The "gas" approximation in the nonlinear stability theory for Korteweg–de Vries solitons

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We consider the nonlinear stability of Korteweg–de Vries solitons in the long-wavelength approximation. We show that the well known Kadomtsev–Petviashvili equation when applied to this problem reduces to the equations of a perfect and, moreover, a monatomic (!) gas and after that to a simple Laplace equation describing the self-focusing of solitons in the nonlinear stage.

1. The aim of the present paper is the study of non-linear perturbations of Korteweg–de Vries (KdV) solitons. The KdV equation

$$u_t+uu_x=sc_0l^2u_{xxx},$$

where $s=\pm 1$ is the sign of the dispersion, c_0 the sound speed, and l the dispersion length, has an exact solution in the form of a KdV soliton:

$$u_1=-su_0\,\mathrm{ch}^{-2}\theta,\quad \theta=(x-ct)/\Delta, \tag{1}$$

$$c=-su_0/3,\quad \Delta=l(12c_0/u_0)^{1/2},$$

and it is well known that perturbations of these solitons are described by the Kadomtsev–Petviashvili (KP) equation

$$(u_t+uu_x-sc_0l^2u_{xxx})_x=-{}^1/_2c_0(u_{yy}+u_{zz}), \tag{2}$$

in the derivation of which one assumes that the inequality $k_{y,z}\ll k_x$ is satisfied. We note that this equation is useful not only for the analysis of the stability of the solitons (1) but is also of great interest in its own right as it is completely integrable in the two-dimensional case (see Ref. 1) by the inverse scattering (IS) method for all initial conditions.

The linear stability theory of the solitons, based on a linearization of the type $u=u_1+\delta u$, was considered in Refs. 2–5. In Refs. 2–4 the "linear long-wavelength approximation" ($k_{y,z}\to 0$) was obtained, while in Ref. 2 a two-dimensional medium was studied, in Ref. 3 a three-dimensional medium, and in Ref. 4 the case of a three-dimensional anisotropic medium.

In particular, in Ref. 2 a dispersion law was found for small perturbations

$$\omega_{\mathrm{KP}}^2=-{}^2/_9sc_0u_0k_y^{\,2},$$

showing the instability of the soliton (1) in a medium with positive dispersion, $s=+1$. Zakharov[5] used the IS method to solve the linearized KP Eq. (2) exactly; this enabled him to find the exact dispersion law

$$\omega^2=-|\,\omega_{\mathrm{KP}}^2\,|(1-k_y\Delta^2/6^{1/2}l)$$

(for $s=+1$) which for a sufficiently small soliton amplitude ($u_0\ll c_0$) describes among other things also the region of maximum growth rate for small perturbations.

The aim of the present paper is, however, a study of large, i.e., nonlinear, perturbations. We note in this connection that the "chains" of two-dimensional solitons discovered in Refs. 6–8 and whose linear stability theory was considered in Ref. 9 can also be particular solutions of Eq. (2). On the basis of the general formulae of Ref. 5 one can, in particular, consider the example of an unstable (for $s=+1$) soliton (1), from which the above-described chain splits off, so that the nonlinear perturbations of the solitons (1) can have very diverse shapes each of which must be considered separately if we are interested in the details of their behavior in the region of the maximum growth rate. This example shows the impossibility of a general description of nonlinear perturbations with arbitrary $k_\perp$. An attempt at such a general consideration on the basis of a variational principle with a Lagrangian was made in Refs. 10 and 11, but it was shown in Ref. 12 that the method used in Refs. 10 and 11 does not allow one correctly to describe the maximum growth rate region.

Only far from that region, when $k_\perp\ll k_\perp^{\max}$, does it turn out that a general nonlinear consideration is nonetheless possible, as we shall show in what follows. In fact, we shall show on the basis of Whitham's method,[13] which was also used in Refs. 10 and 11, that in the "long-wavelength nonlinear wave approximation" perturbations of the KdV solitons (1) behave as clusters of a perfect and, moreover, monatomic (!) gas with an adiabatic index $\gamma=5/3$, with an effective pressure which is positive in the stable case ($s=-1$) and negative in the unstable case ($s=+1$). In the unstable case $s=+1$ these gas equations reduce simply to the Laplace equation which can easily be solved (we indicate the simplest two spontaneous solutions). It is curious that although the problem of the stability of KdV solitons may be assumed to have been well studied by now, nonetheless this useful and obvious gas analogy had not been noted before in other studies.

We note also that it is shown in Ref. 14, which is essentially a continuation of the present paper, that a similar gas approximation is applicable also to other kinds of solitons (e.g., two-dimensional KP solitons, cnoidal KdV waves, nonlinear Schrödinger equation (NSE) solitons, and sine-Gordon equation solitons), while in Ref. 15 many other (about 20) examples are given of similar quasi-gas media which make a simple unified description possible.

