Determination of the internal structure of relativistic astrophysical objects in the wave approximation

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Spherically symmetric inverse problems for the scattering of quantum particles by a static gravitational field are considered in the framework of the general theory of relativity. Methods are developed for recovering the metric tensor from scattering data at fixed energy or zero angular momentum for the Klein-Fock-Gordon equation in the Schwarzschild metric. The connection between the S matrix and the operator of the square of the 4-momentum in curved space is investigated. The main links of the developed algorithm are two definite nonlinear ordinary differential equations of third and fourth order constructed from the scattering data. On the one hand, the investigated inverse problems generalize the previously solved classical inverse problems for the gravitational field to the quantum case; on the other hand, they generalize the Marchenko and Regge-Newton methods known in quantum scattering theory to include the case of gravitational fields. A certain analogy is established between the motion of a scalar particle in a strong gravitational field and motion in a field with a potential that depends on the angular momentum or the energy in nonrelativistic quantum mechanics. This analogy makes it possible to model relativistic problems in the general theory of relativity by nonrelativistic problems. The results are also valid for electromagnetic waves and may be topical for direct determination of the internal structure of neutron stars by probing in the range of radio waves.

1. INTRODUCTION

Spherically symmetric inverse problems for the scattering of classical particles by a static gravitational field have been formulated and solved by one of the present authors[1-3] in the framework of the general theory of relativity. For massless weakly interacting particles, the inverse problem consisted of recovering the form of the gravitational field (the 4-space metric) from a null geodesic asymptote behavior that depends on the impact parameter. For massive particles, three algorithms were established for recovering the metric for the cases when the scattering data are given for fixed angular momentum, fixed energy, and fixed impact parameter. The solutions of all these problems reduce to definite nonlinear ordinary differential equations of second order constructed from the scattering data. In the Newtonian limit, these algorithms give the well-known inversion methods of Hoyt[4] and Firsov[5,6] and the method proposed by the present authors[7,8] for inverse problems of the simplest potential scattering.

In the present paper, we propose a quantum generalization of the previously considered classical inverse problems in the theory of gravitation. This generalization will be done initially for spinless relativistic particles of mass m (for example, mesons), i.e., for the Klein-Fock-Gordon equation. The gravitational field is assumed to be classical and to satisfy Einstein's equations.

Although the exposition is given in this paper in corpuscular language, the results can mostly be translated to the case of electromagnetic waves. Of practical interest here may be the range of radio waves whose wavelength is comparable with the diameter of neutron stars.

From the theoretical point of view, it is interesting to trace how the inversion algorithms change with increasing complexity of the interaction on going from the electromagnetic to the gravitational field. The methods developed here are a generalization of the algorithms that exist in quantum scattering theory for solving inverse problems in flat space to the case of curved space. We are referring to Marchenko's well-known algorithm for the inverse problem for fixed angular momentum and the Regge-Newton algorithm for the inverse problem for fixed energy. Reviews of the theory of inverse wave problems can be found in the papers of Faddeev,[9,10] and the monographs of Agranovich and Marchenko,[11] de Alfaro and Regge,[12] Newton,[13] and Chadan and Sabatier.[14] It is natural to expect that the Marchenko and Regge-Newton integral equations can be derived from their gravitational analogs in the Newtonian limit.

Several studies have been devoted to the generalization of the nonrelativistic methods for inverse problems to the case of relativistic particles in flat space. For example, for the Klein-Fock-Gordon equation the first results were obtained by Corinaldesi,[15] de Alfaro,[16] and Verde.[17] They relate to the Gel'fand-Levitan and Jost-Kohn [18] methods. Marchenko's method for the Klein-Fock-Gordon equation was developed by Weiss and Scharf.[19] Finally, relativistic inverse problems at fixed energy were investigated by Coudray and Coz.[20,21] They generalized the Regge-Newton method to scalar and spinor particles, i.e., to the Klein-Fock-Gordon and Dirac equations. However, in these studies the gravitational field was not considered at all.

The aim of the present paper is to investigate the problem of the connection between the S matrix and the operator of the square of the 4-momentum in the gravitational field. We shall use the relativistic system of units with $\hbar = c = 1$. As in the classical problems,[1-3] we assume that the structure of the gravitational field is determined by the interior Schwarzschild metric[22]

$$ds^2=e^\nu dt^2-e^\mu dr^2-r^2(d\theta^2+\sin^2\theta d\chi^2),\quad \nu<0,\quad \mu>0,\qquad (1)$$

which we extend to the entire space. Further, we assume that

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the metric functions $\nu(r)$ and $\mu(r)$ are regular at the origin and that the space with the metric (1) is asymptotically flat, i.e., that ν and μ together with all their derivatives decrease sufficiently rapidly as $r\to\infty$.

The corresponding system of three Einstein equations in the framework of a static hydrodynamic model is given in Ref. 23. In such a model, the gravitating matter is characterized by radial distributions of the density, $\rho(r)$, and pressure, $p(r)$, these being connected by the Oppenheimer-Volkoff integrodifferential equation (condition for a static solution). In these approximations, the system of Einstein equations can be integrated in quadratures; its solution, with allowance for the boundary conditions, has the form

$$f\equiv e^{-\mu}=1-\varkappa r^{-1}\int\limits_0^r d\xi\,\xi^2\rho,$$

$$\nu=\int\limits_r^\infty d\xi\,\xi^{-1}f^{-1}(f-\varkappa\xi^2p-1),$$

where $\varkappa$ is the Einstein gravitational constant. The inverse formulas

$$\rho=(1-f-rf')/\varkappa r^2,\quad p=(f-1+rf\nu')/\varkappa r^2 \tag{2}$$

make it possible to recover the density and pressure of the fluid from the metric tensor.

