Dynamic chaos in the interaction between external monochromatic radiation and a two-level medium, with allowance for cooperative effects

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The interaction between monochromatic radiation and an ensemble of two-level systems is discussed. It is shown that, when the density of the medium exceeds a certain threshold, the dynamics of the system is significantly influenced by the self-consistent radiation field. The semiclassical approximation is used to find analytically and numerically the conditions for the approach of the system to stochasticity, which can be satisfied for a particular choice of the external field amplitude and its detuning from the frequency of two-level transitions. Estimates are given of parameter values for which this effect can be observed in gaseous media.

1. INTRODUCTION

Much attention is being devoted at present to dynamic chaos in classical and quantum-mechanical nonlinear systems.[1–4] An interesting new topic is the dynamic chaos that can occur when atoms and molecules interact with their own radiation fields (cooperative effects),[5–12] and studies are being carried out of the associated stochastic dynamics, including both relaxation processes (transient chaos and strange attractors)[5–7] and stochasticity extending over time intervals much shorter than the characteristic relaxation time (Hamiltonian systems).[8–12] In this paper, we shall confine our attention to the dynamics of Hamiltonian systems.

When the resonant interaction between atoms and a field is described, the analysis is usually confined to the two-level approximation, and the nonresonant interaction term is neglected. This corresponds to the so-called rotating wave approximation (RWA)[13–15] in which the energy is periodically transferred from the atoms to the field and back again (the oscillations are nonlinear).[16–19] It is shown in Ref. 8 (see also the subsequent papers[10,11]) that inclusion of the nonresonant term (departure from the RWA) causes the oscillations in the populations of the two-level system and in the self-consistent radiation field to become stochastic. A significant point for the onset of chaos in the system is the existence of a threshold condition for the interaction constant between the two-level ensemble and its own radiation fields. The population of high-lying states of nonequidistant multilevel systems during the development of stochastic instability is investigated numerically in Ref. 9. An ensemble of multilevel systems interacting with its own radiation field and with an external monochromatic field of constant amplitude is examined in Ref. 5 in the case where the frequency of the external field is exactly equal to the frequency of transmissions between the two bottom levels of the ensemble. As in Ref. 8, it is found that this model leads to global chaos when a certain kinetic threshold value of the interaction constant between the atoms and the field, corresponding to a departure from the RWA, is exceeded. The interaction between an ensemble of almost equidistant three-level systems and two modes of the electromagnetic field is discussed in Ref. 12 in the RWA. Dynamic chaos is found to arise in Ref. 12 when the dipole moments of the $1 \to 2$, $2 \to 3$, and $1 \to 3$ are commensurate (one- and two-photon transitions). Since all the results reported in Ref. 12 were obtained in RWA, chaos is possible in principle for arbitrarily small values of the atom-field interaction constants. We also note that all the investigations mentioned above are based on the semiclassical approximation (the self-consistent radiation field is described classically using the Maxwell equations; the criterion for the validity of this approximation is discussed in Ref. 20) and the homogeneous approximation (i.e., the characteristic size of the specimen containing the atoms is assumed much shorter than the wavelength of the radiation).

In this paper, we investigate the dynamics of the interaction between an ensemble of two-level systems (atoms, molecules, impurities in crystals) and an external monochromatic field of constant amplitude, taking into account cooperative effects (self-consistent radiation fields). We shall carry out our analysis in the rotating wave approximation and the slowly-varying-amplitude (SVA) approximation.[13–15] In contrast to Ref. 9, we shall take into account the finite detuning Δ of the external field frequency from the transition frequency in the two-level system. The criterion for the onset of chaos will be found analytically and numerically. It is important to note that the onset of chaos in this model does not require a critical value of the interaction constant between atoms and the self-consistent radiation field that corresponds to a departure from RWA to be exceeded. It is shown that the condition for the onset of stochasticity can be satisfied for arbitrarily low values of the interaction constant between the atoms and the self-consistent field if the detuning Δ and the amplitude of the external field are suitably chosen.

Our paper is organized as follows. In Section 2, we derive the equations describing the interaction between an external magnetic field of constant amplitude and an ensemble of two-level systems, taking cooperative effects into account (self-consistent radiation field). In Section 3, we obtain an approximate analytic criterion for the fluctuations in the populations of the two-level systems and the amplitude of the self-consistent radiation field to become stochastic. The results of a numerical solution of the complete set of equations are presented in Section 4. Estimates of physical parameters for which dynamic chaos can be observed in optical experiments are given in the concluding Section.

