Peculiarities in the dynamics of superfluid ^{3}He-A: analog of chiral anomaly and of zero-charge

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Peculiarities in the dynamics of ^{3}He-A (anomalous current, non-analyticity of the gradient expansion, nonconservation of the superfluid current at $T = 0$, the orbital-angular-momentum paradox) are due to the vanishing of the gap in the energy spectrum of the fermionic excitations at two points on the Fermi sphere. Near these points, the Bogolyubov equations for the fermions become linearized and are transformed into the Weyl equations for zero-mass right-hand "electrons" that move in an "electromagnetic" field and in a "gravitational" field; the fields are formed by fluctuations of the order parameter. This makes it possible to relate the peculiarities in the ^{3}He-A dynamics with the chiral anomaly, the zero-charge, the nonlinear polarization of the vacuum, and production of electron-positron pairs in strong fields in quantum electrodynamics with zero-mass chiral fermions. The local gauge invariance and the general covariance of the obtained Weyl equations permit a substantial simplification of the derivation of various action peculiarities for ^{3}He-A, including that of the Wess-Zumino action.

INTRODUCTION

The dynamics of ^{3}He has at low temperatures a number of unusual features due to the vanishing of the energy gap in the fermion-excitation spectrum at two points on the Fermi surface. These are the presence of an anomalous current, the orbital-angular momentum paradox, the non-analyticity of the gradient expansion, the nonconservation of the superfluid flow at $T = 0$, the existence of a normal component at $T = 0$, and others (see Sec. I). We consider here the analogy between these phenomena and those in quantum electrodynamics (QED). The connection between ^{3}He-A and QED was first noted by Combescot and Dombre (see Ref. 1 and the citations therein), who pointed out that the fermion-excitation spectrum in the field of the anisotropy vector $\mathbf{l}$ is similar to the spectrum of a charged Dirac particle in a magnetic field. This has enabled them to calculate correctly the density of states on the Fermi surface, and by the same token the density of the normal component at $T = 0$. It was shown in later papers[2–7] that the dynamics of fermion excitations that interact in ^{3}He-A with collective boson modes of a multicomponent order parameter is similar in many respects to the dynamics of chiral zero-mass fermions that interact with electromagnetic, weak, and gravitational fields. This has made it possible to relate the peculiarities of the ^{3}HE-A dynamics to the zero-charge and chiral anomaly phenomena, a relation which we discuss here in greater detail.

It is shown in Sec. II that the Bogolyubov equation for fermion excitations reduces near the zeros on the Fermi surface (boojums) to the Weyl equation for chiral fermions. These chiral fermions are located in an order-parameter field, and some of the components of the order parameter act on the fermion in the same manner as an electromagnetic or gravitational field. As a result, the Bogolyubov equations acquire near the boojums a local gauge invariance and a general covariance, thereby substantially simplifying the derivation of the features connected with the zeros of the gap.

In Sec. III is discussed, as applied to ^{3}He-A, the zero-charge phenomenon in a system of fermions interacting with a gauge field.[8] It is shown that this phenomenon leads to a non-analytic expansion of the ^{3}He-A energy at $T = 0$ in terms of the spatial and temporal gradients of the order parameter.

In Sec. IV we track the connection between the chiral anomaly in QED with zero-mass fermions (nonconservation of the chiral current, see, e.g., Ref. 9), and the nonconservation of the superfluid flow in ^{3}He-A at $T = 0$. It is shown that the Schwinger source of the current in QED is the analog, in ^{3}He-A of momentum transfer from the superfluid to the normal subsystem of the fermion excitation, a transfer that takes place even at $T = 0$, in view of the nonzero density of the fermion states in the presence of a magnetic field.

In Sec. V is discussed, for the ^{3}He-A dynamics, the Wess-Zumino action that describes the chiral anomaly and the orbital-momentum paradox in ^{3}He-A, and also the connection of this action with the θ-term in QED.

I. ANOMALIES IN THE DYNAMICS OF ^{3}He-A AT $T=0$

Cooper pairing takes place in superfluid ^{3}He in a state with orbital momentum $l = 1$ relative to the motion of the ^{3}He atoms in the pair (p-pairing) and with a pair spin $s = 1$ (see the review in Ref. 10). Superfluid phases differ from one another in their symmetry which governs the possible spin and orbital-momentum projections. The state established in ^{3}He-A has a projection $m = 1$ of the orbital momentum l on a quantization axis whose direction is designated by the unit vector $\mathbf{l}$ and with zero projection of the spin s on a quantization axis whose direction is labeled by the unit vector $\mathbf{d}$. The vectors $\mathbf{l}$ and $\mathbf{d}$ define respectively the axes of the orbital (liquid-crystal) and magnetic (spin) anisotropy in ^{3}He-A. In addition, the vector $\mathbf{l}$ indicates the common rotation direction of the Cooper pairs about their axes. Owing to this internal rotation, the entire liquid has, even in a homogeneous state, an angular momentum $(1/2)N\hbar\mathbf{l}$, where N is the number of the ^{3}He atoms ($N/2$ pairs, each with an angular momentum $\hbar$ in the $\mathbf{l}$ direction).

The simplest system having the same internal symmetry as ^{3}He-A is a Bose condensate of isolated boson molecules

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having an angular momentum $\hbar\mathbf{l}$. We consider initially the dynamics of this system, which is free of anomalies, and then proceed to a realistic model for ^{3}He-A, viz, a Fermi gas with weak pairing interaction, and examine the anomalies that appear in the orbital dynamics as a result of the appearance of a Fermi surface having points at which the energy gap vanishes.

The dynamics of a Bose condensate of molecules at $T = 0$ is determined by phenomenological equations uniquely obtained either from a Lagrange formalism that uses the internal symmetry of ^{3}He-A (Ref. 11) or from a Poisson-bracket algebra determined by Lie algebra of the symmetry group in ^{3}He-A (see, e.g., Ref. 12). The hydrodynamic variables in ^{3}He-A at $T = 0$ are the density and the soft variables $\mathbf{l}$ and $\mathbf{v}_s$ that describe the coherent motion of the Bose condensate and are connected with spontaneous violation of the gauge invariance and of the invariance to spatial rotations. The solenoidal, and its curl is expressed in terms of the gradients of the vector $\mathbf{l}$ by the Mermin-Ho relation[13]:

$$\text{rot}\,\mathbf{v}_s=(\hbar/4m_3)e_{ikl}l_i[\nabla l_k,\ \nabla l_l], \tag{1.1}$$

where m_3 is the mass of the ^{3}He atom. The superfluid motion is not potential because of the combined rotational-gauge symmetry that is preserved in ^{3}He-A (see Ref. 10), as a result of which $\mathbf{v}_s$ is expressed in terms of orbital variables. The angular part of the wave function of the orbital motion of the molecule with $l = 1$ and $m = 1$ takes the form $n_x + in_y$, where $\mathbf{n}$ is the direction of the radius vector between the atoms in the molecule, while x and y are axes perpendicular to $\mathbf{l}$. The directions of these axes are designated $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$:

$$|\mathbf{\Delta}_1|^2=|\mathbf{\Delta}_2|^2=1,\quad \mathbf{\Delta}_1\mathbf{\Delta}_2=0,\quad [\mathbf{\Delta}_1\mathbf{\Delta}_2]=\mathbf{l},$$

so that $n_x + in_y = \mathbf{n}(\mathbf{\Delta}_1 + i\mathbf{\Delta}_2)$. The complex vector $\mathbf{\Delta}_1 + i\mathbf{\Delta}_2$, which is the same for all Bose-condensate molecules, is the orbital part of the order parameter in ^{3}He-A. The superfluid velocity is expressed in terms of this order parameter as follows:

$$\mathbf{v}_s=(\hbar/2m_3)\Delta_{1i}\nabla\Delta_{2i}, \tag{1.2}$$

from which follows relation (1.1).

