Hydrodynamics of the boundary of *A* and *B* phases of superfluid ^{3}He

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A hydrodynamic theory of the *A-B* phase boundary of ^{3}He at finite temperatures is developed. It consists of the surface conservation laws and of the boundary conditions that relate the hydrodynamic flows with the disequilibrium of the thermodynamic quantities on the surface. The ensuing transport coefficients are estimated. The spectra of the natural oscillations of the *A* and *B* phase boundary are calculated and the problem of passage of a fourth-sound wave through this boundary is solved. The possibility of experimentally observing the considered phenomena is discussed.

I. INTRODUCTION

It became clear from practically the very start of the research into superfluid ^{3}He that the phase transition between phases A and B is of first order, and therefore the phase diagram of ^{3}He contains a curve on which the phases A and B can coexist. This stimulated papers on the properties of the interface between phases A and B.[1–3] The authors of these papers were interested only in the static properties of the interface, such as the surface energy or the supercooling of phases A and B. The question at present is that of investigating the dynamics of the interface. Problems connected with dynamics can be arbitrarily divided into two categories. First is to obtain the general relations between the fluxes of the hydrodynamic quantities of phases A and B and to introduce phenomenological kinetic coefficients based on the general requirements of thermodynamics. The second is the calculation of phenomenological kinetic coefficients on the basis of a microscopic theory. A microscopic calculation of a number of kinetic coefficients was reported in the papers of Yip and Leggett,[4] and also in Ref. 5. In the present paper we dwell mainly on a solution of the first class problems and discuss only estimates of the ensuing transport coefficients.

The hydrodynamics of the A and B phase boundary was already investigated in Ref. 6 at $T=0$, i.e., in the absence of dissipation. Introduction of the dissipative function and of the kinetic coefficient A of the boundary growth [see (21) below] leads to the fact that only one of the two modes predicted in Ref. 6, only one is realized at $T=0$, and the particular mode is determined by the value of A. Surface hydrodynamics for the other boundaries of two phases was already considered for $T\neq 0$. The first to obtain a complete set of equations for the free surface of superfluid ^{4}He at finite temperature were Andreev and Kompaneets.[7] Their approach was later generalized by Iordanskiĭ, Korshunov, and Larkin[8] as applied to the boundary between liquid and solid ^{4}He.

Using the method of Refs. 7 and 8, we derive in Sec. II linearized surface-hydrodynamics equations for the *A-B* interface, starting from the conservation laws and the second law of thermodynamics as applied to this interface. The boundary conditions obtained are then simplified, bearing in mind solutions of specific hydrodynamic problems. We estimate next a number of transport coefficients predicted by the hydrodynamics, and compare them with available results by others. In Sec. III we investigate the spectrum of the natural oscillations of the *A-B* interface and solve the problem of passage of a fourth-sound wave through this boundary. The question of experimental observation of the phenomena considered will be dwelled upon briefly in the Conclusion.

II. BOUNDARY CONDITIONS

1. Phenomenological derivation

In the phenomenological derivations of the surface-hydrodynamics equations we shall assume the interface between the two phases to be infinitely thin and specify the interface equation in the form $z=\zeta(x_\alpha,t)$, $\alpha=1,2$. We assume that at equilibrium the boundary is planar and satisfies the equation $z=0$. Phase A occupies the half-space $z<0$, and phase B the half-space $z>0$. To derive the boundary conditions on the interface we use the conservation laws and the second law of thermodynamics for the surface. To this end, we express some hydrodynamic conserved quantity (say, the energy E) as a sum of a surface part ε and the bulk parts E^A (A phase) and E^B (B phase) parts:

$$\int ds\left\{\varepsilon+\int_{-\infty}^{\zeta(x,y,t)} E^A\,dz+\int_{\zeta(x,y,t)}^{\infty} E^B\,dz\right\},$$

where $\int ds$ denotes an integral over the surface. We differentiate next this expression with respect to time and use the law of bulk-energy conservation

$$\dot{E}+\operatorname{div}\mathbf{Q}=0,$$

where $\mathbf{Q}$ is the bulk energy flux. The total energy of the system is conserved. Equating the time derivative to zero, using the Gauss equation, and assuming that at infinity ($z\to\pm\infty$) the energy flux is zero, we obtain the surface-energy conservation law in integral form:

$$\int ds\{\dot{\varepsilon}+Q_z^B-Q_z^A+(E^A-E^B)\dot{\zeta}\}=0. \qquad (1)$$

The law holds if the integrand is equal to some two-dimensional divergence, so that (1) can be rewritten in differential form:

$$\dot{\varepsilon}+\nabla_\alpha\psi_\alpha+(Q_z^B-E^B\zeta)-(Q_z^A-E^A\dot{\zeta})=0. \qquad (2)$$

In exactly the same manner we obtain the conservation laws for the other hydrodynamic quantities. For the amount of matter we have

$$\nabla_\alpha j_\alpha^s+(j_z^B-\rho^B\zeta)-(j_z^A-\rho^A\dot{\zeta})=0. \qquad (3)$$

Equation (3) should contain the time derivative of the surface mass ρ^s, since the tangential component of ρ^s is not equal to zero. This derivative, however, is determined by the dynamics of the texture of the interface, is extremely small (see Chap. III), and we neglect it.

The conservation law for the z projection of the momentum of the tangential component of the momentum and the entropy take respectively the forms

$$\nabla_\beta \pi_{z\beta} + \Pi_{zz}{}^B - \Pi_{zz}{}^A = 0, \tag{4}$$

$$\partial j_\alpha{}^s/\partial t + \nabla_\beta \pi_{\alpha\beta} + \Pi_{\alpha z}{}^B - \Pi_{\alpha z}{}^A = 0, \tag{5}$$

$$\dot{s} + \nabla_\alpha f_\alpha + (F_z{}^B - S^B\dot{\zeta}) - (F_z{}^A - S^A\dot{\zeta}) = R/T^s. \tag{6}$$

In the last equation it is taken into account that the entropy is not a conserved quantity, and therefore the entropy source—the dissipative function R—is added in the right-hand side of (6). Here and elsewhere the superscripts A, B, and s will label respectively quantities pertaining to phases A and B and to interface. Greek subscripts run through values 1 and 2 and are used to designate surface quantities, while Latin subscripts run through values 1, 2, and 3 and are used for bulk quantities; j^s_α, $\pi_{i\beta}$, ε, ψ_α, s, f_α are respectively the surface momentum, momentum flux density tensor, energy, energy flux, entropy, and entropy flux; $\rho, j_i, \Pi_{ij}, E, Q_i, S, F_i$ are respectively the bulk density, momentum, momentum-flux tensor, energy, energy flux, entropy, and entropy flux.

When deriving the linearized surface-hydrodynamics equations it is necessary, as is done in the derivation of the equations of bulk hydrodynamics, to retain in the equations for the matter flux, momentum, and entropy, only the terms of first order in the deviation from equilibrium, and to take into account in the energy conservation law only the terms quadratic in these deviations. The expressions for the bulk fluxes are well known. In the B phase they take, in the corresponding approximation, the form[9]

$$\begin{gathered}\Pi_{zz}{}^B = p^B + \tau_{zz}{}^B, \quad \Pi_{z\alpha}{}^B = \tau_{z\alpha}{}^B,\\ F_z{}^B = S^B v_{nz}{}^B + r_z{}^B, \quad Q_z{}^B = \mu^B j_z{}^B + S^B T^B v_{nz}{}^B + q_z{}^B.\end{gathered} \tag{7}$$

It is necessary to add in the equations for the A-phase fluxes terms that take into account the dynamics of the liquid-crystal anisotropy vector **1** (see, e.g., Ref. 10):

$$\begin{gathered}\Pi_{zz}{}^A = p^A - \frac{\hbar}{4m}\rho^A(\mathbf{l}\operatorname{rot}\mathbf{v}_n{}^A) + \tau_{zz}{}^A,\\ \Pi_{z\alpha}{}^A = \tau_{z\alpha}{}^A, \qquad F_z{}^A = S^A v_n{}^A + r_z{}^A,\\ Q_z{}^A = \mu^A j_z{}^A + S^A T^A v_{nz}{}^A + \frac{\hbar}{4m} j_z{}^A(\mathbf{l}\operatorname{rot}\mathbf{v}_n{}^A)\\ + \dot{l}_i\frac{\partial E_\nabla}{\partial \nabla_z l} - \frac{\hbar}{4m}\rho^A[\mathbf{v}_n{}^A\mathbf{l}]_z + q_z{}^A.\end{gathered} \tag{8}$$

Here p, μ, T, and v_n are the pressure, chemical potential, temperature, and velocity of the normal component, τ_{zi}, r_z, and q_z are the dissipative z-components of the momentum flux density tensor, of the entropy flux, and of the energy flux, **1** is the liquid-crystal-anisotropy vector of the A phase, and E_∇ is the gradient energy connected with the texture of the vector **1**.