2. One of the equations in which we are interested can be easily gotten from simple obvious considerations by following the linear approach shown in Ref. 16. To do this it is sufficient to take into account that the total soliton energy is conserved when its front is twisted, and since the instantaneous energy density equals $u_0^2\Delta\sim u_0^{3/2}$, the following equation must hold:

$$\partial u_0^{3/2}/\partial t+\mathrm{div}(u_0^{3/2}\mathbf{v}_\perp)=0, \tag{3}$$

where $\mathbf{v}_\perp$ are the transverse components of the group velocity, for which we can, in turn, obtain approximate equations

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of the geometrical-optics type.

For a more rigorous derivation we note that an oblique soliton

$$u_1 = \frac{-su_0}{\operatorname{ch}^2\theta}, \quad \theta = \frac{x-x^0}{\Delta}, \quad x^0 = x^0(t,y,z) = c_1 t - \alpha y - \beta z, \tag{4}$$

where the constants u_0, Δ, α, β, c_1 are connected through the relations

$$c_1 = -{}^1/_3 su_0 + {}^1/_2 c_0(\alpha^2+\beta^2), \quad \Delta = l(12c_0/u_0)^{1/2} \tag{5}$$

is a particular solution of the KP equations. We note that this solution is exact and α and β in it are arbitrary but, since in the KP equations themselves we assume that $k_\perp \ll k_\parallel$, we must assume also in (4) that $\alpha, \beta \ll 1$. In what follows, considering the twisting of the front to be small, we shall assume that the function $x^0(t,y,z)$ is arbitrary and that the parameters u_0 and Δ also are no longer constant, as in (4), but are functions which depend weakly on t, y, and z, while the following inequalities hold:

$$\Delta_t \ll x_t^0, \quad \Delta_y \ll x_y^0, \quad \Delta_z \ll x_z^0.$$

Under these conditions the approximate nonlinear equations for the soliton perturbations can be obtained by a method which is close to Whitham's method,[13] which considered not a single soliton but modulated wave packets.

With this in view we introduce the KP [Eq. (2)] a potential φ, putting $u = \varphi_x$ and rewriting (2) in the form

$$\varphi_{tx} + \varphi_x\varphi_{xx} - sc_0 l^2 \varphi_{xxxx} + {}^1/_2 c_0(\varphi_{yy}+\varphi_{zz}) = 0. \tag{6}$$

In this form the equation can be obtained from the Lagrangian

$$\delta \int \mathscr{L}\, dx\, dy\, dz\, dt = 0,$$

$$\mathscr{L} = \varphi_t\varphi_x + {}^1/_3\varphi_x^3 + sc_0 l^2 \varphi_{xx}^2 + {}^1/_2 c_0(\nabla_\perp\varphi)^2, \tag{7}$$

in which we substitute as the minimizing test function the potential

$$\varphi_1 = -su_0(t,y,z)\Delta(t,y,z)\operatorname{th}\theta, \quad \theta = (x-x^0)/\Delta, \tag{8}$$

corresponding to the particular solution (4). In this case, taking into account the previously indicated inequalities, we shall retain derivatives of only the function $x^0(t,y,z)$; this leads to the result

$$\mathscr{L} \approx \frac{u_0^2}{\operatorname{ch}^4\theta}\left[-x_t^0 + \frac{c_0}{2}(\nabla_\perp x^0)^2 + \frac{s}{3}u_0\left(1 - \frac{2}{\operatorname{ch}^2\theta}\right)\right]. \tag{9}$$

Moreover, by analogy with Whitham's method, we evaluate the integral

$$L = \int_{-\infty}^{+\infty} \mathscr{L}\, dx$$

$$= 8l\left(\frac{c_0}{3}\right)^{1/2}\left\{\left[-x_t^0 + \frac{c_0}{2}(\nabla_\perp x^0)^2\right]u_0^{3/2} - \frac{s}{5}u_0^{5/2}\right\}, \tag{10}$$

which one must also consider as a new Lagrangian describing the perturbation of the functions $x^0(t,y,z)$ and $u_0(t,y,z)$ in the soliton.

The Euler-Lagrange equations then lead to the equations

$$-x_t^0 + \frac{c_0}{2}(\nabla_\perp x^0)^2 = \frac{s}{3}u_0, \quad \frac{\partial u_0^{3/2}}{\partial t} = \operatorname{div}(u_0^{3/2} c_0 \nabla_\perp x^0), \tag{11}$$

and if we introduce the notation $\mathbf{v}_\perp = -c_0\nabla_\perp x^0$, we get

$$\frac{\partial}{\partial t}u_0^{3/2} + \operatorname{div}(u_0^{3/2}\mathbf{v}_\perp) = 0, \quad \frac{\partial \mathbf{v}_\perp}{\partial t} + (\mathbf{v}_\perp\nabla)\mathbf{v}_\perp = \frac{s}{3}c_0\nabla_\perp u_0 \tag{12}$$

with the additional condition $\partial v_y/\partial z = \partial v_z/\partial y$ indicating that the flow is potential.