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2. BASIC EQUATIONS

Consider an ensemble of two-level systems interacting with an electromagnetic field E. The two-level systems will be described quantum-mechanically and the field classically by Maxwell equations (semiclassical approach).[13–15] The inclusion of the self-consistent field is found to be significant for concentrations ρ of the two-level atoms for which the reaction of the medium to the field has to be taken into account.[17–19] The condition for this is

$$\omega_c \gtrsim G, \tag{1}$$

where $\omega_c = (2\pi\rho d^2\omega_0/\hbar)^{1/2}$ is the cooperative frequency, $G = d\varepsilon_0/\hbar$ is the Rabi frequency at exact resonance, $\rho = N/V$ is the number of atoms per unit volume, d is the matrix element of the dipole transition, ω_0 is the frequency of the two-level transition, and ε_0 is the characteristic amplitude of the electric fields. We shall consider a tenuous gas in which we can neglect the field-induced dipole-dipole interaction between the atoms compared with the energy of an atomic dipole in the external fields. The condition for this is[21]

$$\rho d^2/\hbar \ll G \quad \text{or} \quad \omega_c^2 \ll G\omega_0. \tag{2}$$

We shall also assume that the total field $E(z, t)$ acting on the two-level systems consists of the self-consistent field $E_s(z, t)$ and an external monochromatic field taken in the form of a linearly polarized plane wave of amplitude ε_0 and frequency Ω:

$$E(z,t) = E_s(z,t) + \varepsilon_0 \cos(\Omega t - k'z), \quad \Omega = ck'. \tag{3}$$

We note that a pump of given constant amplitude has also been examined in theoretical and experimental papers on cooperative Raman scattering of light.[22,23] The self-consistent field $E_s(z, t)$ satisfies the Maxwell equation

$$\frac{\partial^2 E_s}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_s}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 P}{\partial t^2}, \tag{4}$$

where $P(z, t)$ is the polarization produced by the two-level medium. The self-consistent field $P_s(z, t)$ and the polarization $P(z, t)$ will be sought in the form

$$E_s(z, t) = \varepsilon_1(t)\cos(\omega t - kz) + \varepsilon_2(t)\sin(\omega t - kz), \tag{5}$$

$$P(z, t) = P_1(t)\cos(\omega t - kz) + P_2(t)\sin(\omega t - kz), \quad \omega = ck. \tag{6}$$

This can be done by assuming, for example, that the specimen containing the ensemble of two-level systems is placed in a single-mode ring resonator with proper frequency ω. We shall assume in what follows that $\omega = \omega_0$. We now introduce the following variables describing the atomic subsystem:

$$R_k^{\pm}(t) = \frac{1}{N_s}\sum_{j\in\Delta V}^{N_s} \exp[\pm i(kz - \omega_0 t)]\, s_j^{\pm},$$

$$R_k^{z}(t) = \frac{1}{N_s}\sum_{j\in\Delta V}^{N_s} s_j^{z}, \quad s_j^{\pm} = s_j^x \pm i s_j^y, \quad R_k^{\pm} = R_k^x \pm i R_k^y, \tag{7}$$

where $\Delta V = (\Delta z)\pi r^2$ is a physically infinitesimal volume, z is the coordinate of the center of a layer of thickness $\Delta z \ll \lambda$, $\lambda = 2\pi/k$ is the wavelength of the radiation, r is the characteristic radius of a specimen containing the two-level gas, N_s is the number of two-level systems in the volume ΔV ($N_s \gg 1$), and the pseudospin variables s_j^x, s_j^y, s_j^z of the jth individual atom are related to the population amplitude of the upper (a) and lower (b) levels of the jth atom, as follows:[13]

$$s^x = a^* b + a b^*, \quad s^y = -i(a^* b - a b^*), \quad s^z = |a|^2 - |b|^2. \tag{8}$$