To obtain the equations of the surface hydrodynamics by starting from the interrelation of Eqs. (2)–(6), we must know the dependence of the surface energy ε on the other quantities that describe the surface. We can write down right away the contribution made to the surface energy by the quantities connected with surface tension [first term of (9)], with the surface mass [second term in (9)], and with the surface energy of the liquid-crystal anisotropy [third term in (9)]. In the quadratic approximation we have

$$\varepsilon_1 = \frac{1}{2}\alpha_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}\frac{\partial\zeta}{\partial x_\beta} + \frac{\nu}{2\rho^A\rho^B}\left(\frac{j_z{}^A + j_z{}^B}{2} - \frac{\rho^A + \rho^B}{2}\dot{\zeta}\right)^2 + \frac{\beta}{2}\left(l_z - l_\alpha\frac{\partial\zeta}{\partial x_\alpha}\right)^2. \tag{9}$$

We have assumed in (9) for definiteness that at equilibrium the vector **1** lies in the plane of the boundary; $\beta \sim \xi\Delta$ is the orientational energy; ν is the surface mass.

Generalization to the case of an arbitrary equilibrium position of **1** relative to the normal to the boundary is not particularly difficult. Since **1** determines the chosen direction on the interface, the surface tension $\alpha_{\alpha\beta}$ is anisotropic and is, generally speaking, a tensor quantity. We have also assumed that the surface kinetic energy [second term of (9)] is the internal energy of the boundary and should therefore be defined in a coordinate frame that is immobile relative to the boundary itself.

Besides ε_1, there should exist also terms that are connected with other independent thermodynamic properties of the surface, viz., the surface temperature and the velocity of the normal surface excitations. Under complete thermodynamic equilibrium the temperature and the tangential component of the velocity of the thermal excitations are equal respectively to the temperature and the tangential components of the velocities of phases A and B, and the normal velocity component of the bulk excitations is equal to the velocity of the interface. In the case of deviation from equilibrium, these quantities, generally speaking, need not necessarily equal one another. Each independent thermodynamic variable should correspond to its own thermodynamic conjugate. It is convenient to transform to a coordinate frame moving with velocity $1/2(\mathbf{v}_s^A + \mathbf{v}_s^B)$. In this frame, the differential $d\varepsilon_2$ of that part of the surface energy which is connected with the surface temperature and the velocity of the normal excitations, can be represented in the form

$$\begin{gathered}d\varepsilon_2 = T^s ds + [V_\alpha - {}^1/_2(v_{s\alpha}{}^A + v_{s\alpha}{}^B)]\,dj_{0\alpha}{}^s,\\ j_{0\alpha}{}^s = j_\alpha{}^s - \rho^s(v_{s\alpha}{}^B + v_{s\alpha}{}^A)/2,\end{gathered} \tag{10}$$

where **V** is the velocity of surface excitations, and $j^s_{0\alpha}$ is the momentum in the moving coordinate frame. We have taken it into account in (9) and (10) that all the bulk quantities should be symmetric relative to the substitutions $A \leftrightarrow B$. Strictly speaking, it would be necessary to take into account in the surface energy also the contribution from the term of type $v^A_{s\alpha} - v^B_{s\alpha}$, but the thermodynamic conjugate of the latter does not correspond to any observable physical quantity and will not be considered here. Taking the differential of (9), we obtain after a Galilean transformation a final expression for the surface-energy differential $d\varepsilon = d\varepsilon_1 + d\varepsilon_2$ in the lab:

$$d\varepsilon = \alpha_{\alpha\beta}\frac{\partial\zeta}{\partial x_\alpha}d\frac{\partial\zeta}{\partial x_\beta} + \frac{\nu}{\rho^A\rho^B}\left(\frac{j_z{}^A + j_z{}^B}{2} - \frac{\rho^A + \rho^B}{2}\dot{\zeta}\right) \times d\left(\frac{j_z{}^A + j_z{}^B}{2} - \frac{\rho^A + \rho^B}{2}\dot{\zeta}\right) + \beta\left(l_z - l_\alpha\frac{\partial\zeta}{\partial x_\alpha}\right)d\left(l_z - l_\alpha\frac{\partial\zeta}{\partial x_\alpha}\right)$$

$$+T^s\,ds+V_\alpha\,dj_\alpha{}^s+\frac{1}{2}\,(j_\alpha{}^s-\rho^s V_\alpha)\,d(v_{s\alpha}{}^B+v_{s\alpha}{}^A). \tag{11}$$

We differentiate next the surface energy (11) with respect to time and express the time derivatives of the thermodynamic quantities in terms of the spatial derivatives with the aid of the surface conservation laws. We need for this purpose:

a) linearized equations of motion of the superfluid components in the phases A and B:

$$\begin{gathered}\frac{\partial}{\partial t}\mathbf{v}_s{}^B+\nabla\,(\mu^B+h^B)=0,\\ \frac{\partial}{\partial t}\mathbf{v}_s{}^A+\nabla\left[\mu^A+\frac{\hbar}{4m}\,(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n{}^A)+h^A\right]=0,\end{gathered} \tag{12}$$

where h^A and h^B are the dissipative terms in the equations of the superfluids;

b) the thermodynamic law for the surface quantities in the linear approximations:

$$\frac{\rho^s}{2}\,d\left[\mu^A+\mu^B+\frac{\hbar}{4m}\,(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n{}^A)\right]=-s\,dT^s+dp^s,$$

in which we stipulate that the chemical potential μ^s of the surface be equal to the arithmetic mean of the chemical potentials of the phases A and B;

c) an expression for the dissipative part of the bulk energy flux, in terms of the dissipative terms in the entropy, momentum, and superfluid-motion fluxes;

$$q_z=Tr_z+h(j_z-\rho v_{nz})+\tau_{zi}v_{ni}.$$

The operations indicated lead to terms under the two-dimensional divergence sign, which determine the expressions for the surface energy flux ψ_α, and to terms not under the two-dimensional divergence sign, which lead to an expression for the dissipative function R. Setting the dissipative function equal to zero yields an expression for the reversible parts of the surface fluxes. After rather lengthy but straightforward calculations we find that the surface fluxes are equal to

$$\begin{gathered}\psi_\alpha=-\alpha_{\alpha\beta}\dot{\zeta}\,\frac{\partial\zeta}{\partial x_\beta}+T^s f_\alpha+V_\beta\,(\pi_{\alpha\beta}-p^s\delta_{\alpha\beta})\\ +\frac{1}{2}\,j_\alpha{}^s\left[\mu^A+\mu^B+\frac{\hbar}{4m}(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n{}^A)\right]\\ +\frac{1}{2}\,(j_\alpha{}^s-\rho^s V_\alpha)\,(h^A+h^B)+\beta l_\alpha\left(l_z-l_\beta\frac{\partial\zeta}{\partial x_\beta}\right)\dot{\zeta},\\ \pi_{z\alpha}=\alpha_{\alpha\beta}\partial\zeta/\partial x_\beta+\beta l_\alpha(l_z-l_\gamma\partial\zeta/\partial x_\gamma),\\ \pi_{\alpha\beta}=p^s\delta_{\alpha\beta}+\tau_{\alpha\beta}{}^s,\qquad f_\alpha=sV_\alpha+r_\alpha{}^s,\end{gathered} \tag{13}$$