3. If u_{00} is the unperturbed soliton amplitude, it is convenient to introduce the dimensionless instantaneous energy density $\rho = (u_0/u_{00})^{3/2}$ and to rewrite (12) in the form of the gas equations

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\rho\mathbf{v}_\perp = 0, \quad \frac{\partial\mathbf{v}_\perp}{\partial t} + (\mathbf{v}_\perp\nabla)\mathbf{v}_\perp = -\frac{1}{\rho}\nabla_\perp p_{eff}, \tag{13}$$

where the "effective pressure" equals

$$p_{eff} = -sp_{00}\rho^\gamma, \quad \gamma = {}^5/_3, \quad p_{00} = {}^2/_{15}c_0u_{00} > 0. \tag{14}$$

From this it is clear that perturbations of KdV solitons behave as a monatomic (since $\gamma = 5/3$) perfect gas, if the medium has a negative dispersion ($s = -1$) and $p_{\text{eff}} > 0$. In this case we get in the linear approximation the wave equation

$$\partial^2\rho_1/\partial t^2 - c_{00}^2\Delta_\perp\rho_1 = 0, \quad c_{00} = {}^1/_3(2c_0u_{00})^{1/2} \tag{15}$$

for the perturbations $\rho = 1 + \rho_1$, $\rho_1 \ll 1$ which, hence, do not grow. Such solitons are stable but the analogy with the gas shows that when we take into account that the amplitude is finite, effects such as wave breaking (which, however, go beyond the framework of our approximations) are possible here.

In media with a positive dispersion ($s = +1$) solitons are unstable, as has already been established before both in linear and in non-linear approximations (see Refs. 2–5). This instability leads to the soliton[2] self-focusing, which is very simply described by our Eqs. (13). For instance, for an axially symmetric perturbation Eqs. (13) take for $s = +1$ the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{r\,\partial r}r\rho v = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = c_{00}^2\rho^{-1/3}\frac{\partial\rho}{\partial r} \tag{16}$$

and have, in particular, a self-similar simple solution

$$v = \frac{3r}{5\tau}, \quad \rho = \left[\rho_0^{2/3} - 2\left(\frac{r}{5\tau c_{00}}\right)^2\right]^{3/2}, \quad \rho_0 = \left|\frac{\tau_*}{\tau}\right|^{6/5}, \tag{17}$$

where $\rho_0(\tau)$ is the density on the $r = 0$ axis; $\tau = t - t_0 < 0$, and τ_* is an additional parameter. Here $-\infty < t < t_0$ and at time $t = t_0$ ($\tau \to -0$) the soliton is self-focused at the point $r = 0$.

4. Of great interest is the possibility to obtain a general solution in the case of a two-dimensional medium when, as we have already noted, the KP equation can be integrated completely by means of the IS method. In the most interesting two-dimensional unstable case Eqs. (13) take the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial y}\rho v = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = c_{00}^2\rho^{-1/3}\frac{\partial\rho}{\partial y} \tag{18}$$

and can also be completely integrated using the method indicated in Ref. 15 which allows us to reduce the set (18) to a simple Laplace equation.

To do this we perform first the hodograph transformation, introducing the inverse functions $t(\rho,v)$ and $y(\rho,v)$ for which we get from (18) the linear equations

$$y_\rho = vt_\rho + c_{00}^2\rho^{-1/3}t_v, \quad y_v = vt_v - \rho t_\rho, \tag{19}$$

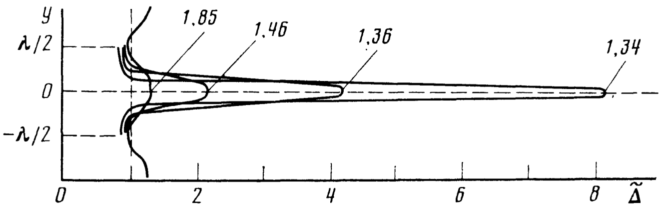

FIG. 1. Evolution of the reduced thickness of a KdV soliton in the self-focusing process. The numbers at the curves show the time in units of $-\gamma t$ for $-\infty < t < 0$.