From the Oppenheimer-Volkoff equation, which follows from the field equations, one can derive the differential equation

$$(\nu'+2/r)f'+(2\nu''+\nu'^2-2\nu'/r-4/r^2)f+4/r^2=0, \tag{3}$$

which connects the functions f and ν. Thus, in the framework of the hydrodynamic model the metric tensor components

$$g_{00}\equiv e^\nu \text{and} g_{11}\equiv f^{-1}$$

are not independent. This is a consequence of the problem's being static.

The quantum (wave) inverse problems studied in the present paper consist of recovering the metric functions f and ν (and then the pressure and matter density) from meson scattering data that depend both on the angular momentum l for fixed energy E as well on E for fixed l.

In Sec. 2, we consider the Klein-Fock-Gordon equation in the Schwarzschild metric; in Sec. 3, we solve the inverse scattering problem for fixed energy; and in Sec. 4, we solve the inverse scattering problem for fixed angular momentum.

2. WAVE EQUATION

To take into account the interaction of classical meson and gravitational fields, it is necessary to construct the generally covariant generalization of the Klein-Fock-Gordon equation. This question is considered in detail in the book of Grib, Mamaev, and Mostepanenko.[24] The simplest approach is to replace the ordinary derivatives in the free Klein-Fock-Gordon equation by covariant derivatives.[22,25] The wave equation obtained in this manner is called the equation with minimal coupling. However, for massless particles this equation does not possess the property of conformal invariance. A different approach leads to a more complicated equation with conformal coupling, which is free of this shortcoming.

For simplicity, we shall restrict ourselves in what follows to the Klein-Fock-Gordon equation with minimal coupling,

$$\Box\Psi+m^2\Psi=0, \tag{4}$$

since we shall consider the scattering of massive mesons. Equation (4) contains the covariant Laplace-Beltrami operator[26]

$$\Box=(-g)^{-1/2}\partial_i[(-g)^{1/2}g^{ij}\partial_j],$$

where $g\equiv\det(g_{ij}),i,j=0,1,2,3$, and contractions over repeated indices are understood. In the Schwarzschild metric (1), the stationary wave equation (4) takes, after separation of the variables in terms of spherical functions,

$$\Psi=e^{-iEt}r^{-1}R_l(r)Y_{lm}(\theta,\chi),$$

the form

$$R_l''+u^{-1}u'R_l'+f^{-1}[E^2e^{-\nu}-m^2-l(l+1)r^{-2}]R_l=(ru)^{-1}u'R_l, \tag{5}$$

where $u=f^{1/2}e^{\nu/2}$. Replacement in (5) of the radial wave function in accordance with $R_l\to\psi_l\equiv u^{1/2}R_l$ gives the final equation for the partial wave,

$$\psi_l''+f^{-1}[E^2e^{-\nu}-m^2-l(l+1)r^{-2}]\psi_l=U\psi_l, \tag{6}$$

$$U(r)\equiv\frac{1}{2}\frac{d^2\ln u}{dr^2}+\frac{1}{4}\left[\frac{d\ln u}{dr}\right]^2+\frac{1}{r}\frac{d\ln u}{dr}, \tag{7}$$

which no longer contains a first derivative. Equation (7) determines a spherically symmetric field $U(r)$ that vanishes in the limit $r\to\infty$ by virtue of the condition of an asymptotically flat nature of space ($u(\infty)=1$, $u'(\infty)=0$, $u''(\infty)=0$). In the free case, when there is no gravitating matter $\nu=\mu=0$, $f=1$, $u=1$, $U=0$. For a weak gravitational field ($m\to\infty$) we put in (6) $U\approx0$, $f\approx1$, $\nu\approx2V/m\ll1$, where V/m is the Newtonian gravitational potential and $e^{-\nu}\approx1-\nu$, and we obtain the ordinary Schrodinger equation in the potential field $V(r)$.

3. INVERSE SCATTERING PROBLEM FOR FIXED ENERGY

In this section, we consider the regular solution $\varphi_1(r)$ of the Klein-Fock-Gordon equation (6). In the limit $r\to0$, such a solution has the regular asymptotic behavior $\varphi_l\sim C_l r^{l+1}$. In the limit $r\to\infty$, the radial wave function behaves as

$$R_l\to\varphi_l\sim A_l\sin(kr+\delta_l-l\pi/2), \tag{8}$$

since the space is asymptotically flat. The wave number is $k=(E^2-m^2)^{1/2}$.

The inverse scattering problem for fixed energy consists in this case of recovering the metric tensor (the components g_{00} and g_{11}), i.e., the "potential" of the gravitational field, from the infinite sequence of phase shifts δ_l, given for all 1. It is clear that the complete set $\{\delta_l\}$ can be readily calculated from the angular dependence of the scattering amplitude given at a certain energy.

We express Eq. (6) in terms of V_1 and V_2:

$$V_1(r) \equiv U+E^2(1-f^{-1}e^{-\nu})-m^2(1-f^{-1}), \quad V_2(r) \equiv r^{-2}(f^{-1}-1), \tag{9}$$

and we then obtain

$$\varphi_l''+[k^2-V_1-l(l+1)V_2]\varphi_l=l(l+1)r^{-2}\varphi_l, \tag{10}$$

where $V_2>0$ and $V_1, V_2 \to 0$, $r\to\infty$. In such a form, the Klein-Fock-Gordon equation for fixed energy is mathematically equivalent to the Schrödinger equation with central potential V_1 and potential $l(l+1)V_2$, which depends on the square of the angular momentum. A Schrödinger equation with L^2-dependent potentials is encountered in nuclear physics.