where $\tau^s_{\alpha\beta}$ and r^s_α are the dissipative parts of the surface-impedance flux tensor and of the surface-entropy flux. In the limit as $T\to 0$ the expressions for the fluxes go over into the expressions given in Ref. 6. The dissipative function takes the form

$$R=-\sum_{i=1}^{11}X_iY_i, \tag{14}$$

$$X_1=j_z{}^B-\rho^B\dot{\zeta}+{}^1\!/_2\nabla_\alpha j_\alpha{}^s,$$

$$Y_1=\frac{\nu}{\rho^A\rho^B}\frac{\partial}{\partial t}\left(j_z{}^B-\rho^B\dot{\zeta}+\frac{1}{2}\,\nabla_\alpha j_\alpha{}^s\right)$$

$$+\mu^B+h^B-\mu^A-h^A+\frac{\hbar}{4m}\,(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n{}^A),$$

$$X_2=S^B(v_{nz}{}^B-\zeta)+r_z{}^B,\qquad Y_2=[T^B+(\tau_{zz}{}^B-\rho^Bh^B)/S^B]-T^s,$$

$$X_3=S^A(v_{nz}{}^A-\zeta)+r_z{}^A,\qquad Y_3=T^s-[T^A+(\tau_{zz}{}^A-\rho^Ah^A)/S^A],$$

$$X_4=(S^B)^{-1}(\rho^Bh^B-\tau_{zz}{}^B),\qquad Y_4=r^B,$$

$$X_5=(S^A)^{-1}(\rho^Ah^A-r_{zz}{}^A),\qquad Y_5=-r^A,$$

$$X_6=\tau_{\alpha z}{}^B,\qquad Y_6=V_\alpha-v_{n\alpha}{}^B,$$

$$X_7=\tau_{\alpha z}{}^A,\qquad Y_7=v_{n\alpha}{}^A-V_\alpha,$$

$$X_8=r_\alpha{}^s,\qquad Y_8=\nabla_\alpha T^s,$$

$$X_9=\tau_{\alpha\beta}{}^s+{}^1\!/_2\rho^s(h^A+h^B)\delta_{\alpha\beta},\qquad Y_9=-\nabla_\alpha V_\beta,$$

$$X_{10}=l_z,\qquad Y_{10}=\beta(l_z-l_\alpha\partial\zeta/\partial x_\alpha)+\partial E_\nabla/\partial\nabla_z l_z,$$

$$X_{11}=l_\alpha,\qquad Y_{11}=(\hbar/4m)\rho^A e_{z\beta\alpha}v_{n\beta}{}^A+\partial E_\nabla/\partial\nabla_z l_\alpha.$$

For the dissipative function to be positive, we must have in the principal approximation a linear relation between X_i and Y_i; this yields boundary conditions that relate the fluxes X_i on the interface with the deviations of the thermodynamic quantities Y_i from their values at equilibrium:

$$X_i=-\sum_{j=1}^{11}\gamma_{ij}Y_j. \tag{15}$$

The first five thermodynamic forces Y_i ($i=1$–5) are connected with the deviation from equilibrium in the bulks of phases A and B, while the next four Y_i ($i=6$–9) are due to deviations of the surface values from equilibrium, and the last two Y_i ($i=10,11$) are related to the dynamics of the liquid-crystal anisotropy vector. The coefficients γ_{ij} must satisfy the conditions that the quadratic form $\gamma_{ij}\,Y_iY_j$ be positive definite and be, in accordance with the Onsager principle, symmetric with respect to permutation of the indices. Thus, the most complete description of the interface calls for introduction of about fifty coefficients γ_{ij}, which together with the conservation laws (2)–(6) constitute the complete set of boundary conditions.

2. Kinetic coefficients

Not all the kinetic coefficients, however, should be considered on a par in the equations of motion. Purely surface quantities enter into the boundary conditions in derivative form, whereas the bulk fluxes enter into these boundary conditions directly. Therefore retention of the dissipative terms r^s_α, $\tau^s_{\alpha\beta}$ in the expressions for the surface fluxes requires that account be taken in the bulk fluxes of terms of higher order of smallness in the mean free path than the ordinary viscosity and thermal-conductivity approximation, and constitutes in our approach an exaggeration of the accuracy. It can be justified, as noted in Ref. 7, only in the case of exceedingly low temperatures, under the condition that the surface excitations freeze out much more slowly than the bulk ones. In practice, therefore, where bulk hydrodynamics is applicable, it is possible to regard the surface fluxes f^s_α and $\pi_{\alpha\beta}$ as nondissipative and discard the eighth and ninth terms of (14).

Furthermore, when solving a number of hydrodynamic problems (see Sec. III), we can neglect entirely, with good accuracy, the surface entropy s and the surface momentum $\mathbf{j}^s$. This allows us to combine the second and third terms of (14), and also the sixth and seventh, into one; The surface temperature T^s and the surface-excitation velocity V drop in this case out of the expression for the dissipative function. For the diagonal terms in (15) we obtain in the former case the Kapitza relation between the heat flux through the surface and the effective temperature difference:

$$[T^A+(\tau_{zz}{}^A-\rho^A h^A)/S^A]-[T^B+(\tau_{zz}{}^B-\rho^B h^B)/S^B]=KQ_z. \quad (16)$$

(K is the Kapitza coefficient), and in the second case the connection between the momentum-flux component and the difference between the tangential velocities of the bulk excitations on the A-B boundary:

$$v_{n\alpha}{}^B-v_{n\alpha}{}^A=B\tau_{\alpha z}. \quad (17)$$

The Kapitza coefficient is proportional to the probability of passage of a ^{3}He quasiparticle from one phase to the other. Owing to the exponential decrease of the density of the above-condensate particles with decrease of the temperature in the B phase, and owing to the elastic reflection of quasiparticles with energy lower than Δ_B, one can expect, in collisions with the boundary in the A phase, this probability to decrease exponentially with temperature like $\exp(-\Delta_B/T)$. We have therefore for the Kapitza coefficient the simple estimate

$$K\sim(\xi/\varkappa)e^{\Delta_B/T},$$

where ξ is the coherence length and $\varkappa$ the bulk thermal conductivity of ^{3}He. Our estimate implies that the quasiparticle manages to become thermalized over the width of the boundary, i.e., $\tau<\xi/v_F$. The estimate goes over into the result obtained by Yip[4] for the ballistic region, when the effective time τ between the quasiparticle collisions becomes equal to ξ/v_F.

The coefficient B has heretofore not been considered in the literature. The tangential component of the momentum is preserved in collisions with the surface, both for Andreev reflection[11] and for an ordinary elastic process. These processes therefore do not contribute to the coefficient B. Tangential momentum is transferred only when quasiparticles pass from one phase to the other. The probability of this process, just as in energy transfer, has a barrier of order Δ_B. This leads therefore to the estimate

$$B\sim(\xi/\eta)e^{\Delta_B/T},$$

where η is the bulk viscosity.

The fourth and fifth terms of (14) yield the connection between the dissipative heat fluxes q_z and the effective momentum $(\tau_{zz}-\rho h)$ on the boundary:

$$q_z=D(\tau_{zz}-\rho h)/TS. \quad (18)$$

The fourth and fifth terms of (14) contain a product of two dissipative fluxes, and are thus quantities of second order in kl. They must be included, however, when account is taken of the temperature surface wave in a superfluid,[12] for in this case δT and v_n are independent quantities in bulk hydrodynamics.