the compatibility condition of which is an equation for the time

$$\rho^{4/3} t_{\rho\rho} + 2\rho^{1/3} t_\rho + c_{00}^2 t_{vv} = 0, \tag{20}$$

which can also be reduced to the Laplace equation, if we introduce instead of ρ and v new dimensionless variables

$$r = \rho^{1/3}, \quad z = -v/3c_{00}, \tag{21}$$

and replace the time $t(\rho,v)$ by a new function which has to be found, $\psi(r,z) = tr^{3/2}$; this gives

$$\psi_{rr} + \frac{1}{r}\psi_r - \frac{m^2}{r^2}\psi + \psi_{zz} = 0, \quad m = \frac{3}{2}. \tag{22}$$

Finally, we can, for convenience, add to r and z a fictitious "angle" φ and consider r,φ,z as cylindrical coordinates in some "phase" space. For the auxiliary function—the "potential" $\Psi = \psi(r,z)\cos m\varphi$ we get the Laplace equation

$$\Delta\Psi(r, \varphi, z) = 0. \tag{23}$$

We can indicate for it two very interesting solutions if we restrict the consideration to only those spontaneous perturbations which vanish in the limit as $t \to -\infty$ so that $\rho = 1$, $v = 0$. This means that the "potential" must be produced only by "charges" which are positioned on the circle of unit radius $r = 1$, $z = 0$ and the Laplace equation must thus be solved in toroidal coordinates ξ,φ,η introduced through the relations

$$r = \sigma \operatorname{sh}\xi, \quad z = \sigma \sin\eta, \quad \sigma = (\operatorname{ch}\xi + \cos\eta)^{-1},$$
$$(dr)^2 + (dz)^2 = (\sigma d\xi)^2 + (\sigma d\eta)^2. \tag{24}$$

The general solution of the Laplace equation can then be written in the form of a series of associated Legendre functions of the second kind, Q_1^n:

$$\Psi = \frac{\cos m\varphi}{r^{1/2}} \sum_{n=0}^{\infty} a_n Q_1^n(\alpha)\cos(n\eta + f_n), \quad \alpha = \operatorname{cth}\xi, \tag{25}$$

each term of which corresponds to a definite multipole.

Of most interest, however, are only the first two terms, the Coulomb and the dipole terms, which lead to the two simplest possible solutions for the time:

$$\gamma t = -r^{-2}Q_1(\alpha), \quad \gamma t = r^{-2}Q_1^1(\alpha)\cos(\eta + f_1). \tag{26}$$

One can check that the Coulomb solution corresponds to a soliton perturbation that is periodic in the y-coordinate, whereas the dipole solution corresponds to an isolated soliton perturbation localized in the y-coordinate. In the dipole solution we can, through choice of a phase f_1 corresponding to a definite orientation of the dipole, obtain three interesting cases: a "hump", a "well", and, finally, a "doublet" in the form of a "hump + well" combination.

In particular, for the Coulomb case of (26) we find, using (19),

$$\tilde{\Delta} = \frac{\Delta}{\Delta_0} = \frac{1}{r} = \frac{\operatorname{ch}\xi + \cos\eta}{\operatorname{sh}\xi},$$

$$\frac{v}{c_{00}} = -3z = \frac{-3\sin\eta}{\operatorname{ch}\xi + \cos\eta},$$

$$\gamma t = -\tilde{\Delta}^2 Q_1, \quad ky = -\eta + \frac{\sin\eta}{\sin\xi}(3\tilde{\Delta}Q_1 - \xi),$$

$$Q_1 = \xi \operatorname{cth}\xi - 1, \tag{27}$$

where $\gamma = kc_{00}$, $k = 2\pi/\lambda$, λ is the wavelength (the only parameter in the problem). In the linear approximation we have from (27)

$$\tilde{\Delta} = 1 + 2e^{\gamma t}\cos ky,$$

and in the nonlinear case the time evolution of the reduced soliton thickness $\Delta(t,y)$ is shown in Fig. 1. We must note that the similar problem of the self-focusing of a wave train of cnoidal waves, which are also solutions of the KP equation, had been considered earlier in Ref. 17, but was described by different equations.

5. In conclusion we emphasize that our gas equations (13) contain only the velocity $\mathbf{v}_\perp$ which is at right angles to the direction of propagation of the soliton (1) and in this way it differs from the "general-geometric ray" equations of Ref. 18, which contain also the component $v_\parallel$ along the normal to the soliton front. We note also that the gas equations (13) can be obtained in the particular two-dimensional case (t,x,y) from the equations of Ref. 10 if in the latter we drop seven terms with derivatives $\Delta_{t,y,z}$, the retention of which was shown in Ref. 12 to lead to inaccurate results in the maximum growth-rate region.

Although our equations, generally speaking, do not describe the maximum growth-rate region, instabilities of a similar type are known, for instance, the Buneman instability, where this region is suppressed by other factors, neglected by us, in particular, by quasilinear effects. We also note a recent paper[19] where the collapse of Langmuir waves described by a NSE is also considered in the long-wavelength approximation. We have used it in Ref. 14 to analyze the stability of NSE solitons.

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