To solve the inverse problem, we use Hooshyar's method.[27] We form the radial functions

$$b(r) \equiv r\exp\left[\int_0^r d\xi\, \xi^{-1}(f^{-1/2}-1)\right], \qquad F(r) \equiv (b')^{-1/2}(r) \tag{11}$$

and introduce the constant

$$\gamma \equiv \lim_{r\to\infty} b(r)/r=b'(\infty). \tag{12}$$

The analog of the Regge-Newton integral equation for the gravitational field has the form

$$K(r,r')=F(r)G(b(r),r')-\int_0^{b(r)} d\xi\, \xi^{-2}K(r,\xi)G(\xi,r'). \tag{13}$$

The symmetric part of the kernel of this last equation is

$$G(r,r')=\sum_\lambda c_\lambda \varphi_\lambda^0(r)\varphi_\lambda^0(r'), \tag{14}$$

where $\varphi_\lambda^0(r) \equiv (kr/\gamma)j_l(kr/\gamma)$ is the free solution of the wave equation (10) with effective momentum k/γ, and the summation is over only half-integer $\lambda \equiv l+\frac{1}{2}$. The constants c_λ in (14) can be calculated given the scattering data; the result of the summation depends parametrically on the constant γ. The kernel of Eq. (13) can be symmetrized by the substitution

$$G(r,r')=rr'\tilde{G}(r,r'), \quad K(r,r')=rr'\tilde{K}(r,r').$$

The integral equation (13) expresses the essence of the connection between the S matrix, given for all λ, and the metric tensor. The function $K(r,r')$ is the triangular kernel of an integral operator of generalized shift, namely, the operator of transformation from the free solution φ_λ^0 to the perturbed φ_λ:

$$\varphi_\lambda(r)=F(r)\varphi_\lambda^0(b(r))-\int_0^{b(r)} d\xi\, \xi^{-2}K(r,\xi)\varphi_\lambda^0(\xi). \tag{15}$$

As can be seen from (11), the region of triangularity of the kernel K is determined by the component g_{11} of the metric.

For the kernel $K(r,r')$ there is a boundary condition at $r'=b$:

$$V_1=k^2-\gamma^{-2}k^2b'^2+F^{-1}[F''-(d/dr)(b^{-2}b'K(r,b)) -b^{-2}b'(d/dr)K(r,b)]. \tag{16}$$

Equation (16) connects the "potentials" V_1 and V_2 on the basis of the scattering data.[1)]

For each fixed r, the linear integral equation (13) is an inhomogeneous Fredholm equation of the second kind and can be solved by the method of determinants.[28] After inversion, we have on the boundary $r'=b$

$$K(r,b,\gamma)=F(r)H(b,\gamma), \tag{17}$$

$$H(b,\gamma) \equiv G(b,b,\gamma)-\int_0^b d\xi\, G(b,\xi,\gamma)\tilde{G}(\xi,b,\gamma,b). \tag{18}$$

Here, $\tilde{G}$ is the Fredholm resolvent for the kernel of the analog of the Regge-Newton equation (13); it depends parametrically on γ and the limit of integration b.

For the metric components g_{00} and g_{11} we can derive a closed system of two differential equations. It is convenient to do this in terms of the functions ν and b. From Eqs. (11), we find

$$f=(b/b'r)^2, \tag{19}$$

and this, after the substitution of the values of f and f' in (3), gives the first equation of the system. We obtain the second equation from the expression (16) by using Eqs. (9), (7), (11), (19), and (17). This system has the form

$$\begin{gathered}2r^2b^2b'\nu''+r^2b^2b'\nu'^2-2rb(rbb''-rb'^2+2bb')\nu' \\ -4rb^2b''+4r^2b'^3+4rbb'^2-8b^2b'=0,\end{gathered} \tag{20}$$

$$\begin{gathered}4r^2b^3b'\nu''+r^2b^3b'\nu'^2-4rb'^2(rbb''-rb'^2-bb')\nu' \\ -16E^2r^4bb'^3e^{-\nu}-8rb^3b''-16r^2(2H-2bH_b \\ -\gamma^{-2}k^2b^3)b'^3+4r^2(4m^2r^2-1)bb'^3+8rb^2b'^2-4b^3b'=0,\end{gathered} \tag{21}$$

where both equations are of second order in b, since the third derivative b''' drops out of (21) (it is contained in F'' and in U). Equation (20), which does not contain ν explicitly, guarantees that our relativistic problem is static, and Eq. (21), which is exact in relativistic quantum mechanics, is the link between the metric tensor and the asymptotic behavior of the wave functions.

The system of equations (20) and (21) can be reduced to a single closed differential equation of third (but not fourth) order for the function $b(r)$. Equations (20) and (21) form a linear algebraic system for the highest derivatives ν'' and b'', and this system is degenerate, i.e., its determinant vanishes identically. By combining (20) and (21) we then get the first-order equation

$$\begin{gathered}r^2b^3\nu'^2-12rb^3\nu'+16E^2r^4bb'^2e^{-\nu}+16r^2(2H \\ -2bH_b-\gamma^{-2}k^2b^3)b'^2-4r^2(4m^2r^2-3)bb'^2-12b^3=0,\end{gathered} \tag{22}$$

which must be solved simultaneously with Eq. (20). Separating ν in (22)

$$\begin{gathered}\nu=-\ln\{(1/16)E^{-2}r^{-4}b^{-1}(b')^{-2}[12b^3+4r^2(4m^2r^2-3)bb'^2 \\ -16r^2(2H-2bH_b-\gamma^{-2}k^2b^3)b'^2+12rb^3\nu'-r^2b^3\nu'^2]\},\end{gathered} \tag{23}$$

differentiating both sides of (23) with respect to r, and substituting ν'' from (20), we arrive at a quadratic equation for ν', from which we determine

$$\nu'=(\alpha_1^2-8\alpha_2)^{1/2}-\alpha_1, \tag{24}$$

where

$$\alpha_1 \equiv -2b''/b' - 4rb^{-3}(H-bH_b)b'^2 + r(2m^2r^2-1)b^{-2}b'^2 + 2\gamma^{-2}k^2rb'^2 + 2b^{-1}b' - r^{-1},$$
$$\alpha_2 \equiv rb^{-4}(H-bH_b+b^2H_{bb}+\gamma^{-2}k^2b^3)b'^3 + 2b^{-3}(H-bH_b)b'^2 - \gamma^{-2}k^2b'^2.$$