It is easy to estimate the coefficient D. We note for this purpose that an A-phase quasiparticle incident on the interface and having an energy ε_q lower than the gap in the B phase, is reflected not by the usual but by the Andreev mechanism,[11] i.e., the quasiparticle has an overwhelming probability of becoming a quasihole; the z-projection of the momentum q_z changes in this case by the small amount[4]:

$$\Delta q_z=\frac{2\varepsilon_q}{v_F\cos\vartheta}\left(1+\frac{v_{nz}-\dot{\zeta}}{v_F\cos\vartheta}\right), \quad (19)$$

where v_F is the Fermi velocity and ϑ the angle between the vector $\mathbf{q}$ and the normal $\mathbf{n}$ to the surface. The first term leads simply to a redefinition of the equilibrium pressure. Of importance to us is the second term. At not too high temperatures, i.e., when $T\ll\Delta$, all the excited quasiparticles have a wave vector $\mathbf{q}$ practically parallel to $\mathbf{l}$ and if $\mathbf{l}$ is not oriented too close to $\pi/2$, we get the estimate[4]

$$\delta p_z{}^A\approx N^A v_F\Delta q_z\sim E_N{}^A(v_{nz}-\zeta)/v_F\cos\vartheta, \quad (20)$$

where N^A is the total number of excitations and E_N^A is the energy of the A-phase excitations. On the other hand, the dissipative heat flux at $T\ll\Delta$ (in this case $K\to\infty$) is equal to

$$q_z{}^A=T^A r_z{}^A=TS^A(v_n{}^A-\zeta).$$

We obtain thus a power-law dependence of D on temperature:

$$D^A\sim v_F N_F T^4/\Delta^2$$

($\Gamma=(TS)^2/D^A$ in the notation of Ref. 4). In the B phase, the number of quasiparticles is exponentially small, and this leads to the rough estimate

$$D^B\sim v_F N_F T^2 e^{-\Delta/T}, \quad T\ll\Delta.$$

The quantity D^B is finite because of the nonzero quantum-mechanical probability of reflection of the particle from the potential well.

Consider now the first term in the expression for the dissipative function (14). We introduce for it the diagonal kinetic coefficient A. The expression for the current through the boundary takes then the form

$$j_z{}^B-\rho^B\dot{\zeta}=-A\left[\frac{\nu}{\rho^A\rho^B}\frac{\partial}{\partial t}(j_z{}^B-\rho^B\dot{\zeta})+\mu^B+h^B-\mu^A-h^A\right.$$
$$\left.+\frac{\hbar}{4m}(\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_n{}^A)\right]. \quad (21)$$

Relation (21) is in fact a generalization of the usual relation between the current through the interface and the difference of the chemical potentials. In our case of a nonzero surface mass, the first term in the right-hand side describes the acceleration acquired by the ^{3}He atoms as a result of the difference between the chemical potential. The last term is due to the finite orbital momentum of the quasiparticles. It will be shown below (Sec. III), one can neglect the dissipative terms h^A and h^B as well as $(\mathbf{l}\times\mathrm{curl}\,\mathbf{v}_n^A)$ in the discussion of the hydrodynamic motions of the boundary.

The coefficient A is by itself dissipative, and motion of the boundary without dissipation is possible only in two cases: either a) A becomes infinite, or b) A vanishes. To each of these possibilities there corresponds a definite undamped mode.[6] It was noted in Ref. 6, however, that on a given interface (given the temperature and frequency) the coefficient A can take on only one fixed value, so that only one mode can be realized.

Since the phases A and B have a single wave function, it is reasonable to assume that at $T=0$, in the linear approximation, the usual superfluidity relation obtains between the current and the gradient of a phase with a certain effective surface mass:

$$j_z^B-\rho^B\dot{\zeta}=(\rho^A\rho^B/\nu)(\varphi^B-\varphi^A). \tag{22}$$

This is precisely the relation obtained from (21) by integrating over time, if A is allowed to tend to infinity, i.e., case a) occurs at $T=0$.

At nonzero temperature the coefficient A must become finite, in view of the dissipative flux of the normal component. The normal-component flux through the boundary is exponentially small at $T\ll\Delta$:

$$j-\rho\zeta\sim mN^Av_Fe^{-\Delta/T},\quad N^A\sim E_N{}^A/T,$$

and, as follows from the discussion of relation (19), the increment to the equilibrium pressure on account of the normal component is $\delta p^A\sim E_N^A$. Therefore

$$j-\rho\dot{\zeta}\sim R^{-1}(\mu^A-\mu^B)\sim\frac{m\rho v_F}{T}e^{-\Delta/T}(\mu^A-\mu^B).$$

Relation (21) can be rewritten at $T\ll\Delta$ in a form similar to the linearized resistive model of a Josephson junction [see (21a) below]. To this end, we divide (21) by A, integrate with respect to frequency, and use the fact that $j_n\ll j_s$. We find thus that

$$A\sim\rho^{A2}\rho^{B2}R/\nu^2\omega^2,\quad T\ll\Delta.$$

Note that the left-hand side of (21) contains the derivative of the total current, since we assume that the second term of Eq. (9) for the surface kinetic energy ε_1 is determined by the total current through the boundary. If it is assumed that only the superfluid current contributes to the surface kinetic energy, so that the second term of (9) is equal to $I(\varphi^A-\varphi^B)^2/2$, the boundary condition (21) is obtained directly in a form similar to the relation of the resistive model (without allowance for the surface currents $\mathbf{j}^s$ and for the dynamics of $\mathbf{l}$):

$$\begin{gathered}j_z=\rho\dot{\zeta}=I(\varphi^B-\varphi^A)+(\mu^A-\mu^B)/R,\\ I=\rho_s{}^A\rho_s{}^B/\nu,\quad R\sim(T/m\rho v_F)e^{\Delta/T},\end{gathered} \tag{21a}$$

where R is a kinetic coefficient analogous to the weak-coupling resistance.

We discuss finally the value of the surface mass. Let $T=0$. Then $\rho_s^A=\rho_s^B=\rho^A=\rho^B=\rho$. We define the surface mass in the hydrodynamic approximation as the complement to what occurs if $\rho=\rho_0=\text{const}$ over the entire width of the transition. This is precisely the surface mass that enters in the equation (10) for the surface momentum j_α^s:

$$\rho^s=\int_{-\infty}^{\infty}[\rho_0-\rho_s(x)]dz.$$

This equation is not quite rigorous, only an estimate, since hydrodynamics is known to be valid only when the size of the inhomogeneity is less than the correlation length ξ.

According to Refs. 13, the presence of inhomogeneities in the A-phase texture leads to anomalous terms in the superfluid current and to the appearance of a normal component even at $T=0$. If the normal component is immobilized (stopped by the walls), the surface mass ρ^s is determined only by the superfluid density. For the superfluid density in the direction $\mathbf{d}$, in which the gap vanishes, we have

$$\rho_{s\|}=[1-a(\hbar v_F/\Delta)|(\mathbf{d}\nabla)\mathbf{d}|]\rho, \tag{23}$$

where a is a certain numerical factor of order unity. On going from phase A to phase B, those points on the Fermi surface at which the gap vanishes move in some special manner counter to one another and vanish in the B phase. This is accompanied by $\mathbf{d}$ gradients of the order of ξ^{-1}. We do not know the exact $\mathbf{d}(z)$ dependence for $T=0$, but can estimate the order of magnitude of ρ^s. We obtain from (23)

$$\rho^s\sim c(\mathbf{l})\rho\xi,$$

where $c(\mathbf{l})$ is a certain numerical coefficient that depends on the $\mathbf{l}$ orientation.

Expression (9) for the surface energy contains another surface mass ν. At finite temperature we have $\rho_s^A\neq\rho_s^B\neq\rho_0$ and the determination of ν depends on the concrete form of the motion. Thus, if we put $v_n^A=v_n^B=\zeta$, then ν can be determined in the hydrodynamic approximation by an expression that does not depend on the choice of the boundary ζ:

$$\nu=\int_{-\infty}^{\infty}dz\frac{[\rho_s(z)-\rho_s{}^A][\rho_s(z)-\rho_s{}^B]}{\rho_s(z)}.$$

The tenth and eleventh terms in (14) give the boundary conditions on the vector $\mathbf{l}$. The boundary conditions relate the angular-momentum flux (second terms in Y_{10} and Y_{11}) and the surface "restoring" force (first term of Y_{10}) to the dynamics of $\mathbf{l}$ (terms X_{10} and X_{11}). The first term in Y_{11} describes the interaction of $\mathbf{l}$ with the normal motion of ^{3}He-A.