We now assume that $\pi r^2 \rho\lambda \gg 1$, which enables us to neglect spatial variations in the field strength and in the pseudospins inside ΔV, and to consider that these values are equal to the averages evaluated over ΔV (Refs. 24 and 25). The polarization components $P_1(t)$ and $P_2(t)$ in (6) are related to the variables R_k^x and R_k^y as follows:

$$P_1(t) = \rho d R_k^x(t), \quad P_2(t) = -\rho d R_k^y(t). \tag{9}$$

When the RWA and SVA inequalities are satisfied,[13–15]

$$|\dot{\varepsilon}_i| \ll \omega|\varepsilon_i|, \quad i = 1, 2; \quad |\dot{R}_k^{\alpha}| \ll \omega_0 |R_k^{\alpha}|, \quad \alpha = x, y, z;$$
$$|\omega_0 - \Omega| \ll \omega_0, \quad \omega_c \ll \omega_0, \quad G \ll \omega_0 \tag{10}$$

the Schrödinger equations for the two-level systems and the Maxwell equations (4) lead to a closed set of equations (the derivation is similar to that given in Refs. 13–15):

$$\begin{aligned}
\dot{R}_k^x &= -\varkappa R_k^z\{-\varepsilon_2(t) + \varepsilon_0 \sin[\Delta(t - z/c)]\},\\
\dot{R}_k^y &= \varkappa R_k^z\{\varepsilon_1(t) + \varepsilon_0 \cos[\Delta(t - z/c)]\},\\
\dot{R}_k^z &= \varkappa\{R_k^x(-\varepsilon_2(t) + \varepsilon_0 \sin[\Delta(t - z/c)])\\
&\quad + R_k^y(-\varepsilon_1(t) - \varepsilon_0 \cos[\Delta(t - z/c)])\},\\
\dot{\varepsilon}_1 &= \beta R_k^y, \quad \dot{\varepsilon}_2 = \beta R_k^x,
\end{aligned} \tag{11}$$

where $\beta = 2\pi\rho d\omega_0$, $\Delta = \Omega - \omega_0$, $\omega = \omega_0$, and $\varkappa = d/\hbar$. Equations (11) describe the interaction dynamics of an ensemble of two-level systems interacting with a monochromatic external field over time intervals much shorter than the characteristic relaxation time. Direct verification will show that the equations given by (11) have associated with them the following conservation law:

$$\begin{aligned}
&[R_k^x(t)]^2 + [R_k^y(t)]^2 + [R_k^z(t)]^2\\
&= [R_k^x(0)]^2 + [R_k^y(0)]^2 + [R_k^z(0)]^2 = 1.
\end{aligned} \tag{12}$$

We note that, for the set of equations given by (11), the z-dependence is significant only when a phase shift $\Delta z/c$ appears in the perturbing field. This shift is small when $|\Delta l/c| < 1$, where l is the characteristic size of the specimen. For $\Delta \sim \omega_c$, which is of interest to us (see Sections 3 and 4), this condition becomes

$$\omega_c l/c < 1. \tag{13}$$

From now on, when we analyze (11), we shall neglect this phase shift and assume that (13) is satisfied.

To conclude this Section, let us consider, following Refs. 26 and 27, the condition for a constant external field amplitude ε_0. The amplitude may be looked upon as given if the energy supplied in the characteristic time $T_c = 2\pi/\omega_c$ is greater than the energy removed by the medium from the external field:

$$\varepsilon_0^2 c T_c > \rho\hbar\omega_0 l. \tag{14}$$

Using the definition of cooperative and Rabi frequencies (1), we can rewrite (14) in the form

$$G^2 T_c/\omega_c^2 > l/c. \tag{15}$$

In the case in which we are interested, i.e., $G \sim \omega_c$ (see Sections 3 and 4), this inequality assumes the simple form

$$T_c > l/c. \tag{16}$$

It follows that the imposed-field approximation is valid for distances satisfying (16).

3. STOCHASTICITY CRITERION

Let us now consider the possible behavior of the solutions of (11) for different relationships between the parameters ω_c, G, and Δ.

(1) Suppose that $\varepsilon_0 = 0$. The dynamics of the system under these conditions was investigated in Refs. 16–19. The typical behavior consists of quasiperiodic nonlinear oscillations in the dynamic variables $R_k^x, R_k^y, R_k^z, \varepsilon_1, \varepsilon_2$.

(2) Suppose that $\Delta = 0$. The dynamics of (11) is then analogous to the case where $\varepsilon_0 = 0$.