Eliminating now ν' and ν'' from (20) by means of (24) and differentiating once more and eliminating the irrationalities, we find the required closed equation for the function $b(r)$:

$$\begin{aligned}&2\alpha_2\alpha_3^2 - 2\alpha_1^2\alpha_2\alpha_3 + \alpha_1^2\alpha_3\alpha_6 + 16\alpha_2^2\alpha_3 - 8\alpha_2\alpha_3\alpha_6\\ &-2\alpha_1\alpha_3\alpha_4 - \alpha_1^4\alpha_6 + \alpha_1^3\alpha_5\alpha_6 - 8\alpha_1^2\alpha_2\alpha_5^2\\ &+2\alpha_1^3\alpha_2\alpha_5 - \alpha_1^2\alpha_6^2 + 12\alpha_1^2\alpha_2\alpha_6 - 4\alpha_1^2\alpha_2^2\\ &-16\alpha_1\alpha_2^2\alpha_5 - 8\alpha_1\alpha_2\alpha_5\alpha_6 + 16\alpha_2^2\alpha_5^2 + 2\alpha_1^3\alpha_4\\ &-2\alpha_1^2\alpha_4\alpha_5 + 32\alpha_2^3 + 8\alpha_2\alpha_6^2 - 32\alpha_2^2\alpha_6\\ &-16\alpha_1\alpha_2\alpha_4 + 16\alpha_2\alpha_4\alpha_5 + 4\alpha_4^2 = 0. \qquad (25)\end{aligned}$$

Here

$$\begin{aligned}\alpha_3 \equiv \alpha_1' = &-2b'''/b' + 2b''^2/b'^2 - 8r(H-bH_b)b'b''/b^3\\ &+2r(2m^2r^2-1)b'b''/b^2 + 4k^2rb'b''/\gamma^2\\ &+2b''/b + 4r(3H-3bH_b+b^2H_{bb})b'^3/b^4\\ &-2r(2m^2r^2-1)b'^3/b^3 - 4(H-bH_b)b'^2/b^3\\ &+3(2m^2r^2-1)b'^2/b^2 + 2k^2b'^2/\gamma^2 + r^{-2},\end{aligned}$$
$$\begin{aligned}\alpha_4 \equiv \alpha_2' = &\,3r(H-bH_b+b^2H_{bb}+\gamma^{-2}k^2b^3)b'^2b''/b^4\\ &+4(H-bH_b)b'b''/b^3 - 2k^2b'b''/\gamma^2 - r(4H-4bH_b\\ &+3b^2H_{bb} - b^3H_{bbb} + k^2b^3/\gamma^2)b'^4/b^5\\ &-(5H-5bH_b+b^2H_{bb}-\gamma^{-2}k^2b^3)b'^3/b^4,\end{aligned}$$
$$\alpha_5 \equiv -b''/b' + b'/b - 2r^{-1}, \quad \alpha_6 \equiv -r^{-1}b''/b' + b'^2/b^2 + b'/br - 2r^{-2}.$$

The nonlinear third-order differential equation (25) is the main link in the algorithm for recovering the metric from the phase shifts $\{\delta_l\}$. It can be constructed from the S matrix which determines the form of the function $H(b)$. The equation must be integrated for the initial conditions $b(0)=0$, $b'(0)=1$, $b''(0)=0$, which follow from (11). The highest derivative b''' occurs in Eq. (25) quadratically (it is contained only in α_3) and can be separated. This is helpful in numerical calculations. The integral $b(\gamma, r)$ of Eq. (25) depends parametrically on the constant γ. It satisfies the transcendental equation

$$b'(\gamma, \infty) = \gamma, \qquad (26)$$

which follows from (12). After the calculation of γ, we find the other unknown function $\nu(r)$ of the system using formulas (24) and (23), for which no integrations are required.

We now consider the question of the construction of the coefficients $\{c_\lambda\}$ of the series (14) from the known sequence of phase shifts $\{\delta_\lambda\}$. Substitution of the expansion (14) for G in the analog of the Regge-Newton equation gives in conjunction with the representation (15) an expansion for the kernel

$$K(r,r') = \sum_\lambda c_\lambda \varphi_\lambda(r)\varphi_\lambda^0(r') \qquad (27)$$

with respect to the free and perturbed solutions. Using (27) in the integral representation (15), we obtain

$$\varphi_\lambda(r) = F(r)\varphi_\lambda^0(b(r)) - \sum_{\lambda'} \Lambda_{\lambda\lambda'}(r)c_{\lambda'}\varphi_{\lambda'}(r),$$

$$\Lambda_{\lambda\lambda'}(r) \equiv \int_0^{b(r)} d\xi\, \xi^{-2}\varphi_\lambda^0(\xi)\varphi_{\lambda'}^0(\xi). \qquad (28)$$

The asymptotic form of the system of equations (28) makes it possible to relate the physical phase shifts to the numbers c_λ. If we go in (28) to the limit $r\to\infty$ and take into account the asymptotic behaviors for the free wave function

$$R_\lambda^0 = \varphi_\lambda^0 \sim \sin(kr/\gamma + \pi/4 - \pi\lambda/2)$$

and for the perturbed R_λ (8) and also Eqs. (11) and (12), then we obtain the system of equations

$$\tilde{A}_\lambda \exp i\delta_\lambda = 1 - \sum_{\lambda'} i^{\lambda-\lambda'}\tilde{\Lambda}_{\lambda\lambda'}(\infty)\tilde{c}_{\lambda'}\tilde{A}_{\lambda'}\exp i\delta_{\lambda'}, \qquad (29)$$

with $\tilde{c}_\lambda \equiv (k/\gamma)c_\lambda$, $\tilde{A}_\lambda \equiv \gamma^{1/2}A_\lambda$, and

$$\tilde{\Lambda}_{\lambda\lambda'}(\infty) \equiv (\lambda'^2 - \lambda^2)^{-1}\sin[(\pi/2)(\lambda'-\lambda)].$$

