errors.

It is clear from the figure that the coefficient $\varkappa_T(\xi)$ for the case $\xi \approx 1$ most often met with, differs strongly (by a factor 5–10) from the usually made estimates on the basis of a mixing length, which corresponds to $\varkappa_T$ from (58).

We note that knowing the function $M(1,2)$ [e.g., the series of approximations (47) for it] one can easily write down an expression for all possible correlators of the temperature fluctuations, $\langle T_1(1)T_1(2)\rangle$, and so on.

We did not use in the derivation of Eq. (35) and of the iteration series (47) the assumption that the velocity field has a Gaussian nature, and the method for calculating $\varkappa_T(\xi)$ is thus applicable also for non-Gaussian ensembles $\mathbf{u}(\mathbf{r},t)$.

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