(3) Competition between cooperative and external interactions: $\varepsilon_0 \neq 0$, $\Delta \neq 0$. Let us change the variables, subject to the conservation law (12):

$$R_k^x = \cos\psi, \quad R_k^y = \sin\theta\sin\psi, \quad R_k^z = \cos\theta\sin\psi. \tag{17}$$

The equations given by (11) now reduce to

$$\ddot{\theta} - \sin\theta\sin\psi = -\bar{\varepsilon}_0\bar{\Delta}\sin(\bar{\Delta}\tau), \tag{18}$$

$$\dot{\psi} = [\bar{\varepsilon}_0\sin(\bar{\Delta}\tau) - \bar{\varepsilon}_2(\tau)]\cos\theta, \qquad \dot{\bar{\varepsilon}}_2 = \cos\psi,$$

where

$$\bar{\varepsilon}_i = d\varepsilon_i/\hbar\omega_c, \quad i = 0, 1, 2; \quad \bar{\Delta} = \Delta/\omega_c, \quad \tau = \omega_c t \tag{19}$$

and the dot over the symbols represents differentiation with respect to τ. The field $\bar{\varepsilon}_1(\tau)$ can now be found from (18), using the formula $\bar{\varepsilon}_1(\tau) = \dot{\theta}(\tau) - \bar{\varepsilon}_0\cos(\bar{\Delta}\tau)$. Analytic examination of the behavior of the solutions of (18) for arbitrary relationships between the parameters and for arbitrary initial conditions is a relatively difficult problem. We shall therefore confine our attention to the case

$$\psi(0) = \pi/2, \quad R_k^x(0) = 0; \quad \bar{\varepsilon}_2(0) = 0; \quad \bar{\varepsilon}_0 \ll 1, \quad G \ll \omega_c. \tag{20}$$

These initial conditions correspond to the physically interesting situation in which the self-consistent field and the polarization are zero at the initial time. When (20) is satisfied, the set of equations given by (18) can be approximately reduced to a single equation, i.e., the equation describing a physical pendulum with an external harmonic source of frequency $\bar{\Delta}$ and amplitude $\bar{\varepsilon}_0\bar{\Delta}$ acting upon it:

$$\ddot{\theta} - \sin\theta = -\bar{\varepsilon}_0\bar{\Delta}\sin(\bar{\Delta}\tau). \tag{21}$$

When there is no perturbation ($\bar{\varepsilon}_0 = 0$), the motion of the pendulum is periodic and has two types of singular point on the phase plane, namely, elliptic points with coordinates $\dot{\theta} = 0, \theta = (2k+1)\pi$ ($k = 0, \pm 1, \ldots$), which correspond to the complete population of the bottom levels of the two-level system ($R_k^x = 0$, $R_k^y = 0$, $R_k^z = -1$, $\varepsilon_1 = \varepsilon_2 = 0$), and hyperbolic points with coordinates $\dot{\theta} = 0$, $\theta = 2\pi k$ ($k = 0, \pm 1, \ldots$), that correspond to the complete population of the upper levels of the two-level system ($R_k^x = 0$, $R_k^y = 0, R_k^z = +1, \varepsilon_1 = \varepsilon_2 = 0$). The separatrix of the pendulum (the singular trajectory on the phase plane that separates oscillatory from rotational motion and passes through the hyperbolic points) corresponds to the total transfer of energy from the atoms to the field and back again. When the perturbation is turned on ($\bar{\varepsilon}_0 \neq 0$), there are nonlinear resonances between the harmonics of the eigenfrequency of the nonlinear oscillations of the pendulum and the frequency of the external force $\bar{\Delta}$. Depending on the relationship between $\bar{\varepsilon}_0$ and Δ, we can have two typical cases of dynamic behavior of the system (21) (Ref. 2).

(a) For $\bar{\Delta} \gg 1$ ($\Delta \gg \omega_c$), the overlap of nonlinear resonances in the neighborhood of the separatrix results in the appearance of a narrow stochastic layer, while the remaining part of the phase space is filled mostly with periodic trajectories. Using the results of Section 5.1 of Ref. 2, we estimate the width of the stochastic layer on the energy scale as being

$$|(E_c - E)/E_c| \leqslant \bar{\varepsilon}_0\bar{\Delta}\exp(-\pi\bar{\Delta}), \tag{22}$$

where $E_c = 1$ is the energy on the separatrix.