The conservation laws for the hydrodynamic quantities (2)–(6) must be supplemented by the law of conservation of the spin currents on the A-B phase boundary. This law can be obtained by adding to the energies E^A and E^B a term $\mu_BS^{A,B}H$ connected with the magnetic energy, and by adding to the expression for the energy flux a term $(\hbar^2\rho_s^{A,B}/2m^2)\dot{\theta}\nabla_z\theta_z^{A,B}$, where θ is the order parameter of ^{3}He. The spin and orbit dynamics separate, and we obtain the spin conservation law in the form of two boundary conditions [see Eq. (4) of Ref. 6]. The left-hand side contains the spin current of the A or B phase, and the right hand side the surface source of the spin current (the derivative of the surface part of the spin energy with respect to the components of the order parameter). In the general case of nonzero temperature, the difference between the right-hand and left-hand sides of Eq. (4) of Ref. 6 should give the thermodynamic force, which will equal (with a certain kinetic coefficient) the derivative $\dot{\theta}_z$ of the order parameter on the boundary with respect to time.

We set aside the discussion of the off-diagonal kinetic coefficients. The results obtained in the present section will be used below in the discussion of the surface motion at finite temperatures.

III. HYDRODYNAMIC MOTIONS OF THE *A-B* PHASE BOUNDARY

1. Natural oscillations of the surface

The surface-hydrodynamics equations set described in the preceding section permits an investigation of the dynamic modes existing on the A-B phase boundary. To this end we must supplement the surface equations (2)–(6), (16)–(18), and (21) by the equations of motion of the liquid in the bulk. It will follow from the solution that the velocities of the sought modes will in all cases be lower than that of sound.

This permits the use of the bulk equations of an incompressible liquid. The hydrodynamics of ^{3}He-B in the linear approximation agrees fully with the hydrodynamics of superfluid ^{4}He (Refs. 9 and 14). The equations of motion break up into independent equations of motion of the normal and superfluid components; the variable part of the pressure δp^B at a given point is the sum of pressures $\delta p_n^B + \delta p_s^B$ of the normal and superfluid motions.[15] The superfluid velocity $\mathbf{v}_s^B$, the pressure δp_s^B, and the chemical potential $\delta\mu_s^B$ are given by derivatives of the potential φ^B:

$$v_{si}{}^B=\partial\varphi^B/\partial x_i, \qquad \delta p_s{}^B=-\rho_s{}^B\dot{\varphi}^B, \qquad \delta\mu_s{}^B=-\dot{\varphi}^B, \tag{24}$$

and the potential itself satisfies the Laplace equation

$$\Delta\varphi^B=0. \tag{25}$$

The superfluid and normal densities in ^{3}He-A are tensor quantities, so that in the general case it is possible to separate the normal and superfluid parts of the equation. In an appropriate approximation, however, such a separation is possible. Let, for the sake of argument, the vector **l** be oriented either perpendicular to the interface or be directed along the x axis in the plane of the boundary. The tensors ρ_{sij}^A, ρ_{nij}^A then become diagonal. From the entropy-conservation law we obtain, as before in the incompressible liquid approximation, div $\mathbf{v}_n^A = 0$. Using this to express $\partial v_{nz}^A/\partial z$ in terms of $\partial v_{nz}^A/\partial x$ and substituting in the matter-conservation law, we get

$$\rho_{ij}{}^A\, \partial^2\varphi^A/\partial x_i\, \partial x_j=(\overset{A}{\rho_{nzz}}-\overset{A}{\rho_{nxx}})\,\partial v_{nx}{}^A/\partial x. \tag{26}$$

We shall show below that the ratio of the tangential velocity of the normal velocity to the superfluid one is small. We can therefore neglect the right-hand side of (26). We obtain thus the equation of motion of the ^{3}He superfluid component:

$$\rho_{ij}{}^A\partial^2\varphi^A/\partial x_i\partial x_j=0. \tag{27}$$

We have also used here the fact that the contribution made by the dynamics of **l** to the superfluid current is small compared with the contribution from the term with the phase gradient. In fact, the former current is of the order of $(\hbar/m)\rho k^2\zeta$ and becomes equal to the phase current $\rho\dot{\zeta}$ only when $(\hbar/m)\rho(\nu\alpha^{-1/2}k\sim 1$. (The estimate was obtained using the dispersion law (39).) This takes place, however at too large $k\sim\zeta^{-1}$, at which our theory is no longer valid. The solutions of Eqs. (25) and (27) are of the form

$$\varphi^B=\varphi_0{}^B \exp(i\omega t-ikx-kz),$$

$$\varphi^A=\varphi_0{}^A \exp[i\omega t-ikx+(\rho_x/\rho_z)^{1/2}kz]. \tag{28}$$

The linearized equations of the hydrodynamics of the ^{4}He normal component were written out and solved in Ref. 16. For ^{3}He-B they are

$$i\omega\rho_n\mathbf{v}_n+\eta\Delta\mathbf{v}_n-\nabla p_n=0 \quad \operatorname{div}\mathbf{v}_n=0. \tag{29}$$

We have taken a Fourier transform with respect to time. For ^{3}He-A one can substitute (27) in the momentum conservation law and exclude thereby the superfluid component. In this case, however, the "pressure" tensor of the normal component is no longer described by one scalar quantity as in the B-phase. We obtain thus a system of differential equation with respect to v_i, in which ρ_n^A, η^A, p_n^A in (29) are tensor quantities. We make next some simplifications. First, it can be shown that the first term of (29) is small compared with the remaining ones in both the A and B phases for wavelengths at which the wave vector k satisfies the condition

$$k>k_0=\Delta/\varepsilon_F(v_F\tau).$$

We have used here the dispersion law (39); τ is the effective time between quasiparticle collisions. An estimate for $\tau\sim 10^{-7}$ s yields $k_0\sim 1$ cm^{-1}. It will be shown below that the dispersion law (39) is valid in practice at even higher values of k. In the case of the law (38), the estimated k_0 tends to be even smaller, $k_0\sim 10^{-2}$ cm^{-1}. We can thus always neglect the tensor structures of η^A and p_n^A in ^{3}He-A and regard these quantities as scalars. In this case, η^A must be taken equal to the shear viscosity in a direction perpendicular to **l** if the liquid moves in a plane perpendicular to **l**, and equal to the corresponding "parallel" component of the shear viscosity for motions in a plane containing **l**.[17]

The solutions of the system (29) take the form[16]:

$$\delta p_n{}^{A,B}=P_0{}^{A,B}e^{\pm kz}, \qquad v_{nz}{}^{A,B}=e^{\pm kz}(v_{nz0}^{A,B}+zP_0^{A,B}/2\eta^{A,B}),$$

$$v_{n\mu}{}^{A,B}=-i\frac{k_\mu}{k}e^{\pm kz}\left\{\mp v_{nz0}^{A,B}+\frac{P_0{}^{A,B}}{2\eta^{A,B}}\left(\mp z-\frac{1}{k}\right)\right\}, \tag{30}$$

where $v_{nz0}^{A,B}$ is the value of the z-projection of the normal velocity at $z=0$; the upper sign in all the equations pertains to the A phase, and the lower to the B phase; P_0 is an unknown quantity to be determined from the boundary condition. We satisfy the boundary condition by putting $v_{n\mu}^A(z=0)=v_{n\mu}^B(z=0)=0$, for in this case, as follows from the solution (3), the quantity $\tau_{\mu z}(z=0)$ also vanishes. By the same token, we express P_0 in terms of the z-projection of the normal velocity:

$$P_0{}^{A,B}=\mp 2\eta^{A,B}kv_{nz0}{}^{A,B}. \tag{31}$$

It is necessary next to connect $v_{nz0}^{A,B}$, using the boundary condition (16), with the surface velocity. To this end we express the temperature increment at a given point of the liquid in terms of the increments of the pressures of the normal and superfluid motions (Ref. 15) at $z=0$:

$$S^{A,B}\delta T^{A,B}=\delta p^{A,B}+\rho^{A,B}\partial\varphi^{A,B}/\partial t. \tag{32}$$

Substituting (31) in (32) and next in (16), we get

$$-2k(\eta^A v_{nz0}^{A}+\eta^B v_{nz0}^{B})+i\omega(\rho_{nz}{}^A\varphi_0{}^A-\rho_{nz}{}^B\varphi_0{}^B)$$

$$=KT^BS^{B2}(v_{nz}{}^B-\dot{\zeta}). \tag{33}$$

If $K\to\infty$, the condition (33) is simplified and yields simply $v_{nz}^B=v_{nz}^A=\zeta$ at $z=0$. We shall use (33) later [see (35)].