The infinite system of algebraic equations (29) for the constants $\tilde{c}_\lambda$ (it also contains the unknown amplitudes $\tilde{A}_\lambda$) has exactly the same form as the corresponding system in the potential inverse problem with $V_2 \equiv 0$. The potential case has been rigorously investigated by Newton[29] and Sabatier[30,31] The main results are in the monographs of Newton[13] and Chadan and Sabatier[14]. The system of equations (29) decomposes into two infinite groups of equations:

$$\sin\delta_\lambda = \sum_{\lambda'} a_{\lambda'}M_{\lambda\lambda'}\cos(\delta_{\lambda'}-\delta_\lambda), \qquad (30)$$

$$\tilde{A}_\lambda = \cos\delta_\lambda - (4\lambda)^{-1}\pi a_\lambda - \sum_{\lambda'} a_{\lambda'}M_{\lambda\lambda'}\sin(\delta_{\lambda'}-\delta_\lambda), \qquad (31)$$

where $a_\lambda \equiv \tilde{c}_\lambda\tilde{A}_\lambda$, $M_{\lambda\lambda'} \equiv (\lambda'^2-\lambda^2)^{-1}$ for odd value of $\lambda'-\lambda$ and $M_{\lambda\lambda'} \equiv 0$ for even $\lambda'-\lambda$. The first group of equations (30) is a system of linear equations for the numbers a_λ, while formulas (31) express the amplitudes $\tilde{A}_\lambda$ in terms of a_λ. Thus, the problem of calculating the sequence $\{a_\lambda\}$ is reduced to the inversion of numerical matrices. We determine next the amplitudes $\tilde{A}_\lambda$ and the numbers $\tilde{c}_\lambda$ and c_λ. After this procedure, the function G in the kernel of the analog of the Regge-Newton equation can be constructed.

The problem of uniqueness in this inverse problem resides in the uniqueness of the choice of the function $H(b)$ in the differential equation (25). It is clear that the choice of $H(b)$ is unambiguous if the coefficients c_λ can be chosen uniquely given the phase shifts δ_λ. Therefore, the question reduces to the uniqueness of the solution of the system of linear equations (30). If the phase shifts $\delta_l \to 0$ sufficiently rapidly as $l\to\infty$, then the system (30) has a unique solution.[14] In the potential case, the algorithm then gives a potential with asymptotic behavior $V(r) = o(r^{-3/2})$ as $r\to\infty$. For a unique solution of the gravitational inverse problem it is necessary to require definite rates of decrease of the "potentials" V_1 and V_2 as $r\to\infty$. However, in the metric (1) the space is asymptotically flat, and large distances correspond to the nonrelativistic case. Here, the contribution $V_2 \ll V_1$ and $V_1 \approx m^2\nu$. Therefore, our inverse problem has a unique solution in the class of metric tensors for which $\nu(r) = o(r^{-3/2})$ and $g_{00} = 1 + o(r^{-3/2})$, $r\to\infty$. The rate of decrease of the metric function $\mu(r)$ is immaterial. If ν decreases more slowly, then there is no uniqueness in the solution of the inhomogeneous equations (30). For example, for the case $\delta_l = 0$ for all l we obtain $\nu = \mu \equiv 0$ (flat Minkowski space) and other nonvanishing metric functions. Such metrics may be called "transparent" by analogy with "transparent" potentials.[30] Study of the properties of "transparent" metrics is an interesting theoretical problem.

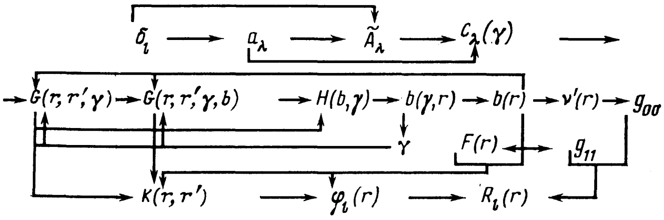
We now present briefly the method developed for recovering the metric tensor from the phase shifts δ_l of mesons scattered in the gravitational field. First of all, given the phase shifts it is necessary to calculate the constants a_λ by solving the linear algebraic system of equations (30) and then, in accordance with the expressions (31), find the amplitudes $\tilde{A}_\lambda$ and the coefficients $\tilde{c}_\lambda$ and c_λ. After summation of the series (14) for the function G, we construct the Fredholm resolvent $\tilde{G}$ for the analog of the linear integral Regge-Newton equation (13). By means of (18) we obtain the function $H(b, \gamma)$, which determines the form of the nonlinear third-order differential equation (25) for the function $b(r)$. The integral of this equation depends parametrically on the unknown constant γ, which is given by the transcendental equation (26). Formula (19) now enables us to calculate the metric function $f(r)$, and formulas (24) and (23) the metric function $\nu(r)$. From the field equations (2) we can then determine the hydrodynamic parameters $\rho(r)$ and $p(r)$ of the matter and, eliminating the radial variable, find the equation of state $p(\rho)$ of the matter.

Finally, knowing the metric, we can obtain the exact wave function R_1 directly without solving the Klein-Fock-Gordon equation. Indeed, we have

$$K(r,r')=F(r)\left[G(b(r),r')-\int_0^{b(r)} d\xi\, G(b(r),\xi)\tilde{G}(\xi,r',b(r))\right]$$

and from the integral representation (15) we calculate $R_l = u^{-1/2}\varphi_l$. Thus, the wave function can be found by means of two quadratures and is completely determined by its own asymptotic behavior.