(b) When $\bar{\Delta} \lesssim 1$ ($\Delta \lesssim \omega_c$), a wider statistical layer appears in the neighborhood of the separatrix and its energy width is[2]

$$|(E - E_c)/E_c| \leqslant \bar{\varepsilon}_0\bar{\Delta}^2. \tag{23}$$

We know (see, for example, Section 5.3 of Ref. 2) that the stochastic layer fills a major part of the phase plane (with the exclusion of the neighborhood of the elliptic point) when the following conditions are satisfied:

$$\bar{\varepsilon}_0\bar{\Delta} \approx 1 \text{ and } \bar{\Delta} \leqslant 1,$$

i.e., when the three characteristic frequencies of oscillations of the dynamic system are commensurable:

$$G \gtrsim \omega_c \gtrsim \Delta. \tag{24}$$

When the initial conditions are chosen inside the stochastic layer, all the dynamic variables become random functions of time with a wide Fourier spectrum. We can then pass from the purely dynamic to the kinetic description.[1–4] For an equation such as (21), we know[1,2] that, provided the stochasticity criterion is satisfied, we observe a diffusion growth of the quantity $|\dot{\theta}(\tau)| = |\bar{\varepsilon}_1(\tau) + \bar{\varepsilon}_0\cos(\bar{\Delta}\tau)|$ with time, i.e., the growth of the self-consistent field $\varepsilon_i(t)$, such that $|\dot{\theta}|_{\max} \sim 3$–$4$, whereas $|\dot{\theta}|_{\max} = 2$ for $\bar{\varepsilon}_0 = 0$.

The transition from (18) to (21) can be justified only if (20) is satisfied. The complete system (11), or (18), with arbitrary relationships between the parameters can be investigated only by numerical methods. Finally, we note that all the conclusions relating to stochastic dynamics, including the "global" chaos criterion (24), were obtained essentially

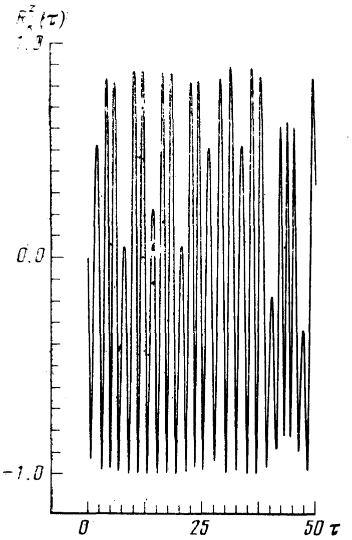

FIG. 1. The process $R_k^z(\tau)$ in the case of chaos: $\bar{\varepsilon}_0 = 2$; $\bar{\Delta} = 1$; $R_k^x(0) = 0$; $R_k^y(0) = 1$; $R_k^z(0) = 0$; $\bar{\varepsilon}_1(0) = \bar{\varepsilon}_2(0) = 0$

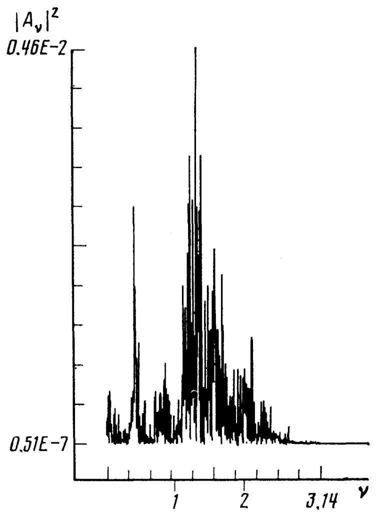

FIG. 3. Fourier spectrum of the process $R_k^z(\tau)$ in the case of chaos: $\bar{\varepsilon}_0 = 2$; $\bar{\Delta} = 1$; $R_k^x(0) = 0$; $R_k^y(0) = 0.866$; $R_k^z(0) = 0.5$; $\bar{\varepsilon}_1(0) = \bar{\varepsilon}_2(0) = 0$

within the framework of perturbation theory and are valid for $\bar{\varepsilon}_0 \lesssim 1$ (Refs. 1 and 2). It follows that the case $\bar{\varepsilon}_0 \gg 1$ requires separate examination, but simple estimates show that when $\bar{\varepsilon}_0 \gg 1$ Eqs. (11) describe only regular oscillations in the populations of atoms in the external field of constant amplitude, i.e., Rabi oscillations.