In the equation of motion of the vector **l**, the term connected with the internal orbital momentum is small like $(\Delta/\varepsilon_F)^2$ (Ref. 18). We omit also the terms that describe the interaction of **l** with the gradients of the normal velocity, and retain only the dissipative term and the texture term:

$$\mu_L\dot{\mathbf{l}}+\delta E_\nabla/\delta\mathbf{l}=0, \tag{34}$$

where E_∇ can be expressed in the form (Ref. 18)

$$E_\nabla=\gamma_1(\operatorname{div}\mathbf{l})^2+\gamma_2[\mathbf{l}\operatorname{rot}\mathbf{l}]^2+\gamma_3(\mathbf{l}\operatorname{rot}\mathbf{l})^2,$$

and the coefficients γ_1, γ_2, and γ_3 are equal, in the weakbinding approximation, to $(\hbar/m)^2\rho$. The orbital viscosity coefficient μ_L that enters in (34) can be estimated at[17] $\mu_L\sim N_F\tau\Delta^2$. To determine those values of the wave vector k at which the gradient term or the term with orbital viscosity is the more significant, we obtain the ratio of the first term in (34) to the second. The order of magnitude of this ratio is

$N_F m^2\Delta^2\omega\tau/\rho_s\hbar^2k^2$.

Taking the dispersion law (37) into account, we note that the ratio becomes of the order of unity at large $k\sim\hbar\tau/m\xi^2\sim10^5$ cm^{-1}. At lower values of k the ratio is always larger than unity. Thus, at $k<10^5$ cm^{-1} the viscous term is always larger than the gradient term. This means in fact that the oscillations of the vector **l** propagate over a length $\sim k_z^{-1}\sim10^{-5}$ cm, which is comparable with the wall thickness. Such distances are infinitely small for hydrodynamics and are not considered.

Using the relations obtained, we can rewrite the conservation laws (3) for matter, (4) for the z component of the momentum flux, and the gradient condition (21) on the momentum flux through the interface by expressing them in terms of the superfluid components ρ_s^A, ρ_s^B, the surface tension tensor α_{ij}, the viscosities η^A and η^B, the transport coefficient A, and the surface mass ν. We assume that the Kapitza coefficient is large enough, so that the boundary condition (33) can be written in the form $v_{nz}^A=v_{nz}^B=\zeta$ (see below for estimates of K and for allowance for its finite value). We obtain then a system of three linear equations for the three unknowns φ^A, φ^B, ζ:

$$k(\rho_s{}^B\varphi^B+\rho_{s1}{}^A\varphi^A)=i\omega(\rho_{s2}{}^A-\rho_s{}^B)\zeta, \tag{35a}$$

$$i\omega(\rho_{s1}{}^A\varphi^A-\rho_s{}^B\varphi^B)-2i\omega(\eta^A+\eta^B)k\zeta-\alpha_{ij}k_ik_j\zeta=0, \tag{35b}$$

$$\rho_s{}^B(k\varphi^B+i\omega\zeta)=-i\omega A[(\nu/\rho_{s2}{}^A)(k\varphi^B+i\omega\zeta)+(\varphi^B-\varphi^A)], \tag{35c}$$

where we have put $\rho_{s1}^A=(\rho_{sz}^A\rho_{sx}^A)^{1/2}$, $\rho_{s2}^A=\rho_{sz}^A$ and used Fourier transforms with respect to the time and the coordinates. The first term in (35b) describes the contribution of the superfluid component and the excess pressure, the second term in (35b) describes the contribution of the normal component, and the third the surface tension. We have retained in (35c) the surface transport coefficient A. The left-hand side of (35c) is the flux through the boundary, and the right is a relation similar to the linearized Josephson relation. It is easily understood that if the normal density is neglected in (29) the normal component of ^{3}He contributes only to the damping of the surface waves. The system of linear equations (35) can be solved in general form, and this leads to the dispersion equation

$$\begin{gathered}(i\omega-\rho_s{}^B\rho_{s2}{}^A/A\nu)k\rho_s{}^B\rho_{s1}{}^A\{\omega^2(\rho_{s2}{}^A+\rho_s{}^B)-k[\alpha_{ij}k_ik_j\\+2i\omega k(\eta^A+\eta^B)]\}-i\omega(\rho_s{}^B\rho_{s2}{}^A/\nu)\{k[\alpha_{ij}k_ik_j\\+2i\omega k(\eta^A+\eta^B)](\rho_{s1}{}^A+\rho_s{}^B)-\omega^2(\rho_s{}^B-\rho_{s1}{}^A)(\rho_s{}^B-\rho_{s2}{}^A)\}=0.\end{gathered} \tag{36}$$

As should be the case, expression (36) is symmetric with respect to the interchange $\rho_s^A\leftrightarrow\rho_s^B$ (in this case we must put $\rho_{s1}^A=\rho_{s2}^A$). The imaginary terms in the dispersion law, as can be easily seen from (36), will result from terms containing the kinetic coefficient A and the viscosity η. An investigation of (36) is in the general case quite cumbersome, and we shall consider only particular cases. We let A tend to infinity [this corresponds to the low-temperature limit, as can be seen from the discussion of (21)]; then

$$\omega^2=\frac{[k\rho_{s2}{}^A(\rho_{s1}{}^A+\rho_s{}^B)+\nu\rho_{s1}{}^Ak^2][\alpha_{ij}k_ik_j+2i\omega(\eta^A+\eta^B)k]}{\nu\rho_{s1}{}^A(\rho_{s2}{}^A+\rho_s{}^B)k+\rho_{s2}{}^A(\rho_s{}^B-\rho_{s1}{}^A)(\rho_s{}^B-\rho_{s2}{}^A)}. \tag{37}$$

This equation is similar in many respects to the expression proposed for the neutral oscillations of the interface between solid and liquid ^{4}He (Ref. 19), with allowance for the finite surface mass (in Ref. 19 was considered the case of zero temperature, and the dispersion law contains therefore no imaginary term). The second term in the square brackets, at all values of $k<\xi^{-1}$, is smaller than the first and can be neglected. The form of the dispersion law (37) is greatly changed on going though a certain value of the wave vector

$$k_0\sim(\rho_s{}^B-\rho_{s1}{}^A)(\rho_s{}^B-\rho_{s2}{}^A)/\rho_s\nu,$$

which is determined by the equality of the two terms in the denominator of (37). At small $k<k_0$, the first term in the denominator is smaller than the second, and the dispersion law takes the form

$$(\mathrm{Re}\,\omega)^2=\frac{\rho_{s1}{}^A+\rho_s{}^B}{(\rho_s{}^B-\rho_{s1}{}^A)(\rho_s{}^B-\rho_{s2}{}^A)}\alpha_{ij}k_ik_jk,$$

$$\mathrm{Im}\,\omega=\frac{\rho_{s1}{}^A+\rho_s{}^B}{(\rho_s{}^B-\rho_{s1}{}^A)(\rho_s{}^B-\rho_{s2}{}^A)}(\eta^A+\eta^B)k^2. \tag{38}$$

Such a dispersion law should be obtained also as $\nu\to0$ for all wave vectors, i.e., when there are no effects whatever connected with the surface mass. Oscillations set in only under the condition that $\rho_s^B\neq\rho_s^A$. Note that the relation (38) contains a dependence of the frequency on the wave vector, similar to the dispersion of capillary waves on the interface of two immiscible liquids,[20] but the physical processes, and hence the coefficients of αk^3 are different in (38) and in Ref. 20.