The hydrodynamics equations for the variables ρ, $\mathbf{l}$, and $\mathbf{v}_s$ of a Bose condensate of isolated molecules include the internal-angular-momentum conservation law $\mathbf{L} = (\hbar/2m_3)\rho\mathbf{l}$ and an equation for $\mathbf{v}_s$:

$$\partial_t\mathbf{L}+\delta F/\delta\mathbf{\theta}=0, \tag{1.3a}$$

$$\partial_t\mathbf{v}_s+\nabla\mu+\frac{\hbar}{2m_3}e_{imn}l_i\nabla l_m\partial_t l_n=0. \tag{1.3b}$$

Here F is the free energy that depends on ρ, $\mathbf{v}_s$, and $\mathbf{l}$, and $\mu = \delta F/\delta\rho$ is the chemical potential. Variation with respect to $\delta\theta$ denotes variation with respect to an infinitely small rotation of the order parameter $\mathbf{\Delta}_1 + \mathbf{\Delta}_2$, such that

$$\delta\mathbf{l}=[\delta\mathbf{\theta},\mathbf{l}],\quad \delta\mathbf{v}_s=-(\hbar/2m_3)l_i\nabla\delta\theta_i.$$

Equation (1.3a) contains both the particle-number conservation law, which is obtained by scalar multiplication of the equation by $\mathbf{l}$:

$$\partial_t\rho+\nabla\mathbf{j}=0,\quad \mathbf{j}=\delta F/\delta\mathbf{v}_s, \tag{1.4}$$

and an equation obtained for the variable $\mathbf{l}$ from that part of (1.3a) which is transverse to $\mathbf{l}$:

$$\frac{\hbar}{2m_3}\rho\partial_t\mathbf{l}+\left[\mathbf{l},\frac{\delta F}{\delta\mathbf{l}}\bigg|_{\mathbf{v}_s}\right]+\frac{\hbar}{2m_3}(\mathbf{j}\nabla)\mathbf{l}=0. \tag{1.5}$$

The particle flux $\mathbf{j}$ is fully coherent at $T = 0$ and consists of two parts: one transported with supersonic velocity $\mathbf{v}_s$, and the other (the orbital current) caused by the incomplete cancellation of the internal rotational motion of the molecules if $\mathbf{l}$ is not uniform:

$$\mathbf{j}^{coh}=\rho\mathbf{v}_s+{}^1/_2\,\text{rot}\,\mathbf{L},\quad \mathbf{L}=(\hbar/2m_3)\rho\mathbf{l}. \tag{1.6}$$

The liquid flow is equal to the momentum density, and should therefore be conserved. Equation (1.3) ensures both momentum conservation

$$\partial_t j_i^{coh}+\nabla_k\pi_{ik}=0, \tag{1.7}$$

and energy conservation, and is thereby completely closed.

We proceed now to real ^{3}He-A, where the isolated Bose molecules are replaced by Cooper pairs produced on the Fermi surface. In this system, the hydrodynamics equations, at least in linerized form, can be obtained either from the Gor'kov equations, or from the matrix transport equation (see Ref. 14). This gives rise to unexpected deviations from Eqs. (1.3), the reason being that in the energy spectrum $E(\mathbf{k})$ of the Fermi quasiparticles

$$E^2(\mathbf{k})=\varepsilon^2(k)+|\Delta(\mathbf{k})|^2,\quad \varepsilon=k^2/2m_3-k_F{}^2/2m_3,$$
$$\Delta(\mathbf{k})=(\Delta_0/k_F)\mathbf{k}(\mathbf{\Delta}_1+i\mathbf{\Delta}_2) \tag{1.8}$$

the gap $\Delta(\mathbf{k})$ vanishes at two points on the poles of the Fermi surface, $\mathbf{k} = \pm k_F\mathbf{l}$ (here Δ_0 is the maximum gap obtainable on the equator of the Fermi sphere). Those poles at which the gaps vanish are vortical singularities (boojums) of the phase Φ of the gap:

$$\Delta(\mathbf{k})=|\Delta(\mathbf{k})|e^{i\Phi(\mathbf{k})}, \tag{1.9}$$

namely, Φ changes by 2π on circling around the pole on the Fermi surface. At the pole itself the phase Φ is indeterminate, so that the modulus of the gap must vanish. Boojums are topologically stable to small stirrings of the order parameter,[15] therefore the zeros in the gaps do not vanish at small deviations from the A-phase in dynamic processes, and the singularities in the ^{3}He-A dynamics are stable.

The following phenomena are caused by the zeros of the gap.

1. Non-analyticity in the expansion of the free energy F in terms of the gradients of the vector $\mathbf{l}$. Namely, the coefficient K_3 in the expansion

$$F\{\mathbf{l}\}=\int d^3x\{K_1(\nabla\mathbf{l})^2+K_2(\mathbf{l}\,\text{rot}\,\mathbf{l})^2+K_3[\mathbf{l}\,\text{rot}\,\mathbf{l}]^2\} \tag{1.10}$$

goes to infinity logarithmically as $T\to 0$ (Ref. 16):

$$K_3=\frac{k_F{}^3\hbar^2}{24\pi^2m_3}\ln\frac{\Delta_0{}^2}{T^2}, \tag{1.11}$$

implying the presence, in the gradient expansion, of a non-analytic term[17]

$$[\mathbf{l}\,\text{rot}\,\mathbf{l}]^2\ln\frac{\Delta_0{}^2}{[\mathbf{l}\,\text{rot}\,\mathbf{l}]^2}. \tag{1.12}$$

2. The coherent current of particles contains an additional anomalous term (see, e.g., Ref. 18):

$$\mathbf{j}^{coh}=\rho\mathbf{v}_s+{}^1/_2\,\text{rot}\,\mathbf{L}+\mathbf{j}_{an},$$

$$\mathbf{j}_{an}=-{}^1/_2C_0\mathbf{l}(\mathbf{l}\,\text{rot}\,\mathbf{l}),\quad C_0=k_F{}^3/3\pi^2. \tag{1.13}$$

The anomalous current is expressed in terms of the difference of the mixed derivatives with respect to the phase $\Phi(\mathbf{k},\mathbf{r})$, Ref. 19:

$$\mathbf{j}_{an}=\sum_{k<k_F}\mathbf{k}\left(\frac{\partial}{\partial\mathbf{k}}\frac{\partial}{\partial\mathbf{r}}-\frac{\partial}{\partial\mathbf{r}}\frac{\partial}{\partial\mathbf{k}}\right)\Phi(\mathbf{k},\mathbf{r}), \tag{1.14}$$

and this difference is not equal to zero only at the boojums points, where Φ has a vortical singularity.

3. The coefficient of $\partial_t\mathbf{l}$ in (1.5) contains not the pair density $\rho/2m_3$, but the quantity $L_0=1/2\hbar(\rho/m_3-C_0)$, which is small to the extent that the gap is small compared with the Fermi energy. Neglecting the Fermi-liquid corrections, this quantity takes the form[16,20]

$$L_0=\frac{\hbar}{2}\left(\frac{\rho}{m_3}-C_0\right)=-\frac{\Delta_0^2}{2}\sum_{\mathbf{k}}\frac{(\mathbf{kl})^2}{k_F^2}\frac{\partial}{\partial\varepsilon}\left(\frac{1}{E}\right)$$
$$\approx\frac{\hbar}{8m_3}\rho\frac{\Delta_0^2}{\varepsilon_F^2}\ln\frac{\varepsilon_F}{\Delta_0}. \tag{1.15}$$

As a result, Eqs. (1.4) and (1.5) do not combine to form the general equation (1.3a); this attests to the nonlocality of the dynamic angular momentum. If it is assumed that, as before, the coherent angular momentum of the Cooper pairs is equal to $\mathbf{L}=\hbar\rho\mathbf{l}/2m_3$, then Eq. (1.3a) acquires a right-hand side (source of the angular momentum):

$$\partial_t\mathbf{L}+\delta F/\delta\theta=\frac{\hbar}{2}C_0\partial_t\mathbf{l}. \tag{1.16}$$

4. Equations (1.16) and (1.3b) are no longer closed, since they do not conserve the coherent current (1.13). The equation for the current acquires a right-hand part (momentum source)[19]:

$$\partial_t j_i^{coh}+\partial_k\pi_{ik}^{coh}=-{}^3/_2\hbar C_0 l_i(\partial_t\mathbf{l},\operatorname{rot}\mathbf{l}). \tag{1.17}$$

Nonconservation of the coherent momentum at $T=0$ has led to the conclusion that ^{3}He-A should have even at $T=0$ a normal component transported with normal velocity $\mathbf{v}_n$ together with the momentum. This is a system of fermion excitations that are produced in coherent motion of the condensate as a result of the vanishing of the gap.[19]