The scheme for recovering the metric tensor has the form



To illustrate the method, we take the example discussed by Newton[29] and Hooshyar.[27] Suppose all $c_\lambda = c_{\lambda'}$, $\delta_{\lambda\lambda''}$, where $c_{\lambda'} \neq 0$, $\delta_{\lambda\lambda'}$ is the Kronecker delta, and there is no summation. In this case

$$G(r,r')=c_{\lambda'}\varphi_{\lambda'}{}^0(r)\varphi_{\lambda'}{}^0(r'),$$

and the expressions (27), (28), and (17) give[2)]

$$H(b)=c_{\lambda'}[1+c_{\lambda'}\Lambda_{\lambda'\lambda'}(b)]^{-1}\varphi_{\lambda'}{}^{02}(b).$$

The differential equation (25) for $b(r)$ therefore contains spherical Bessel functions, is still very complicated, and cannot be solved in quadratures. Thus, the simple example of Newton does not lead in the gravitational inverse problem to a result that has an explicit form.

We consider the Newtonian approximation (nonrelativistic limit) for the established algorithm. Here[22] $m\to\infty$, $g_{00} \approx 1+2V/m$, $g_{11}\approx 1$, and then $U\approx 0$, $V_1\approx 2mV$, $V_2\approx 0$, i.e., the "potential" that depends on the angular momentum can be ignored, and then $b(r)\approx r$, $F(r)\approx 1$, $\gamma\approx 1$. Formula (17) for the kernel of the transformation operator takes the trivial form $K(r,r)\approx H(r)$, and the differential equation (21) gives

$$2mV=-\frac{2}{r}\frac{d}{dr}\frac{K(r,r)}{r}, \tag{32}$$

since $\nu\approx 2V/m \ll 1$. The expression (32) is the well-known (Ref. 14, p. 239 of the Russian translation) connection between the potential[3)] and the kernel K from the integral equation

$$K(r,r')=G(r,r')-\int_0^r d\xi\,\xi^{-2}K(r,\xi)G(\xi,r'),$$

which follows from (13). This is the Regge-Newton equation, which solves the inverse problem for fixed energy in flat space.

4. INVERSE PROBLEM FOR FIXED ANGULAR MOMENTUM

To analyze this inverse problem, we restrict ourselves to the case of S waves. The $r\to\infty$ behavior of the physical wave function is

$$R_0(k,r)\equiv R(k,r)\to\psi_0(k,r)\equiv\psi(k,r)\sim C(k)\sin(kr+\delta(k)), \tag{33}$$

and in the limit $r\to 0$ this function is regular. The inverse scattering problem for $l=0$ consists of recovering the metric tensor from the energy dependence of the S-wave phase shift $\delta(k)$.

We express the coefficients in the wave equation (6) for $l=0$ in terms of

$$W_1(r)\equiv U+m^2f^{-1}(1-e^{-\nu}),\quad W_2(r)\equiv f^{-1}e^{-\nu}-1, \tag{34}$$

and we then obtain

$$\psi''-W_1\psi+k^2W_2\psi=-k^2\psi, \tag{35}$$

where $W_2>0$ and W_1, $W_2\to 0$, $r\to\infty$. In such form, the Klein-Fock-Gordon equation for fixed $l=0$ is equivalent to the Schrödinger equation with a central "potential" W_1 and a potential k^2W_2, that depends on the square of the momentum.

Energy-dependent potentials may be of topical importance in nuclear theory. However, if we discount some remarks of Marchenko,[32] the only study hitherto made of the inverse problem for energy-dependent potentials is the S-wave case for a potential with linear dependence on k (Ref. 14, p. 185 of the Russian translation). This problem has been more or less completely solved by Jaulent and Jean.[33]

Thus, it is necessary to develop an algorithm of the inverse problem for a k^2-dependent potential in nonrelativistic quantum mechanics. Such an algorithm will be a generalization of Marchenko's well-known algorithm in scattering theory.

We give only the main results. We shall assume that gravitating matter is distributed in the exterior of an absolutely inpenetrable sphere of some finite radius r_0. There is no gravitational field within such a hollow sphere, and the metric is Galilean.[22] The wave function is $\psi(k,r_0)=0$. We form the radial functions

$$\beta(r)\equiv r-\int_r^\infty d\xi\,(f^{-1/2}e^{-\nu/2}-1),\qquad \Phi(r)\equiv[\beta'(r)]^{-1/2}=f^{1/4}e^{\nu/4}. \tag{36}$$

Then r_0 is a root of the transcendental equation

$$r=\int_r^\infty d\xi\,(f^{-1/2}e^{-\nu/2}-1),$$

and, thus, in the region $r>r_0$ of space in which are interested the monotonically increasing $\beta(r)>0$. In the inverse problem, r_0 is not known *a priori* and must itself be determined.

We introduce a linear integral equation, the analog of the Marchenko equation for the gravitational field:

$$B(r,r')=\Phi(r)D(\beta(r)+r')+\int_{\beta(r)}^\infty d\xi\,B(r,\xi)D(\xi+r'). \tag{37}$$

The symmetric kernel D of this equation is constructed from the scattering data.

The $k\to\infty$ behavior of the solution of the Klein-Fock-Gordon equation (35) can be found in the standard manner.[34] It has the form

$$\psi(k,\ r)\sim u^{1/2}(r)\sin k\beta(r), \tag{38}$$

and the wave function oscillates rapidly. In (38), we go to the limit $r\to\infty$ and note that $\beta(r)\sim r$; then it can be seen from the asymptotic behavior (33) that $C(\infty)=1$ and $\delta(\infty)=0$. It is interesting to note that for the Klein-Fock-Gordon equation in flat space the high-energy limit of the phase shift is nonzero.[19] The connection between the kernel D and the phase shift is given (in the absence of bound states) by

$$D(r)=(2\pi i)^{-1}\int_{-\infty}^\infty dk\{\exp 2i\delta(k)-1\}e^{ikr}. \tag{39}$$

The integral equation (37) is the principal link between the S matrix, which is given for all k, and the metric g_{ij} of space. This equation determines a triangular integral operator of the transformation from the free to the perturbed solution. Let $h(k,r)$ be the Jost solution of the wave equation (35), i.e., $h\sim\exp ikr$, $r\to\infty$; then

$$h(k,r)=\Phi(r)\exp[ik\beta(r)]+\int_{\beta(r)}^\infty d\xi\,B(r,\xi)e^{ik\xi},\quad r>r_0. \tag{40}$$

This is the gravitational analog of Levin's representation.[35] The region of triangularity of the kernel B is given by the determinant $g_{00}g_{11}$ of the radial-temporal part of the metric tensor.