For large $k>k_0$, expression (37) yields a linear dispersion law

$$(\mathrm{Re}\,\omega)^2=\frac{\rho_{s2}{}^A}{\rho_{s1}{}^A}\frac{\rho_{s1}{}^A+\rho_s{}^B}{\rho_{s2}{}^A+\rho_s{}^B}\frac{\alpha_{ij}k_ik_j}{\nu},$$

$$\mathrm{Im}\,\omega=\frac{\rho_{s2}{}^A}{\rho_{s1}{}^A}\frac{\rho_{s1}{}^A+\rho_s{}^B}{\rho_{s2}{}^A+\rho_s{}^B}\frac{(\eta^A+\eta^B)k}{\nu}. \tag{39}$$

Waves described by (38) exist as weakly damped if $\mathrm{Im}\,\omega/\mathrm{Re}\,\omega<1$, and this requires that the wave vector satisfy the condition $k<\rho\alpha/\eta^2$. At $T\sim\Delta$ the damping is weak only at small $k\sim1$ cm^{-1}. Waves in the regions where (39) is valid are never weakly damped at these temperatures, inasmuch as they satisfy in the hydrodynamic regime the condition

$$\mathrm{Im}\,\omega/\mathrm{Re}\,\omega\sim\eta/\nu^{1/2}\alpha\sim\varepsilon_F\tau/\hbar\sim10^3.$$

At $kv_F\tau>1$, however, a ballistic regime sets in. Clearly, the expressions for the real parts (37)–(39) of the spectrum remain exactly the same as before (since they do not contain any dissipative coefficients whatever), and the imaginary parts will be of the correct order of magnitude (with a certain redefinition of the shear-viscosity coefficient). Indeed, $\eta\sim\rho_nv_Fl$, where ρ_n decreases exponentially as $T\to0$, and l (the mean free path) increases exponentially. In the ballistic regime, the quantity l in the estimate of the shear viscosity should be replaced by some temperature-independent characteristic dimension k^{-1} (k is the reciprocal of the wavelength). The effective viscosity in this region is thus $\eta_{\mathrm{eff}}\sim\rho_n\sim\exp(-\Delta/T)$. At a temperature lower by an order of magnitude, the imaginary parts of (38) and (39) is decreased by five orders and we go over to the weak-damping region.

As $T\to0$ we have $\rho_s^A\to\rho_s^B\to\rho_0$, therefore $k_0\to0$ and in the entire range of wave vectors we have the linear dispersion law (39), as predicted in Ref. 6. Assuming that ρ_n^A

$\sim\rho(T/\Delta)^2$ (for a perpendicular orientation plane of the vector l) and $\nu\sim\rho\xi$ we find that the transition from the 3/2 law to the linear law takes place at $k\sim 10\,\text{cm}^{-1}$ if $T\sim 0.1T_c$.

If the coefficient A is allowed to tend formally to zero, this leads to the usual dispersion law for the interface of two immiscible liquids,[20] which was obtained by us for $T=0$ [see Eq. (7) of Ref. 6].

We return now to a discussion of the boundary condition (33). The approach used by us calls for satisfaction of the inequality $(v_n-\zeta)/\zeta\ll 1$. An estimate of the values of K at which this relation holds, when T yields for the reciprocal wavelength k the following expressions:

$$k<KTS^2/2\eta\sim\xi m k_B{}^2/a\rho\hbar^2 v_F\tau^2\sim 10\,\text{cm}^{-1},$$

where k_B is the Boltzmann constant and a is the interatomic distance.

It follows, however, from the discussion of the boundary condition (16) that the Kapitza coefficient increases exponentially with decrease of temperature, in such a way that at $T=0.1T_c$ it is 10^5 times larger than its value at $T=T_c$. Therefore in the temperature region in which one can expect only weakly damped natural modes, we get the ratio $(v_n-\zeta)/\zeta\ll 1$ if $k<10^5\ \text{cm}^{-1}$, i.e., actually for all wave vectors.

The energy dissipation rate $\dot{E}$ on account of the surface transport coefficient K, as follows from (14) and (16), is equal to

$$\dot{E}=TK[S(v_n-\zeta)]^2. \tag{40}$$

Using the ratio of $\dot{E}$ to the total wave energy $E\sim\rho v^2/k$ and the boundary condition (33), we estimate the ratio of the imaginary and real parts of the spectrum:

$$\frac{\dot{E}}{\omega E}\sim\frac{\hbar\rho a v_F{}^6 k^2\tau^3}{k_B{}^2}e^{-\Delta/T}\sim 10^{-4}k^2$$

at $T\sim 0.1T_c$. In this region, the equations for all the damping mechanisms (14) should have an exponential temperature dependence.

The spin degrees of freedom do not alter the dispersion law. As an estimate of the influence of the spin motion on the damping, we note that $\dot{E}\sim D_s(\chi\Omega/\gamma^2c_s)\dot{\theta}^2$, where D_s is the spin-diffusion coefficient, χ the susceptibility, Ω the dipole frequency, and c_s the spin-wave velocity (Ω/c_s is the reciprocal spin-oscillation penetration length). Recognizing that $\theta\sim k\zeta$ in order of magnitude, and that the spin-diffusion coefficient is estimated for $T\sim T_c$ at $D_s\sim v_F^2\tau$, we obtain for the energy dissipation per period [for the dispersion law (39)]

$$\frac{\dot{E}}{\omega E}\sim\frac{D_s\chi}{\gamma^2\rho}\frac{\Omega}{c_s}\frac{k^3}{\omega}\sim\frac{\hbar}{\Delta}\Omega v_F\tau a k^2,$$

which is small compared with the damping due to viscosity and thermal conductivity at all values of k.

2. Passage of sound through the ^{3}He *A-B* phase boundary

Since the bulk densities and velocities of the first sound are equal in the A and B phases of superfluid ^{3}He [accurate to the small quantity $(\Delta/\varepsilon_F)^2$], the first-sound wave passes almost completely through the boundary without reflection at not too low temperatures. It suffices for this purpose, for example, for the growth coefficient A to have the upper bound

$$A<\rho^3/u_1(\rho^B-\rho^A)^2\sim(\rho/u_1)(\varepsilon_F/\Delta)^4,$$

where u_1 is the first-sound velocity. The second sound is reflected and converted into first sound. Of greatest interest to us is the passage of fourth-sound wave, which it lends itself to experimental investigation.[21] Let the interface be located inside a channel in which the conditions for fourth-sound propagation are met, and let its coordinate be $x=0$. The volume is occupied by the A phase on the left ($x<0$) and by the B phase on the right ($x>0$). We assume that the liquid-crystal vector l is fixed by the walls and does not participate in the sound motion. Clearly, all the equations will contain the superfluid-density tensor component parallel to the channel axis. Let a fourth-sound wave be incident on the boundary from the A-phase side. We have then for the phases φ_1 of the incident, φ_1' of the reflected, and φ_2 of the transmitted waves

$$\begin{aligned}\varphi_1&=\Phi_1\exp\{i\omega(-t+x/u_4{}^A)\},\\ \varphi_1'&=\Phi_1'\exp\{i\omega(-t-x/u_4{}^A)\},\\ \varphi_2&=\Phi_2\exp\{i\omega(-t+x/u_4{}^B)\}.\end{aligned} \tag{41}$$