5. At $T=0$ there exists an additional incoherent current of fermion excitations

$$\mathbf{j}^{inc}=\sum_{\mathbf{k}}\mathbf{k}\nu(E(\mathbf{k})+\mathbf{k}(\mathbf{v}_s-\mathbf{v}_n)), \tag{1.18}$$

where $\nu(E)=\theta(-E)$ is a step-function fermion distribution function at $T=0$ and differs from zero at arbitrarily small $\mathbf{v}_s-\mathbf{v}_n$, owing to the absence of a gap in the spectrum. Calculation of the current (1.18) leads to the following value of the normal-component density at $T=0$ in the case of a constant field $\mathbf{l}$ (Refs. 10 and 21):

$$\mathbf{j}^{inc}=\overleftrightarrow{\rho}_n(\mathbf{v}_n-\mathbf{v}_s), \tag{1.19a}$$

$$(\rho_n^{(1)})_{ij}=\rho l_i l_j(k_F/\Delta_0)^2(\mathbf{l}(\mathbf{v}_n-\mathbf{v}_s))^2, \tag{1.19b}$$

and if $\mathbf{l}$ is not uniform the normal density ρ_n acquires one more contribution[9,17,22,23]:

$$(\rho_n^{(2)})_{ij}={}^3/_4\rho l_i l_j(\hbar v_F/\Delta_0)\,|[\mathbf{l}\operatorname{rot}\mathbf{l}]|. \tag{1.19c}$$

Investigtion of the dynamics of the excitations near boojums in the hydrodynamic regime has indeed shown[1] that the source of the coherent current $\mathbf{I}$ in the right-hand side of (1.17) is the drain for the incoherent current:

$$\partial_t j_i^{inc}+\partial_k\pi_{ik}^{inc}=-I_i,\quad \mathbf{I}=-{}^3/_2\hbar C_0\mathbf{l}(\partial_t\mathbf{l},\operatorname{rot}\mathbf{l}), \tag{1.20}$$

so that the total momentum of the coherent and incoherent motions is conserved.

All the phenomena described here are the results of vanishing of the gap at two points on the Fermi sphere. To clarify the physical meaning of these phenomena it is therefore necessary to investigate the dynamics of the fermion excitations near the boojums where, as will be shown in the next section, the fermions in the field of the inhomogeneous order parameter in ^{3}He-A are perfectly similar to chiral fermions in electromagnetic and gravitational fields.

II. CHIRAL FERMIONS IN ^{3}He-A

The model BCS action that describes triplet Cooper pairing in ^{3}He is given by

$$S\{\Psi_\alpha\}=\int d^3r\,dt\Big\{\Psi_\alpha^+(i\hbar\partial_t+\mu)\Psi_\alpha-\frac{1}{2m_3}(\hat{\mathbf{p}}\Psi_\alpha)^+\hat{\mathbf{p}}\Psi_\alpha$$
$$+\lambda({}^1/_2\Psi_\alpha\hat{\mathbf{p}}\Psi_\beta)^+_*({}^1/_2\Psi_\alpha\hat{\mathbf{p}}\Psi_\beta)_*\Big\},\quad \hat{\mathbf{p}}=-i\hbar\nabla, \tag{2.1}$$

where $\Psi_\alpha(\mathbf{r},t)$ is the spinor annihilation operator of the ^{3}He atom, λ is the constant of the p-harmonic of the pairing interaction, and the asterisk denotes symmetrization with respect to the spin indices.

Below the temperature T_c of the superfluid transition, a coherent superfluid state sets in, with a nonzero quasi-average

$$\boldsymbol{\Delta}_{\alpha\beta}=\lambda\langle{}^1/_2\Psi_\alpha\hat{\mathbf{p}}\Psi_\beta\rangle, \tag{2.2}$$

that is symmetric in the spinor indices and can therefore be expressed in terms of a symmetric combination of the Pauli matrices $\hat{\sigma}^\mu$ and of the metric spinor $\hat{g}$ as follows:

$$\boldsymbol{\Delta}_{\alpha\beta}=\frac{1}{k_F}(\hat{g}\hat{\sigma}^\mu)_{\alpha\beta}\mathbf{A}_\mu,\quad \hat{g}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{2.3}$$

The complex 3×3 matrix $\mathbf{A}_\mu\equiv A_{\mu i}$ is called the order parameter. It transforms like a vector with respect to the second (Latin) subscript in spatial rotations and like a vector with respect to the first (Greek) subscript in spin rotations. The spin-orbit interaction in superfluid ^{3}He is exceedingly small and does not influence the considered effects, so that the two rotations can be treated independently.

Transformation from pure fermion fields to coherent boson fields and to fermion excitations against the background of a new superfluid vacuum is possible, for example, by functional integration (see Ref. 24) on introduction of an additional Gaussian integral over the Bose fields:

$$\int d\boldsymbol{\Delta}^+d\boldsymbol{\Delta}\exp\left(\frac{1}{\lambda}\int d^3r\,dt\,\boldsymbol{\Delta}^+\boldsymbol{\Delta}\right).$$

A shift $\boldsymbol{\Delta}_{\alpha\beta}\to\boldsymbol{\Delta}_{\alpha\beta}-1/2\lambda\Psi_\alpha\hat{\mathbf{p}}\Psi_\beta$ is carried out in this integral to cancel out fourth-order terms in the action (2.1); the effective fermion-boson action takes then a form quadratic in the fermions:

$$S\{\Psi_\alpha,\Delta_{\alpha\beta}\}=\int d^3r\,dt\Big\{\Psi_\alpha^+(i\hbar\partial_t+\mu)\Psi_\alpha - \frac{1}{2m_3}(\hat{\mathbf{p}}\Psi_\alpha)^+(\hat{\mathbf{p}}\Psi_\alpha) - {}^1\!/_2\Delta_{\alpha\beta}\Psi_\alpha^+\hat{\mathbf{p}}\Psi_\beta^+ - {}^1\!/_2\Delta_{\alpha\beta}^+\Psi_\alpha\hat{\mathbf{p}}\Psi_\beta+\frac{1}{\lambda}\Delta^+\Delta\Big\}. \qquad (2.4)$$

This action describes the fermion quasiparticles in the field of an 18-component matrix $A_{\mu i}$, which is itself a dynamic variable. The most interesting is interaction between fermion and boson fields in the case when the superfluid ^{3}He is in the A phase, i.e., the vacuum manifold is described by matrices of the form:

$$A_{\mu i}=\Delta_0 d_\mu(\Delta_{1i}+i\Delta_{2i}). \qquad (2.5)$$

The vacuum manifold of the A phase is five-dimensional (two angles of the magnetic-anisotropy $\mathbf{d}$ and three orbital-subsystem angles that describe the rigid-body rotation of the unit vectors $\mathbf{\Delta}_1$, $\mathbf{\Delta}_2$, and $\mathbf{l}=\mathbf{\Delta}_1\times\mathbf{\Delta}_2$). Thus, five of the 18 boson variables in ^{3}He-A are Goldstone variables, and the remaining 13 have mass, since they lead out of the vacuum manifold.

We shall consider hereafter only those boson variables that are directly connected with anomalies in the orbital dynamics, i.e., we confine ourselves to the vacuum manifold (25), and fix the vector $\mathbf{d}$, leaving only the three degrees of freedom of the unit vectors $\mathbf{\Delta}_1$, $\mathbf{\Delta}_2$ and $\mathbf{l}$. We have excluded thereby from consideration the "W bosons" [5] and the "gravitons",[7] which have mass.

If the vector $\mathbf{d}$ is directed along the y axis, the action for the fermions (2.4) breaks up into two independent actions, separate for fermions with spin along z and for fermions with opposite spin, i.e., along $-z$. We write down the action for each of the spin directions, leaving out the spin subscripts:

$$S_\uparrow=S_\downarrow=\int d^3r\,dt\Big\{\Psi^+(i\hbar\partial_t+\mu)\Psi-\frac{1}{2m_3}(\hat{\mathbf{p}}\Psi)^+(\hat{\mathbf{p}}\Psi) -\frac{\Delta_0}{2k_F}(\mathbf{\Delta}_1+i\mathbf{\Delta}_2)\Psi^+\hat{\mathbf{p}}\Psi^+-\frac{\Delta_0}{2k_F}(\mathbf{\Delta}_1-i\mathbf{\Delta}_2)\Psi\hat{\mathbf{p}}\Psi\Big\}. \qquad (2.6)$$

Introducing, finally, the Bogolyubov spinor in the particle-hole space

$$\chi=2^{-1/2}\begin{pmatrix}\Psi\\ \Psi^+\end{pmatrix}$$

and the corresponding Pauli matrices τ^a, we obtain

$$S_\downarrow=S_\uparrow=\int d^3r\,dt\,\chi^+(i\hbar\partial_t-\hat{H})\chi,$$

$$\hat{H}=\tau^3\left(\frac{\hat{\mathbf{p}}^2}{2m_3}-\mu\right)+\frac{\Delta_0}{2k_F}\tau^1(\Delta_1{}^i\hat{p}_i+\hat{p}_i\Delta_1{}^i)+\frac{\Delta_0}{2k_F}\tau^2(\Delta_2{}^i\hat{p}_i+\hat{p}_i\Delta_2{}^i). \qquad (2.7)$$