For the kernel $B(r,r')$ there is a boundary condition at $r'=\beta$:

$$\Phi W_1=\Phi''-\beta''B(r,\ \beta)-2\beta'dB(r,\ \beta)/dr. \tag{41}$$

This last formula connects the "potentials" W_1 and W_2 through information on the phase shift.

The gravitational analog of the Marchenko equation (37) is a linear integral Fredholm equation of the second kind, which can be inverted.[28] If $\tilde{D}$ is the Fredholm resolvent of the kernel D (it depends parametrically on the limit of integration β), then

$$B(r,r')=\Phi(r)\Big[D(\beta(r)+r')+\int_{\beta(r)}^\infty d\xi\,D(\beta(r)+\xi)\tilde{D}(\xi,r',\beta(r))\Big] \tag{42}$$

and on the boundary $r'=\beta$

$$B(r,\ \beta)=\Phi(r)\omega(\beta), \tag{43}$$

$$\omega(\beta)\equiv D(2\beta)+\int_\beta^\infty d\xi\,D(\beta+\xi)\tilde{D}(\xi,\beta,\beta). \tag{44}$$

For the metric tensor, one can derive a closed system of two differential equations. We do this in terms of the functions ν and β. From the definition (36), we have

$$f=e^{-\nu}/\beta'^{\,2} \tag{45}$$

and, substituting f and f' in (3), we obtain the first equation of the system. We find the second equation from the boundary condition (41), taking into account Eqs. (34) (7), (36), (45), and (43). The resulting system

$$r^2\beta'\nu''-r(r\beta''+2\beta')\nu'+2\beta'^{\,3}e^\nu-2r\beta''-2\beta'=0, \tag{46}$$

$$m^2r\beta'^{\,3}e^\nu-\beta''+2r\omega_\beta\beta'^{\,3}-m^2r\beta'^{\,3}=0 \tag{47}$$

contains the relativistic condition (46) for a static situation and the connection (47) between the metric and the kernel of the transformation operator. The first equation does not contain β, and the second does not contain ν' and ν''; the third derivatives β''' from Φ'' and from U cancel.

We now establish a closed differential equation for the function $\beta(r)$. In (47) we separate

$$\nu=\ln[m^{-2}r^{-1}(\beta''/\beta'^{\,3}+r\omega_1)], \tag{48}$$

where $\omega_1(\beta)\equiv m^2-2\omega_{\beta,}$, and, differentiating (48) with respect to r twice, we eliminate ν, ν', and ν'' from Eq. (46). We then obtain the required equation for $\beta(r)$:

$$\begin{aligned}
&r^3\beta'^{\,2}\beta''\beta^{(4)}+r^4\omega_1\beta'^{\,5}\beta^{(4)}-r^3\beta'^{\,2}\beta'''^{\,2}-4r^3\beta'\beta''^{\,2}\beta'''\\
&-10r^4\omega_1\beta'^{\,4}\beta''\beta'''-2r^2\beta'^{\,2}\beta''\beta'''+4r^4\omega_2\beta'^{\,6}\beta'''\\
&-4r^3\omega_1\beta'^{\,5}\beta'''+6r^3\beta''^{\,4}+15r^4\omega_1\beta'^{\,3}\beta''^{\,3}\\
&+(5r^2+2m^{-2})\beta'\beta''^{\,3}-12r^4\omega_2\beta'^{\,5}\beta''^{\,2}\\
&+3r(3r^2+2m^{-2})\omega_1\beta'^{\,4}\beta''^{\,2}+r\beta'^{\,2}\beta''^{\,2}\\
&-2r^2(r^2-3m^{-2})\omega_1^{\,2}\beta'^{\,7}\beta''-2r^4\omega_3\beta'^{\,7}\beta''\\
&-2r^5\omega_4\beta'^{\,10}+2m^{-2}r^3\omega_1^{\,3}\beta'^{\,10}+4r^4\omega_1\omega_2\beta'^{\,9}-2r^3\omega_1^{\,2}\beta'^{\,8}=0.
\end{aligned} \tag{49}$$

Here

$$\omega_2(\beta)\equiv\omega_{\beta\beta},\ \omega_3(\beta)\equiv\omega_{\beta\beta\beta},\ \omega_4(\beta)\equiv2\omega_{\beta\beta}^{\,2}+m^2\omega_{\beta\beta\beta}-2\omega_\beta\omega_{\beta\beta\beta}.$$

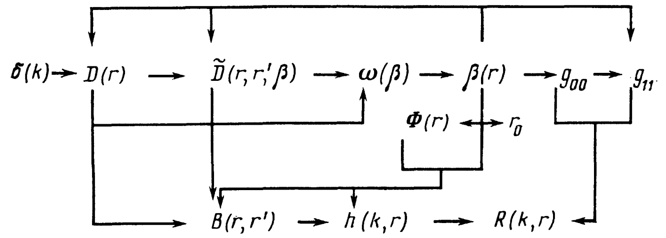
The nonlinear fourth-order differential equation (49) is the final result in the algorithm for recovering the metric from the phase shift $\delta(k)$. It can be constructed from the S matrix, which determines the form of the function $\omega(\beta)$. Equation (49) must be integrated subject to the boundary condition $\beta(r)\sim r$, $r\to\infty$, whence $\beta'(\infty)=1$, $\beta''(\infty)=0$, and $\beta'''(\infty)=0$. The highest derivate $\beta^{(4)}$ occurs linearly in (49). The other unknown function of the system, $\nu(r)$, can be found in accordance with (48) without any integrations at all.