Here Φ_1, Φ_1', Φ_2 are the amplitudes of the corresponding waves, u_4^A and u_4^B are the velocities of the fourth-sound waves in phases A and B. We write for the interface motion $\zeta=\zeta_0\exp(-i\omega t)$. The velocity of the superfluid component and the chemical potential are expressed in the usual manner in terms of the phase (24). We recognize that in the fourth-sound wave the increments of the chemical potential and of the pressure are connected by the relation

$$\delta p=(u_1{}^2/u_4{}^2)\rho_s\delta\mu. \tag{42}$$

To calculate the amplitude reflection coefficient $Z=\Phi_1'/\Phi_1$ we must use the conservation laws for the material (3) and the momentum1) (4), and also the boundary condition (21), which is equivalent to the energy conservation law. Differentiating (24) and (42) and substituting in (3), (4), and (21), we get

$$(\rho_s{}^A/u_4{}^A)(\Phi_1-\Phi_1')+\rho^A\zeta_0=(\rho_s{}^B/u_4{}^B)\Phi_2+\rho^B\zeta_0, \tag{43a}$$

$$(\rho_s{}^A u_1{}^{A2}/u_4{}^{A2})(\Phi_1+\Phi_1')=(\rho_s{}^B u_1{}^{B2}/u_4{}^{B2})\Phi_2, \tag{43b}$$

$$\frac{\rho_s{}^B}{u_4{}^B}\Phi_2+\rho^B\zeta_0=A\left[i\omega\frac{\nu}{\rho_s{}^A\rho_s{}^B}\left(\frac{\rho_s{}^B}{u_4{}^B}\Phi_2+\rho^B\zeta_0\right)-(\Phi_1+\Phi_1'-\Phi_2)\right]. \tag{43c}$$

Since, as noted above, $\rho^A=\rho^B$, the variable ζ_0 is eliminated from (43a) and (43b) and these two relations suffice to calculate the coefficient Z. Expression (43c) determines next the amplitude of the interface. Simple calculations yield

$$Z=(u_4{}^A u_1{}^{B2}-u_4{}^B u_1{}^{A2})/(u_4{}^A u_1{}^{B2}+u_4{}^B u_1{}^{A2}). \tag{44}$$

We can simplify (44) even more. We recognize also for this purpose that $u_1^A=u_1^B$, and that the fourth-sound velocities are expressed in terms of the velocities u_1 and u_2 of the first and second sounds by the relation

$$u_4{}^2=(\rho_s/\rho)u_1{}^2+(\rho_n/\rho)u_2{}^2.$$

We have $u_2\ll u_1$ in the entire temperature interval. Therefore, with good accuracy, $u_4=(\rho_s/\rho)^{1/2}u_1$. Equation (44) takes then the form

$$Z=[(\rho_s{}^A)^{1/2}-(\rho_s{}^B)^{1/2}]/[(\rho_s{}^A)^{1/2}+(\rho_s{}^B)^{1/2}]. \tag{45}$$

In this approximation, the boundary moves in such a way that the normal and superfluid components flow through the boundary independently, and the total flow through the boundary is zero. The amplitude of the boundary oscillations is $\zeta_0 = (\rho_s^B/\rho)(\Phi_2/u_4)$ [from Eq. (43c)]. The result is independent of the surface mass and of the coefficient A. The sound-wave intensity reflection coefficient is equal to Z^2. Since there are no energy losses on the boundary in the considered approximation, the sum of the squares of the reflection and transmission coefficients is equal to unity.

In fact, the periodic motion of the normal component through the interface is accompanied by periodic emission and absorption of heat on the boundary itself. In addition, owing to the boundary condition (33), a finite temperature jump should take place on the *A-B* boundary. This leads to energy dissipation, first on account of the onset of a damped temperature wave in the volumes of the phases *A* and *B*, and second on account of the surface dissipation coefficient K. The dissipation $Q_{\text{dis}}^{(1)}$ due to bulk absorption the penetration depth q_T of the temperature wave, and the amplitude δT of the temperature oscillations, are estimated as follows:

$$Q_{dis}^{(1)} \sim \int \frac{\varkappa(\nabla\delta T)^2}{T}dx, \qquad q_T \sim \frac{\varkappa}{C_v\omega}, \qquad \delta T \sim \frac{Q_0}{(\omega\varkappa C_v)^{1/2}},$$

where $Q_0 \sim T(S^A - S^B)\zeta$ is the heat released on the surface and C_v is the heat capacity of ^{3}He. For the ratio of the dissipation $Q_{\text{dis}}^{(1)}$ to the incident energy flux $Q \sim u_4\rho_s v_s^2$ we have

$$\frac{Q_{dis}^{(1)}}{Q} \sim \frac{T(S^A-S^B)^2\dot{\zeta}^2}{\rho u_4(\varkappa C_v\omega)^{1/2}v_s^2} \sim \frac{C_v k_B T}{\rho v_F^2(\omega\tau)^{1/2}} \sim 10^{-3}\omega^{-1/2}$$

at $T \sim \Delta$. This ratio is small at all reasonable values of ω.

An estimate of the dissipation $Q_{\text{dis}}^{(2)}$ due to the finite Kapitza jump yields $Q_{\text{dis}}^{(2)} \sim KTS^{B2}\zeta^2$. The ratio of the dissipation to the incident energy flux Q is then

$$\frac{Q_{dis}^{(2)}}{Q} \sim \frac{KTS^{B2}\dot{\zeta}^2}{\rho u_4 v_s^2} \sim \frac{C_v\xi k_B T}{\rho v_F^3\tau} \sim 10^{-7}$$

at $T \sim \Delta$, which is negligibly small. The energy dissipation for the latter mechanism increases exponentially with K at low temperatures and can reach values of the order of unity at very low temperatures. Apparently, however, in this region it is meaningless to speak of hydrodynamic fourth-sound waves.

IV. CONCLUSION

The boundary conditions on the *A-B* phase boundary of superfluid ^{3}He were obtained on the basis of conservation laws and the second law of thermodynamics. The resultant transport coefficients can be arbitrarily divided into two classes, viz., "internal" surface coefficients that determine the connection between the surface generalized forces and the surface flows, and "external" coefficients that determine the connection between the bulk generalized forces and the flows through the surface. Off-diagonal cross kinetic coefficients are also possible. As explained in the text, the quantities of the first class should be neglected in the region where bulk hydrodynamics holds. With decrease of temperature, however, the bulk normal excitations fade away, since the density of the surface normal component remains finite even at zero temperature.[13] The surface hydrodynamics should include in this region the eighth and ninth transport coefficients from (14).

We have calculated the spectrum of the oscillations of the *A-B* phase boundary and the passage of fourth sound through this boundary. The natural oscillations of the boundary can apparently excited by thermal pulses. At $T \sim T_c$, unfortunately, these oscillations are strongly damped. On going through the ballistic region, however, the imaginary part of the spectrum should decrease exponentially with decrease of temperature. Estimates give grounds for hoping to observe weakly damped oscillations at $T \sim 0.1T_c$. (This temperature can be reached for the *A-B* phase boundary of ^{3}He if a magnetic field $\sim 10^4$ G is applied.) Surface absorption of fourth sound, on the contrary, is small at $T \sim T_c$. An investigation of the fourth-found reflection and transmission coefficients in this region is therefore an interesting experimental problem.

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[1)] Strictly speaking, account should be taken in the momentum-flux conservation law, of the viscous terms in the momentum-flux tensor. They are, however, small. In fact, the off-diagonal component $\pi_{yx} \sim \eta v_{nx}/a$($a$ is the channel thickness), $v_{nx} \sim (a/q^2)^2 v_{sx}$,$q^2 \sim \eta/\rho\omega$ is the viscous-wave penetration depth, and $\delta p \sim \rho\delta\mu \sim \rho(\omega/k)v_{sx}$. Therefore $\pi_{xy}/\delta p \sim ka$ is less than unity for the fourth-sound wave.

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