If the order parameter $\mathbf{\Delta}_1+i\mathbf{\Delta}_2$ is homogeneous in space, the fermion spectrum is obtained by squaring the Hamiltonian:

$$E^2(\mathbf{k})=\hat{H}^2(\mathbf{k})=(k^2/2m_3-\mu)^2+(\Delta_0{}^2/k_F{}^2)\left[(\mathbf{\Delta}_1\mathbf{k})^2+(\mathbf{\Delta}_2\mathbf{k})^2\right]. \qquad (2.8)$$

The energy of the Bogolyubov quasiparticles $E(\mathbf{k})$ vanishes at two points on the Fermi sphere: $\mathbf{k}_\pm=\pm k_F\mathbf{l}$, where $k_F^2/2m_3=\mu$. Near these points, which we shall distinguish by a subscript e that takes on the values $+1$ and -1 (i.e., $\mathbf{k}_e=ek_F\mathbf{l}$), the spectrum takes the form

$$E_e{}^2(\mathbf{k})=(\mathbf{e}_1,\mathbf{k}-e\mathbf{A})^2+(\mathbf{e}_2,\mathbf{k}-e\mathbf{A})^2+(\mathbf{e}_3,\mathbf{k}-e\mathbf{A})^2 =g^{ij}(k_i-eA_i)(k_j-eA_j), \qquad (2.9)$$

where

$$\mathbf{A}=k_F\mathbf{l},\quad \mathbf{e}_1=c_\perp\mathbf{\Delta}_1,\quad \mathbf{e}_2=c_\perp\mathbf{\Delta}_2,\quad \mathbf{e}_3=c_\parallel\mathbf{l},$$

$$g^{ij}=\sum_{a=1}^{3}e_a{}^ie_a{}^j=c_\parallel{}^2l^il^j+c_\perp{}^2(\delta^{ij}-l^il^j),\quad c_\parallel=\frac{k_F}{m_3},\quad c_\perp=\frac{\Delta_0}{k_F}. \qquad (2.10)$$

This corresponds to a spectrum of zero-mass particles that move in an electromagnetic field (with vector potential $\mathbf{A}$) and in a gravitational field (metric tensor g^{ij} made up of the triads e_a^i) with the fermions having positive "charge" $e=+1$ near the north pole and negative $e=-1$ near the south pole.

A similar expansion in $\hat{\mathbf{p}}-e\mathbf{A}$ can be carried out directly in the Hamiltonian (2.7). Neglecting then the term quadratic in $\hat{\mathbf{p}}-e\mathbf{A}$ and rotating through an angle $\pi(1-e)/2$ about the third axis in the particle-hole space (the Nambu space), we obtain for the Hamiltonian (2.7)

$$\hat{H}={}^1\!/_2e\tau^a[e_a{}^i(\hat{p}_i-eA_i)+(\hat{p}_i-eA_i)e_a{}^i]. \qquad (2.11)$$

The equations for the fermion field χ near the poles

$$i\hbar\partial_t\chi=\hat{H}\chi \qquad (2.12)$$

are thus Weyl equations for charged chiral fermions, and owing to the factor e in the Hamiltonian (2.11) the particles close to different poles have different chiralities, viz., right-hand for positively charged particles and left-hand for negatively charged. Since the sign of the charge is the same as that of the parity, the chiral current in such a QED coincides with the electric current.

Compared with the general Bogolyubov equation far from the poles, the Weyl equation near the poles has a property that will be found very useful: it has both local gauge invariance and invariance to general transformations of the coordinates in the sense that the transformations of the vector potential $\mathbf{A}$ and of the triads e_a^i can be so defined that they cancel out the local tranformations of the spinor χ. Of course, such local transformations are not invariant to boson action, but nonetheless the fact that invariance to fermion action does take place is very important for the following reason.

The effective boson action obtained by integrating over the fermion vacuum can be broken up into two parts. Some terms in this action are obtained by integration over all the fermions, icluding those far from the poles, where the Bogolyubov Hamiltonian (2.7) differs from the Weyl Hamiltonian (2.11). There is no invariance for such terms. The action terms of interest to us, however, which describe the chiral anomaly and the zero-charge, are obtained by integrating exclusively over the fermions near the poles of the Fermi sphere; the contribution of the remaining regions of the Fermi surface is of no importance for them. In this case they can be obtained by using the Weyl equation, therefore these terms in the boson action will of necessity be expressed in terms of such combinations of A_i and e_a^i which are invariant to transformations that cancel out local transformations of the spinor.

To determine these combinations, Eqs. (2.12) and (2.11) must be rewritten in an explicitly covariant form[7]:

$$e_a{}^\alpha \gamma^a \nabla_\alpha \chi = 0, \qquad (2.13)$$

where a and α run now through the four values 0, 1, 2, and 3. The matrices γ^a are equal to

$$\gamma^a = (e, \tau^1, \tau^2, \tau^3), \qquad (2.14)$$

the triads e_a^i are augmented to tetrads by introducing the components $e_0^i = e_1^0 = e_2^0 = e_3^0 = 0$ and $e_0^0 = 1$, so that the metric tensor becomes four-dimensional:

$$g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab}, \quad \eta^{11} = \eta^{22} = \eta^{33} = -\eta^{00} = 1. \qquad (2.15)$$

The covariant derivative

$$\nabla_\alpha = \partial_\alpha + {}^1/_8 \omega_{\alpha,\, ab} [\gamma^a, \gamma^b] - ieA_\alpha \qquad (2.16)$$

is expressed in terms of the electromagnetic potential A_α and in terms of the connection coefficient ω (see, e.g., Ref. 25):

$$\omega_{\alpha,\, ab} = e_a{}^\nu (\partial_\alpha e_{\nu b} - \Gamma_{\alpha\mu}{}^\gamma e_{\gamma b}), \qquad (2.17)$$

where Γ are Christoffel symbols expressed in the usual fashion in terms of the metric tensor. The electromagnetic potential A differs from the previously introduced $A_\alpha^{(0)} = (0, k_F \mathbf{l})$ by the value of the torsion $A_{\gamma,\, \mu\nu}$:

$$A_\alpha = A_\alpha^{(0)} + {}^1/_8 (-g)^{-1/2} g_{\alpha\beta} e^{\beta\gamma\mu\nu} A_{\gamma,\, \mu\nu},$$

$$A_{\gamma,\, \mu\nu} = e_\gamma{}^a (\partial_\mu e_{\nu a} - \partial_\nu e_{\mu a}). \qquad (2.18)$$

In the principal approximation in $\mathbf{l}$ and $\mathbf{v}_s$, i.e., neglecting their gradients, the gauge field takes the form

$$A_\mu = (A_0, \mathbf{A}), \; A_0 = k_F \mathbf{l} \mathbf{v}_s, \; \mathbf{A} = k_F \mathbf{l}. \qquad (2.19)$$

Note that A_μ is an axial gauge field: A_μ is transformed by space (P) and time (T) inversions, as follows:

$$TA_0 = A_0, \quad PA_0 = -A_0, \quad T\mathbf{A} = -\mathbf{A}, \quad P\mathbf{A} = \mathbf{A}. \qquad (2.20)$$

Variation of the action with respect to A_μ leads therefore to a chiral current, i.e, the "electric" charge of the fermions coincides with the chiral charge.

Equation (2.13) is not altered by a local gauge transformation or by general coordinate transformations if it is assumed that A_μ and e_i^μ are transformed like the components of an electromagnetic field and like tetrads in general relativity theory. Integration over the fermions near the poles should lead therefore to an effective boson action that depends only on locally invariant variables, such as the electric field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

or on the Reimann curvature (see Ref. 7). This property permits a substantial simplification of the calculation of various anomalies in the dynamics of ^{3}He-A.

Thus, a local gauge invariance and a general covariance have been produced simultaneously in a locally noninvariant medium consisting of uncharged atoms, such as liquid ^{3}He. This takes place only for low-energy levels and its cause is that the Bogolyubov equation becomes linear in the spatial and temporal gradients near the gap. Superfluid ^{3}He is no exception in this respect; there exist also other systems in which the symmetry is not broken but, conversely, increases in the low energy limit (see Ref. 26).