4. NUMERICAL CALCULATIONS

Numerical calculations based on (15) have been carried out using the dimensionless variables (19). The calculation was performed by the Hamming predictor-corrector method.[28] The accuracy of the calculation was monitored by checking that the conservation law (12) was satisfied. In all cases, this was obeyed to within a few tenths of a percent. To find the difference between the caustic and quasiperiodic trajectories of the dynamic system (11), we computed not only the trajectories themselves but also the Fourier spectrum and the local instability. The local instability was defined as the logarithm of the separation between two initially close trajectories:

$$U(\tau) = \ln\Big\{ \sum_{\alpha=\{x,y,z\}} [R_k^{(\alpha)}(\tau) - R_k^{(\alpha)'}(\tau)]^2 + \sum_{i=1,2} [\bar{\varepsilon}_i(\tau) - \bar{\varepsilon}_i'(\tau)]^2 \Big\}^{1/2}, \tag{25}$$

where the prime represents the trajectory with similar initial conditions. The Fourier spectrum of the process $X(\tau)$ was determined as

$$A_q = \frac{1}{N} \sum_{n=0}^{N-1} X_n \exp[-i2\pi q/N],$$

where $q \in (0, N-1)$, $A_q^* = A_{N-q}$, n is the discrete time corresponding to the continuous τ, and N is the total time. The dimensionless frequencies μ (in units of ω_c) in the spectrum of $X(\tau)$ are related to q by $\nu = 2\pi q/N$.

Figures 1–7 show some typical time dependences, spec-

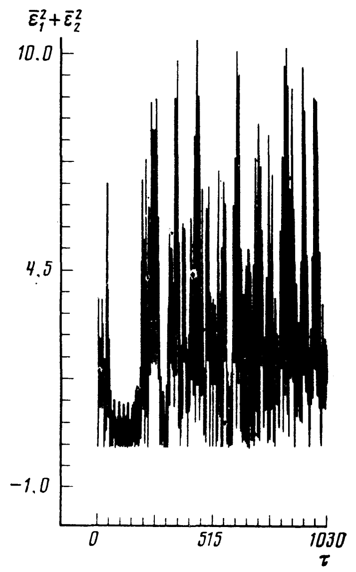

FIG. 2. The self-consistent field $\bar{\varepsilon}_1^2(\tau) + \bar{\varepsilon}_2^2(\tau)$ as a function of time in the case of chaos. The initial conditions and parameter values are the same as in Fig. 1.

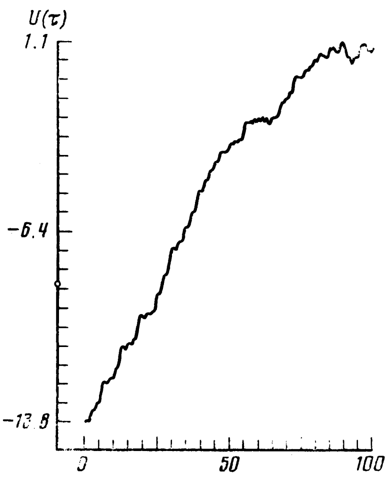

FIG. 4. Local instability. The initial parameters are the same as in Fig. 1.

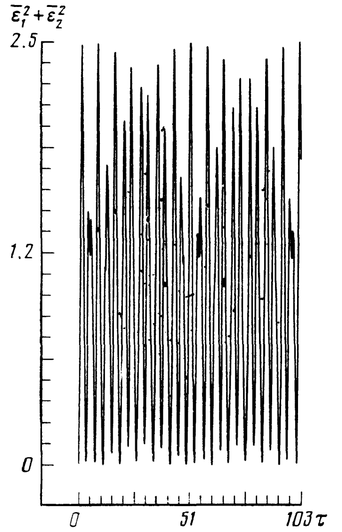

FIG. 5. The self-consistent field as a function of time for quasiperiodic motion: $\bar{\varepsilon}_0 = 2$; $\overline{\Delta} = 3$; $R_k^x(0) = 0$; $R_k^y(0) = 1$; $R_k^z(0) = 0$; $\bar{\varepsilon}_1(0) = \bar{\varepsilon}_2(0) = 0$