Uniqueness of the solution of the inverse problem is ensured by unique choice of the function $\omega(\beta)$ in the differential equation (49). If there are no bound states, the metric is determined by the phase shift $\delta(k)$ uniquely. Otherwise, identical phase shifts and binding energies lead to different g_{ij}. It is interesting to use this circumstance to obtain families of metrics that do not give any scattering at all. Such "transparent" metrics are analogous to the "transparent" potentials for fixed angular momentum.[36,37]

The main stages in the recovery of the form of the gravitational field from the S-wave phase shift $\delta(k)$ of mesons is as follows. It is first necessary to calculate the Fourier transform (39), i.e., the kernel D of the analog of the linear integral Marchenko equation (37), and find the Fredholm resolvent $\widetilde{D}$ for this kernel. Then, using the definition (44), it is necessary to construct the function $\omega(\beta)$, which determines the structure of the nonlinear fourth-order differential equation (49) for the function $\beta(r)$. After integration of this equation, we obtain r_0, which gives the transcendental equation $\beta(r)=0$. Then, using (48), we calculate the metric component g_{00} and, using (45), the component g_{11}. From the field equations (2), we can then determine the pressure and density of the fluid and, then, the equation of state of the matter.

Knowing the recovered metric tensor, we can find the exact meson wave function $R(r)$ by means of quadratures without solving the wave equation. Indeed, having the function $\Phi(r)$, we calculate in accordance with (42) the kernel of the transformation operator $B(r, r')$ and the Jost solution $h(r)$ by means of the analog of Levin's representation (40). Thus, the wave function $R=u^{1/2}h$ is uniquely determined by the phase shift and the parameters of the bound states (if there are any).

The scheme of the method has the form



We consider the Newtonian approximation for this algorithm. In the nonrelativistic limit $m\to\infty$, $v\approx 2V/m$, $\mu\approx 0$, $W_2\approx 2mV$, $W_2\approx 0$, i.e., there is no energy-dependent potential. Then $\beta(r)\approx r$, $\Phi(r)\approx 1$, $r_0\approx 0$, and from Eq. (43) we have $B(r,r)\approx\omega(r)$. The differential equation (47) reduces to

$$2mV=-2dB(r,r)/dr,$$

which expresses the well-known (Ref. 14, p. 101 of the Russian translation) connection between the potential and the kernel B of the transformation operator. The kernel B now satisfies the integral equation

$$B(r,r')=D(r+r')+\int_r^\infty d\xi\, B(r,\xi)D(\xi+r'),$$

which follows from (37). This is the Marchenko equation that solves the inverse problem for fixed angular momentum $l=0$ in flat space.

5. CONCLUSIONS

The key elements of our algorithms are the two definite nonlinear ordinary differential equations for the metric tensor. In the inverse problem for fixed energy, the inversion process leads to an equation of third order, and in the inverse problem for fixed angular momentum to an equation of fourth order. These differential equations are constructed from the scattering data and thereby resolve the problem of relating the S matrix (for all l or for all k) to the covariant Laplace-Beltrami operator in relativistic quantum mechanics.

The wave equations (10) and (35) for the inverse problems for fixed k and l with corresponding boundary conditions induce generalized eigenvalue problems of the form

$$\hat{Q}\psi(r)=q\Omega(r)\psi(r)$$

for linear Hermitian differential operators $\hat{Q}$ of second order with functions $\Omega(r)>0$. The scalar product in the Hilbert space of the eigenfunctions must be determined with weight Ω; then the basic properties of the spectrum of the simplest Hermitian problem (when $\Omega(r)\equiv 1$) are preserved. This weight is $\Omega=r^{-2}g_{11}$ in the inverse problem for fixed k and $\Omega=g_{00}g_{11}$ in the inverse problem for fixed l. Thus, the weight function is unknown *a priori* and can be found from the S matrix.

The analogy between the Klein-Fock-Gordon equation in the Schwarzschild metric for fixed k and the Schrödinger equation with potential that depends on the angular momentum leads to the existence of two identical one-parameter families of solutions $\psi_\lambda(r)$ (relativistic and nonrelativistic) with parameter λ. The metric tensor g_{ij} is in one-to-one correspondence with the equivalent "potentials" V_1 and V_2, and the obtaining of g_{00} from given V_1 and V_2 reduces to integration of a second-order equation. On the other hand, the Klein-Fock-Gordon equation in the Schwarzschild metric for fixed $l=0$ is identical to a Schrödinger equation with energy-dependent potential, and there exist two identical one-parameter families of solutions $\psi(k,r)$ with parameter k. The metric and the "potentials" W_1 and W_2 are also in a one-to-one correspondence, and to obtain g_{ij} it is again necessary to solve a second-order equation. These isomorphisms make it possible to reduce the calculation of the motion of mesons in a strong gravitational field to the calculation of motion in fields with potentials that depend on λ^2 or k^2 in nonrelativistic quantum mechanics and, thus, afford a unique possibility to model relativistic problems in general relativity by nonrelativistic problems.

If the particles have electric charge, a corresponding potential interaction must occur in the inverse scattering problems on the gravitational background. This background can be specified either by a spherically symmetric metric within a star or by the exterior Schwarzschild metric in the vacuum. The case when the exterior gravitational background is known may be of interest for applications, for example, in atomic and nuclear physics. If an electric field is taken into account in the formalism of inverse problems developed here, its unknown scalar potential can be recovered from scattering phase shifts obtained from an experiment by means of phase-shift analysis.

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[1)]In Ref. 27 there is a misprint in Eq. (2.7): in the second term on the right-hand side there should be b'^2 instead of b'.

[2)]In Ref. 27, there is a misprint in Eq. (3.6): in the argument of the second factor on the right-hand side there should be $b(r)$ instead of r.

[3)]In the monograph of Ref. 14, a system of units in which $2m=1$ is adopted.

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