III. ZERO-CHARGE AND NON-ANALYTICITY OF THE GRADIENT EXPANSION IN ^{3}He-A

An example of a locally invariant term in effective boson action for ^{3}He-A is the logarithmically diverging term in the gradient expansion (1.10): since the logarithm builds up on account of levels with arbitrarily low energy, this term is completely determined by integration over the QED fermion vacuum. This is none other than logarithmic polarization of the fermion vacuum or the zero-charge phenomenon.[8]

The external electromagnetic field restructures the fermion vacuum and thus becomes screened. If the fermions have zero mass, the screening is complete, i.e., the initial "bare" charge e decreases logarithmically to zero at large distances or at low frequencies:

$$e_{eff}^2 = 3\pi / \ln \frac{\Lambda^2}{\omega^2}, \qquad (3.1)$$

where Λ is the ultraviolet cutoff parameter. For ^{3}He-A the parameter Λ should be equal to Δ_0 but not to ε_F, since the Weyl equations are valid only at $\omega \ll \Delta_0$. The QED polarized-vacuum electromagnetic energy, which is invariant to general coordinate transformation, should be of the form

$$\mathscr{L}_{em} = -\frac{(-g)^{1/2}}{16\pi e_{eff}^2} F_{\mu\nu} F^{\mu\nu}, \qquad (3.2)$$

where the index is raised and lowered by the metric tensor (2.15). Recognizing that

$$(-g)^{1/2} = 1/c_\parallel c_\perp^2,$$

$$F_{\mu\nu} F^{\mu\nu} = 2k_F^2 c_\perp^2 [c_\perp^2 (\mathrm{rot}\, \mathbf{l})_\parallel^2 + c_\parallel^2 (\mathrm{rot}\, \mathbf{l})_\perp^2 - (\partial_t \mathbf{l})^2], \qquad (3.3)$$

and neglecting the longitudinal (along $\mathbf{l}$) magnetic field $k_F (\nabla \times \mathbf{l})$ compared with the transverse one, since $c_\perp \ll c_\parallel$, we obtain for the electromagnetic energy the expression:

$$\mathscr{L}_{em} = -\frac{\hbar^2}{24\pi^2} \frac{k_F^3}{m_3} \left\{ [\mathbf{l}\, \mathrm{rot}\, \mathbf{l}]^2 - \frac{1}{c_\parallel^2} (\partial_t \mathbf{l})^2 \right\} \ln \frac{\Delta_0^2}{\omega^2}. \qquad (3.4)$$

The first term in (3.4), i.e., the "magnetic" energy, is precisely the non-analytic term, calculated by Cross,[16] in the gradient expansion (1.10) and (1.11). The second term, which corresponds to the energy of the electric field $\mathbf{E} = k_F \partial_t \mathbf{l}$ (neglecting the superfluid velocity $\mathbf{v}_s$), was obtained for ^{3}He-A in an investigation of the so-called orbital susceptibility, i.e., the response of the system of $\partial_t \mathbf{l}$ (Refs. 27 and 28).

At low temperatures, the electromagnetic Lagrangian (3.4) is dominant in the action that describes the dynamics of the vector $\mathbf{l}$. The remaining noninvariant terms [the first two of (1.10)] can be disregarded in the logarithmic approximation. This Lagrangian determines also the spectrum of the "photons" of the electromagnetic field, i.e, of the collective oscillations of the vector $\mathbf{l}$ (of the so-called orbital waves). In a spatially inhomogeneous medium with $\mathbf{l} = \mathbf{l}^{(0)} = \mathrm{const}$ this spectrum takes the form

$$\omega^2 \approx c_\parallel^2 (\mathbf{q} \mathbf{l}^{(0)})^2. \qquad (3.5)$$

In the static case ($\omega = 0$) the logarithm in (10) and (11) is already cut off at $T = 0$ by the very texture of the vector $\mathbf{l}$ [see (1.2)]. To calculate the "magnetic" energy in the static limit, i.e., the change of the energy of the fermion vacuum under the action of the "magnetic" field $B^i \sim k_F(\nabla\times\mathbf{l})^i$, it is necessary to sum over the negative energies of the fermions (see, e.g., Ref. 9):

$$F_m=\sum_{E<0}[E(B)-E(B=0)]. \tag{3.6}$$

The spectrum of the anisotropic Weyl fermions in ^{3}He-A in the presence of the field B was obtained in Ref. 1 [see Eq. (46) of that reference]. It is convenient to rewrite this spectrum in the following invariant form:

$$E^2_{n,k_B,Q}=(k_iB^i)^2/\tilde{B}^2+2\tilde{B}(n+{}^1/_2-Q), \tag{3.7}$$

where

$$B^i={}^1/_2E^{ikl}F_{kl},\ E^{ikl}=(-g)^{-1/2}e^{ikl},\ F_{kl}=k_F(\partial_kl_l-\partial_ll_k),$$

$$\tilde{B}^2=B_iB^i={}^1/_2F_{ik}F^{ki}=k_F^2c_\perp^2(c_\perp^2(\mathbf{l}\,\mathrm{rot}\,\mathbf{l})^2+c_\parallel^2[\mathbf{l}\,\mathrm{rot}\,\mathbf{l}]^2). \tag{3.8}$$

The spectrum depends on three quantum numbers: the component of the momentum $\mathbf{k}$ along the direction of the field B^i (designated k_B), the number n of the Landau level, and the quantity Q that assumes the role of spin. This is the projection of the Bogolyubov spin in Nambu space, in contrast to the usual ^{3}He nuclear spin which assumes the role of isospin in this analogy between ^{3}He-A and the electric weak interaction (see Ref. 5).

Substitution of (3.7) in (3.6) and summation over the quantum number leads to the following expression (see Ref. 9)

$$F_m=\frac{\tilde{B}(-g)^{1/2}}{2\pi}\int\frac{dk_B}{2\pi}\sum_{E<0,\ n,Q}E_{n,k_B,Q}-\sum_{E<0}E(B=0)$$
$$=\frac{(-g)^{1/2}}{24\pi^2}\tilde{B}^2\ln\frac{\Lambda^2}{\tilde{B}}, \tag{3.9}$$

where the ultraviolet cutoff parameter must again be set equal to Δ_0. Neglecting the longitudinal magnetic field by virtue of the smallness $c_\perp \ll c_\parallel$, we obtain a term analogous to (1.12)

$$F_m=\frac{1}{24\pi^2}\frac{k_F^3}{m_3}[\mathbf{l}\,\mathrm{rot}\,\mathbf{l}]^2\ln\frac{\Delta_0}{c_\parallel|[\mathbf{l}\,\mathrm{rot}\,\mathbf{l}]|}. \tag{3.10}$$

This static limit can be arrived at also with the aid of (3.4) by replacing in it ω^2 under the logarithm sign by the square of the "Larmor" frequency

$$\omega^2\to\omega_L^2=2\tilde{B}\approx2\Delta_0c_\parallel|[\mathbf{l}\,\mathrm{rot}\,\mathbf{l}]|. \tag{3.11}$$

Finally, we write down the static electromagnetic energy (3.9) in a Lorentz-invariant form, including also the dependence on the static "electric" field:

$$F_{em}=-\frac{(-g)^{1/2}}{48\pi^2}(F_{\mu\nu}F^{\mu\nu}g_{00}-4F_{0i}F_0{}^i)\ln\frac{\Lambda^2}{({}^1/_2F_{\mu\nu}F^{\mu\nu})^{1/2}}. \tag{3.12}$$

It is meaningful only at $F_{\mu\nu}F^{\mu\nu}>0$, i.e., at $E<B$. This indicates that in a constant electric field stronger than the magnetic a vacuum of zero-mass fermions loses stability to formation of electron-positron pairs.[29]

We note in conclusion one technical detail. In the calculations of (3.4) and (3.9) we did not sum over the ordinary spins (i.e., over the isospin in the weak-interaction theory). This, however, was offset by the fact that we implicitly took into account twice as many states as in the Bogolyubov equation. Namely, in place of the Bogolyubov vacuum of a Majorana field (see also Ref. 30 on this subject) we have considered a vacuum of fermions and antifermions, which contain jointy twice as many states as the Bogolyubov vacuum.