tra, and local instabilities for different relationships between the parameters. The stochastic solutions, shown in Figs. 1 and 2, have a wide Fourier spectrum (Fig. 3) and are characterized by an exponential divergence between two initially close trajectories (Fig. 4). The quasiperiodic trajectories (Fig. 5) have a discrete spectrum (Fig. 6), and local instability is absent (Fig. 7). Our numerical analysis shows that the approximate criterion (24) for global chaos (the stochastic component occupies most of the phase space) also remains valid for the complete set of equations given by (11). When the self-consistent radiation field is not present at the initial time $[\bar{\varepsilon}_1 = \bar{\varepsilon}_2(0) = 0]$, dynamic chaos is observed mostly in the inverted system $R_k^z(0) \gtrsim 0$, although it is also observed for certain individual values of the parameters in the weakly excited system. The maximum size of the stochastic region in phase space is reached for $\bar{\varepsilon}_0 = 2$, $\overline{\Delta} = 1$. We also note that the magnitude of the self-consistent field generated in the system during stochastic instability may exceed by a substantial factor both the external field $\bar{\varepsilon}_0$ and the self-consistent field generated in the system in the absence of the external pump ($\bar{\varepsilon}_0 = 0$).

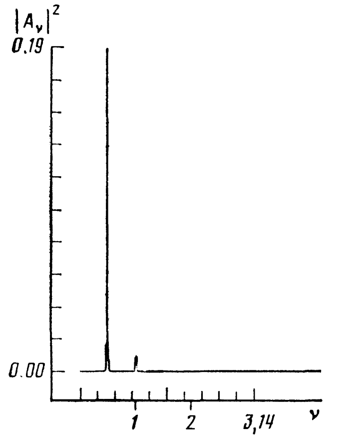

FIG. 6. Fourier spectrum of the quasiperiodic process $R_k^z(\tau)$: $\bar{\varepsilon}_1 = 1$; $\overline{\Delta} = 1$; $R_k^x(0) = 0$; $R_k^y(0) = 0$; $R_k^z(0) = 1$; $\bar{\varepsilon}_1(0) = \bar{\varepsilon}_2(0) = 0$

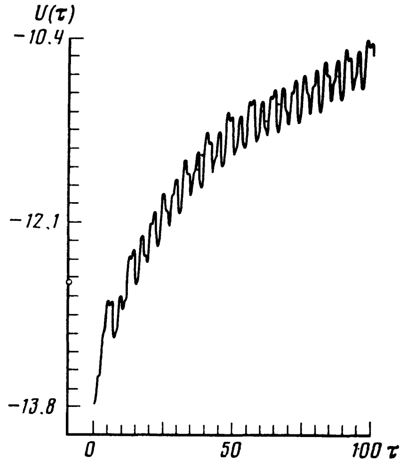

FIG. 7. Form of the function $U(\tau)$ (25) for the case of quasiperiodic motion. The initial conditions and parameters are the same as in Fig. 5.

5. CONCLUSION

We have shown in this paper that, under certain definite conditions (see Section 3), an ensemble of two-level systems interacting with an external monochromatic field may exhibit stochastic instability when cooperative effects are taken into account. A diffusion growth in the self-consistent radiation field may also be observed under these conditions.

Let us estimate the values of physical parameters for which such effects can occur in typical optical experiments. The following conditions must be satisfied.

(1) $G^{-1}, \omega_c^{-1}, t \ll T_2 \sim 10^{-7}$–$10^{-9}$ s, i.e., the criterion for the validity of the nondissipative approach.

(2) The conditions for the validity of RWA and SVA:

$$\omega_c,\ G,\ \Delta \ll \omega_0 \sim 10^{12}\text{–}10^{15}.$$

(3) The condition (16) that the amplitude of the external field be approximately constant: $l < c/\omega_c$.

(4) The global chaos condition (24): $G \gtrsim \omega_c \gtrsim \Delta$.

These conditions can be satisfied, for example, when $\rho \sim 10^{14}$–10^{16} cm^{-3}, $d \sim 10^{-18}$ cgs, $\omega_c \sim 10^{10}$–10^{11} s^{-1}, $G \sim 10^{10}$–10^{11} s^{-1}, $\varepsilon_0 \sim 10$–10^2 cgs, $l \lesssim 0.1$–1 cm.

Dynamic chaos in this system can probably be observed in the microwave range for much lower densities in experiments with so-called Rydberg atoms (see Ref. 29 and references cited therein). This is so because transitions between two closely-spaced Rydberg states have a large dipole moment (larger by a factor of 1000 than the dipole moment of an optical transition in an ordinary atom).

We note in conclusion that our results demonstrate that the transition to chaos in an ensemble of two-level systems is possible for relatively low densities when cooperative effects are taken into account. The necessary condition for developed chaotic motion is merely that the characteristic frequencies of the oscillations in the system be comparable, i.e., $G \gtrsim \omega_c \gtrsim \Delta$.

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