IV. CHIRAL ANOMALY AND NONCONSERVATION OF THE SUPERFLUID CURRENT

Another example of phenomena describable by invariant equations is momentum transfer from coherent superfluid motion into incoherent motion of fermion excitations, since this transfer is effected only near Fermi-surface poles, where the Weyl equations (2.11)–(2.13) are valid. In terms of QED, the role of the Fermi-excitation momentum $\mathbf{j}^{\mathrm{inc}}$ (1.18) is played by the chiral density of the Weyl fermions.[24] Therefore the nonconservation of the superfluid momentum in ^{3}He-A is closely connected with nonconservation of the chiral 4-current J in the presence of an electromagnetic field, known as the chiral anomaly in QED (see Refs. 9 and 31):

$$\partial_\mu J^\mu_{inc}=(e^2/16\pi^2)F_{\mu\nu}F_{\alpha\beta}e^{\mu\nu\alpha\beta}. \tag{4.1}$$

We shall show that this equation coincides with Eq. (1.20) for the momentum of the fermion excitations in ^{3}He-A. According to (2.13) the chiral current J^μ_{inc} obtained by varying the fermion action with respect to the gauge field A_μ, is equal to

$$J^0_{inc}=\sum_e e\langle\chi_e^+\chi_e\rangle\equiv\sum_e e\nu_e,\quad J^i_{inc}=\sum_e e_a{}^i\langle\chi_e^+\tau^a\chi_e\rangle, \tag{4.2}$$

where χ_e and ν_e denote respectively the annihilation operator and the density for right-hand "positrons" and left-hand "electrons," ($e=+1$ and $e=-1$, respectively), and the summation is over both signs of the charge. We recall now that the "positrons" and "electrons" have respectively momentum $k_F\mathbf{l}$ and $-k_F\mathbf{l}$, i.e., the momentum of each of these particles is equal to $ek_F\mathbf{l}$. Therefore, multiplying J^0_{in} by $k_F\mathbf{l}$ we obtain none other than the incoherent momentum of the low-energy excitations:

$$\mathbf{j}^{inc}=\sum_{\mathbf{k}}\mathbf{k}\nu\approx\sum_e ek_F\mathbf{l}\nu_e=k_F\mathbf{l}J^0_{inc}, \tag{4.3}$$

while the equation for $\mathbf{j}^{\mathrm{inc}}$ is obtained by multiplying (4.1) by $k_F\mathbf{l}$

$$\partial_t\mathbf{j}^{inc}+\partial_i\boldsymbol{\pi}^i_{inc}=\frac{k_F\mathbf{l}}{16\pi^2}F_{\mu\nu}F_{\alpha\beta}e^{\mu\nu\alpha\beta}, \tag{4.4}$$

where the momentum-flux tensor of the incoherent motion is given by

$$\boldsymbol{\pi}^i_{inc}=k_F\mathbf{l}J^i_{inc}=k_F\mathbf{l}\sum_e e_a{}^i\langle\chi_e^+\tau^a\chi_e\rangle. \tag{4.5}$$

It is easy to verify that in an approximation quadratic in the spatial and temporal derivatives of the vector $\mathbf{l}$, the right-hand side of (4.4) coincides with the source of the incoherent momentum in (1.20).

As applied to ^{3}He-A, the chiral anomaly means that a description of the low-frequency dynamics of a liquid at $T=0$ only in terms of the coherent vacuum variables ρ, $\mathbf{l}$, and $\mathbf{v}$ is incomplete: the dynamics of the vacuum is accompanied by creation and annihilation of chiral fermion excitations, as a result of which transfer of momentum (of chiral or electric charge in QED) from coherent vacuum (superfluid) motion into incoherent (normal) motion of fermion excitations takes place. The total momentum

$$\mathbf{j}=\mathbf{j}^{coh}+\mathbf{j}^{inc} \tag{4.6}$$

is, on the other hand, conserved, since the momentum sources in Eqs. (4.4) and (1.17) cancel one another. Note that these momentum sources were obtained independently by two opposite approaches. The vacuum-current source was obtained from the general phenomenology of the vacuum motion,[9,12] whereas the incoherent source was obtained from (4.1), i.e., from a microscopic analysis of the dynamics of the Fermion excitations near the poles of the Fermi sphere, a dynamics equivalent to that of fermions in QED.

The situation is similar in QED, where the source of the incoherent chiral current can be represented as the total four-dimensional divergence of a coherent vacuum current J_{coh}; this divergence is expressd in terms of boson variables (see, e.g., Ref. 9), so that the total chiral current is conserved:

$$\partial_\mu(J^\mu_{inc}+J^\mu_{coh})=0,$$
$$J^\mu_{coh}=-\frac{e^2}{4\pi^2}e^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta,\quad J^0_{coh}=-\frac{e^2}{4\pi^2}\mathbf{A}\,\mathrm{rot}\,\mathbf{A}. \tag{4.7}$$

There is, of couse, a substantial difference between the coherent chiral density J^0_{coh} in QED and the superfluid current $\mathbf{j}^{coh}$ in ^{3}He-A. Whereas the sources of these currents and their incoherent parts J^0_{inc} and $\mathbf{j}^{inc}$ are equal (accurate to the factor $k_F\mathbf{l}$), the coherent currents themselves differ from one another. Indeed, if J^0_{coh} is multiplied by k_F and expressed in terms of $\mathbf{l}$, we obtain the expression

$$k_F\mathbf{l}J^0_{coh}=-{}^3/_2C_0\mathbf{l}(\mathbf{l}\,\mathrm{rot}\,\mathbf{l}), \tag{4.8}$$

which not only is not equal to $\mathbf{j}^{coh}$, but differs even by a factor 3 from the anomalous part $\mathbf{j}_{an}$ of the coherent current (1.13).

The reason is that the vacuum currents, unlike their sources, depend subsantially on the vacuum structure, which is different in QED and ^{3}He-A; in the latter, only a small part of the vacuum near the poles imitates the QED vacuum. The value (4.8) is obtained if gauge-invariant cutoff is used in the summation over the Dirac vacuum when the vacuum current is calculated. The value (1.13) is obtained in summation over the entire ^{3}He-A vacuum[2,3,32] that effects by the same token one of the possible gauge-noninvariant regularizations of the divergences in QED.

The incoherent current $\mathbf{j}^{inc}$ of low-energy fermion excitations, on the contrary, is concentrated near the poles at sufficiently low temperatures and frequencies of the motion, and therefore is equal to J^0_{inc} in QED apart from the factor $k_F\mathbf{l}$. Its calculation in ^{3}He-A is therefore of definite interest for QED. The dependence of $\mathbf{j}^{inc}$ in the velocity $\mathbf{v}_n$ of the normal component (provided, of course, that the hydrodynamic aproximation can be used for the normal motion, i.e., $\omega\tau\ll 1$, where τ is the free-path time of the excitations) was obtained in Refs. 19, 17, 23, 10, and 21 [see Eq. (1.19)]. Equations (1.9) can be written in "electromagnetic" variables in the following form:

$$J^{0(1)}_{inc}=\frac{1}{4\pi^2}(A_0-A_{0(n)})(-g)^{1/2}\left(\frac{1}{2}F_{ik}F^{ik}\right)^{1/2},$$
$$J^{0(2)}_{inc}=-\frac{1}{3\pi^2}(-g)^{1/2}(A_0-A_{0(n)})^3. \tag{4.9}$$

(Note that the currents $J^{0(1)}_{inc}$ and $J^{0(2)}_{inc}$ pertain to two different limiting cases and they can, generally speaking, not be added.) Here

$$A_0=k_F\mathbf{l}\mathbf{v}_s,\quad A_{0(n)}=k_F\mathbf{l}\mathbf{v}_n. \tag{4.10}$$

Whereas the scalar potential A_0 is connected with the superfluid flow, the quantity $A_{0(n)}$ introduced in (4.10) is conected with the normal motion, i.e., with the distribution function of the excitations. The distribution function in a quasi-equilibrium (hydrodynamic) regime is characterized by a finite set of hydrodynamic variables, which conserve the collision integral. In other words, the distribution function takes in this regime the appearance of an equilibrium distribution function in the field of supplementary variables such as $\mathbf{v}_n$. The supplementary boson variables have therefore the same structure as the external fields, for example, $A_{0(n)}$ has the same structure as A_0 and is therefore also transformed by a local gauge transformation. The difference $A_0-A_{0(n)}$ is thus a gauge invariant and expressions (4.9) are fully applicable also in QED, if the latter admits of a hydrodynamic regime for the excitations produced in the chiral anomaly. Just as in ^{3}He-A, the existence of J^0_{inc} in the hydrodynamic regime is due to the fact that the density of the fermion states becomes different from zero both in the presence of a magnetic field and in the presence of a "countercurrent" $A_0-A_{0(n)}$ if the fermions have zero mass. The term of type $J^{0(2)}_{inc}$ in (4.9), which describes the nonlinear screening of the electric charge by the electron-positron vacuum, was recently obtained by Gribov[1)] for QED with zero-mass fermions [at $A_{0(n)}=0$].

Expressions (4.9) do not describe the incoherent current completely, since they are not of general-covariant form. To recover the Lorentz invariance it is necessary to add an electric field, and also introduce a vector part of the normal variable $A_{\mu(n)}=(A_{0(n)},\mathbf{A}_{(n)})$, and then

$$J^{\mu(1)}_{inc}=\frac{1}{4\pi^2}(A^\mu-A^\mu_{(n)})(-g)^{1/2}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\right)^{1/2},$$
$$J^{\mu(2)}_{inc}=\frac{1}{3\pi^2}(A^\mu-A^\mu_{(n)})(-g)^{1/2}(A_\nu-A_{\nu(n)})(A^\nu-A^\nu_{(n)}). \tag{4.11}$$

A similar variable was already considered for ^{3}He-A, viz., the distribution function is described not only by the hydrodynamic variable $\mathbf{v}_n$ but also by the quasi-equilibrium vector $\mathbf{l}_n$.[28] This is the orbital-anisotropy vector that characterizes the instantaneous distribution function of the excitations. If the excitations follow without delay the variable vector $\mathbf{l}$, then $\mathbf{l}_n=\mathbf{l}$, otherwise $\mathbf{l}_n$ is an additional dynamic variable. It is this variable that leads to the appearance of $\mathbf{A}_{(n)}$:

$$\mathbf{A}_{(n)}=k_F\mathbf{l}_n. \tag{4.12}$$

Thus, QED with zero-mass chiral fermions is described in the quasi-equilibrium regime by the variables A_μ and $A_{\mu(n)}$. The closed system of equations for these variables should contain, besides the Maxwell equations

$$(4\pi e_{eff}^2)^{-1}\partial_\mu F^{\mu\nu}=J^\nu_{coh}+J^\nu_{inc} \qquad (4.13)$$

also the equation that follows for $A^\mu_{(n)}$ from the transport equation for the excitations.

Attention is called to the fact that (4.11) is meaningful only if $F_{\mu\nu}F^{\mu\nu}$ is positive, i.e., the electric field is weaker than the magnetic one. Otherwise the vacuum is unstable to creation of electron-positron pairs[29] and the fermions no longer have an equilibrium distribution, or else another equilibrium regime sets in. The very same instability is responsible also for the paradox of the orbital angular momentum in ^{3}He-A (Ref. 6), viz., creation of excitations by the electric field $\mathbf{E}=k_F\partial_t\mathbf{l}$ transfers the angular momentum to the noncoherent subsystem.

Note one more analogy between chiral anomalies in QED and ^{3}He-A. In the latter, chiral excitations are produced by flow of the Fermi sea through "openings" in at the poles of the Fermi sphere (see Fig. 1). A positive chiral charge is produced, for example, by flow of particles, in momentum space, from the south pole through the region of large (i.e., far from the poles) momenta to the north pole, where the particles emerge from under the Fermi surface and form chiral excitations that have positive charge ("positrons"). Hole excitations ("anti-electrons"), likewise with positive charge, are then produced near the south pole. In QED, similarly, the chiral anomaly is described as a particle of particles from the large-momentum region of a Dirac vacuum.[33]

V. WESS-ZUMINO ACTION IN ^{3}He-A

The system of equations (1.16) and (1.3b) for the variables ρ, $\mathbf{v}$, and $\mathbf{l}$ in ^{3}He-A at $T=0$ is not closed, since it does not take into account the dynamics of the produced normal excitations that carry away the momentum and the angular momentum. There exists, however, a regime in which this system becomes closed. This occurs if the normal motion is set by external conditions, e.g., in a narrow layer between plates we have $v_n=0$ in the case of ^{3}He-A because of the interaction with the walls. The fact that this system does not conserve the momentum is of no importance in the present situation: the momentum is returned to the walls through the normal subsystem. Nonconservation of the momentum, however, means that the system of equations cannot be obtained via the standard Lagrangian formalism, in which the momentum is automatically conserved. It is therefore necessary to introduce in the Lagrangian action a modification that relaxes the stringent requirements of the usual formalism.

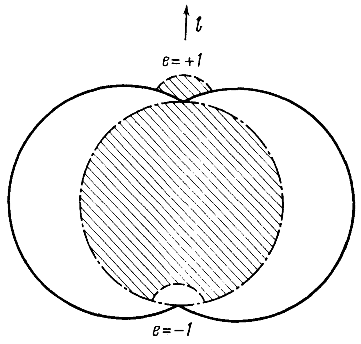

FIG. 1. Formation of chiral charge in ^{3}He-A or of the momentum of fermion excitations. The Fermi particles, shaded, leaking out from the sea inside the Fermi sphere through the poles on the Fermi sphere, where the gap in the spectrum of the fermion excitations (solid line) vanishes. The formation of a positive chiral charge is shown: the excitations with positive charge $e=+1$ ("positrons") are formed near the north pole, whereas near the south pole are formed holes ("antielectrons") which also have positive chiral charge.

Such a relaxation was observed in Ref. 12 in the framework of a Hamiltonian formalism that uses a Poisson-bracket algebra. It was found that the group properties of the dynamic variables, while imposing strong constraints on the structure of the equations, admit of a certain leeway due to introduction of a dynamic invariant C_0 that commutes with all the remaining variables ρ, $\mathbf{v}_s$, and $\mathbf{l}$. The system of equations conserves the momentum only at $C_0=0$ and can be obtained in the framework of the usual Lagrangian formalism (as was done in Ref. 11). It can be shown that for a Lagrangian description of the system of equations at $C_0\neq 0$ it is necesary to supplement the Lagrange action by an action of the Wess-Zumino type[34]:

$$S(\rho,\ \mathbf{l},\ \mathbf{v}_s,\ C_0)=S^{(0)}(\rho,\ \mathbf{l},\ \mathbf{v}_s)+S_{WZ}(\rho,\ \mathbf{l},\ \mathbf{v}_s;\ C_0). \qquad (5.1)$$

Here $S^{(0)}$ is that action which leads to Eqs. (1.3)–(1.7) for a Bose condensate of molecules, i.e., at $C_0=0$, and the additional action S_{WZ} can be written down only using the auxiliary five-dimensional space:

$$S_{WZ}=\frac{\hbar}{2}\int d^3x\,dt\,dx^5\,C_0\mathbf{l}[\partial_t\mathbf{l},\partial_5\mathbf{l}] +\frac{\hbar}{2}\int d^3x\,dt\,C_0(\mathbf{l}\mathbf{v}_s)[\mathbf{l}\,\mathrm{rot}\,\mathbf{l}]. \qquad (5.2)$$

This space is chosen such that its boundary is a physical four-dimensional space-time continuum, and the variation of the action should land outside the boundary. This requirement leads to dynamic invariance of the parameter C_0, i.e., $\partial_t C_0=\partial_5 C_0=0$, and the requirement that the amplitude $\exp(iS/\hbar)$ be independent of the choice of the extension to a five-dimensional space leads to quantization of the quantity $N_0=\int d^3xC_0$ (Ref. 6). These conditions are undoubtedly less stringent than simply the condition $C_0=0$ for a Bose condensate of molecules.

The Wess-Zumino action describes phenomena connected with the anomaly. Variation of S_{WZ} with respect to $\mathbf{v}_s$ leads to an additional anomalous current in (1.13), variation with respect to the order-parameter rotation angle θ leads to a source of angular momentum in the right-hand side of (1.16), and all this alters the dynamics equations in such a way that a momentum source, likewise proportional to the parameter C_0, appears in the right-hand side of (1.17).

This action can be obtained not only from the general phenomenology. Use can be made of the fact that the action S_{WZ}, which describes the transfer of the momentum and of the angular momentum into the excitation subsystem, and is by the same token concentrated near the zeros of the gap, should have a general-covariant and a guage-invariant form and should be completely determined by the "electrodynamic" variables. The only possible invariant, which is a 5-form and is expressed in terms of the field A_μ, is given by

$$S_{WZ}=\frac{\hbar}{48\pi^2}\int d^5x\,e^{\alpha\beta\gamma\mu\nu}A_\alpha F_{\beta\gamma}F_{\mu\nu}. \qquad (5.3)$$

The coefficient is chosen here such that variation of this action with respect to A_μ gives an anomalous chiral current in

QED. For Abelian QED this 5-form is not only closed but also exact, i.e., not only does the variation of the action land on the four-dimensional boundary, but the action itself reduces to a four-dimensional integral over the physical continuum. Actually, by choosing for the electromagnetic field an extension to a five-dimensional space such as to satisfy requirements of gauge invariance and of closure of the 5-form:

$$A_5=\partial_5\theta, \quad \partial_5 A_\mu=0 \quad (\mu=0, 1, 2, 3), \tag{5.4}$$

we obtain the usual four-dimensional integral (the so-called θ-term)

$$S_{WZ}\to S_\theta=\frac{\hbar}{16\pi^2}\int d^4x\, e^{\alpha\beta\gamma\mu}F_{\alpha\beta}F_{\gamma\mu}\theta. \tag{5.5}$$

This is precisely the term that must be added to the electromagnetic action to obtain expression (4.1) for the anomaly in the chiral current. It is assumed that the gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$ transforms also θ into $\theta + \alpha$. From this, according to the Noether theorem, the variation of S_θ with respect to θ leads to the source of the chiral current in (4.1).

The action (5.3) should be applicable also to ^{3}He-A, where $\mathbf{A} = k_F\mathbf{l}$, $A_0 = k_F\mathbf{l}\mathbf{v}_s$. For the form expressed in terms of $\mathbf{v}_s$ and $\mathbf{l}$ to be closed, it is necessary to extend the variable to the five-dimensional region in the following manner:

$$A_5=0, \quad (\partial_5\mathbf{A}, \operatorname{rot}\mathbf{A})=0. \tag{5.6}$$

It then is easy to verify, however, that the action (5.3) reduces to the Wess-Zumino action for ^{3}He-A, i.e., to Eq. (5.2). Thus, in the "helium" regularization of quantum electrodynamics the θ term goes over into a true Wess-Zumino action (5.2), which has a nontrivial topology caused by the fact that $\mathbf{l}$ varies on a sphere or that the Fermi sphere has the topology of a sphere. It is this which leads to quantization of $N_0 = \int d^3x C_0$ (Ref. 6). Direct calculation of S_{WZ} for ^{3}He-A was carried out with the aid of the Bogolyubov equations in Ref. 35.

CONCLUSION

The basic phenomena connected with vanishing of the gap at two points on the Fermi surface in ^{3}He-A evolve in the vicinity of these points. The Bogolyubov equations for fermion quasiparticles are linearized there and are transformed into Weyl equations for right-hand "positrons" and left-hand "electrons" moving in an "electromagnetic" field and in a "gravitational" field. This allows us to relate the singularities in the dynamics of ^{3}He-A to phenomena such as chiral anomaly, zero-charge, nonlinear polarization of the vacuum, and production of electron-positron pairs in an electric field. This is useful in two respects. On the one hand, the local gauge invariance and general covariance of the Weyl equations permit a drastic simplification of the derivation of the various singularities in the action for ^{3}He-A, including the Wess-Zumino action and the non-analytic terms in the gradient expansion. On the other hand, the phenomenology of ^{3}He-A with its two-component hydrodynamics permits a description of the chiral anomaly in QED with zero-mass fermions in terms of a coherent (vacuum) and a noncoherent (above-vacuum) motion.

We have considered here the singularities only in the orbital dynamics, which touches upon only three out of the possible 18 order parameter components that act on fermions as an Abelian gauge field. No less interesting, obviously, is the dynamics of the remaining 15 modes, some of which correspond to gravitons and W bosons in their action on chiral fermions, which can lead to non-Abelian chiral anomalies to related effects in ^{3}He-A. What should be next investigated are the singularities in the dynamics of superfluids in which the gap vanishes on a line, as in the polar phase, and on one of two Fermi spheres, as in ^{3}He-A.

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[1)] V. N. Gribov, private communication.

[1] R. Combescot and T. Dombre, Phys. Rev. **B33**, 79 (1986).
[2] G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 294 (1985) [JETP Lett. **42**, 363 (1985)].
[3] A. V. Balatskiĭ, G. E. Volovik, and A. A. Konyshev, Zh. Eksp. Teor. Fiz. **90**, 2038 (1986) [Sov. Phys. JETP **63**, 1194 (1986)].
[4] G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 428 (1986) [JETP Lett. **43**, 551 (1986)].
[5] G. E. Volovik, *ibid.* **43**, 535 (1986) [**43**, 693 (1986)].
[6] G. E. Volovik, *ibid.* **44**, 144 (1986) [**44**, 185 (1986)].
[7] G. E. Volovik, *ibid.* **44**, 388 (1986) [**44**, 498 (1986)].
[8] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 1177 (1954).
[9] K. Huang, *Quark, Leptons, and Gauge Fields*, World Scientific, 1982.
[10] G. E. Volovik, Usp. Fiz. Nauk **143**, 73 (1984) [Sov. Phys. Usp. **27**, 32 (1984)].
[11] V. V. Lebedev and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **73**, 153 (1977) [Sov. Phys. JETP **46**, 808 (1977)].
[12] G. E. Volovik and A. V. Balatsky, J. Low Temp. Phys. **58**, 1 (1985).
[13] N. D. Mermin and T.-L. Ho, Phys. Rev. Lett. **36**, 594 (1977).
[14] K. Nagai, J. Low Temp. Phys. **38**, 677 (1980). Progr. Theor. Phys. **65**, 793 (1981).
[15] G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **83**, 1025 (1982) [Sov. Phys. JETP **56**, 579 (1982)].
[16] M. C. Cross, J. Low Temp. Phys. **21**, 525 (1975).
[17] R. Combescot and T. Dombre, Phys. Rev. **B28**, 5140 (1983), **B32**, 2690 (1985).
[18] N. D. Mermin and P. Musikar, *ibid.* **121**, 980 (1980).
[19] G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **81**, 989 (1981) [Sov. Phys. JETP **54**, 524 (1981)].
[20] G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 234 (1975) [JETP Lett. **22**, 108 (1975)].
[21] K. Nagai, J. Low Temp. Phys. **55**, 233 (1984).
[22] T. Dombre and R. Combescot, Phys. Rev. **B30**, 3765 (1984).
[23] P. Muzikar and D. Rainer, *ibid.* **27**, 4243 (1983).
[24] V. Alonso and V. N. Popov, Zh. Eksp. Teor. Fiz. **73**, 1445 (1977) [Sov. Phys. JETP **46**, 760 (1977)].
[25] L. D. Faddeev, Usp. Fiz. Nauk **136**, 435 (1982) [Sov. Phys. Usp. **25**, 130 (1982)].
[26] H. B. Nielsen, Field Theories Without Fundamental (Gauge) Symmetries, Preprint NBI-HE-83-41, Nils Bohr institute, 1983.
[27] G. E. Volovik, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 412 (1975) [JETP Lett. **22**, 198 (1975)]. G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **71**, 1129 (1976) [Sov. Phys. JETP **44**, 591 (1976)].
[28] A. J. Leggett and S. Takagi, Ann. Phys. (N.Y.) **110**, 353 (1978).
[29] A. B. Migdal, Fermions and Bosons in Strong Fields [in Russian], Nauka, 1978. V. N. Gribov, New Hypothesis on Quark and Gluon Confinement, Preprint KFKI-1986-20A, Central Institute for Physics Research, Budapest, 1986.
[30] A. Garg, V. P. Nair, and M. Stone, Ann. Phys. (N. Y.) **173**, 149 (1987).
[31] J. Schwinger, Phys. Rev. **82**, 664 (1951). S. Adler, Phys. Rev. **177**, 2496 (1969). J. S. Bell and R. Jackiw, Nuovo Cimento **A260**, 47 (1969).
[32] A. V. Balatskiĭ and V. A. Monyshev, Zh. Eksp. Teor. Fiz. **92**, 841 (1987) [Sov. Phys. JETP **65**, 474 (1987)].
[33] V. N. Gribov, Anomalies as a Manifestation of the High Momentum Collective Motion in the Vacuum, Preprint KFKI-1981-66, Central Institute of Physical Research, Budapest, 1981.
[34] J. Wess and B. Zumino, Phys. Lett. **B37**, 95 (1971). E. Witten, Nucl. Phys. **B223**, 422 (1983). A. P. Malachandran, V. P. Nair, and C. G. Trahern, Phys. Rev. **D27**, 1369 (1983).
[35] A. V. Balatskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 515 (1986) [JETP Lett. **44**, 663 (1986)].

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