The involvement of a passive mode in turbulent motion

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We indicate the possibility of the appearance of turbulent flows in bounded volumes after two Hopf bifurcations, as a result of the simultaneous action of active modes on a passive mode. We study the structure of the strange attractor formed in the phase space of the system. We give the results of numerical experiments confirming the realizability of the given scenario for a transition to turbulence and we find analytical estimates for the parameters of the model for which a transition to chaos may occur.

1. In the analysis of the features of the appearance of turbulent flows in bounded volumes two scenarios proposed by Landau[1] and by Ruelle and Takens[2] have mainly been considered. According to Ref. 1 turbulent motion is the limiting case of a quasi-periodic motion and arises as the result of successive self-excitation of an infinite number of modes of the system. However, from the results of studies given in Ref. 2 it follows that such a development of turbulence is not typical for a general class of systems. The mutual influence of the modes can lead to the destruction of quasi-periodic motion that sets in after a finite number of Hopf bifurcations and to the formation of a strange attractor in the phase space of the system.

One may take it as experimentally established that, in a transition to turbulence in bounded volumes, one observes in the spectrum of the pre-turbulence state of the system at least two discrete spectral components[3,4] corresponding to the excitation of two modes of the system. The problem of the further path followed by the transition to chaos has not yet a unique solution and requires a detailed study[5] both theoretically and experimentally. The Ruelle-Takens scenario assumes that the turbulent motion arises not earlier than after the third Hopf bifurcation.[6] We show in the present paper that turbulence may develop already after the second of such bifurcations. The mechanism for the transition to chaos may in that case be due to the appearance in the turbulent motion of a passive mode[1] that is far from the threshold for self-excitation.

2. Let two Hopf bifurcations establish in the general case a quasi-periodic motion with two close frequencies ω_1 and ω_2; to be specific, let $\omega_1 < \omega_2$. We choose a passive mode with eigenfrequency ω_0 close to the frequencies ω_1, ω_2 and consider the effect of the active modes on its behavior. We shall then neglect the reaction the passive mode on the active ones and also the mutual influence of the active modes. We note that an assumption similar to the last one was made in Ref. 7 in the study of the transition to turbulence during the action of an infinite number of self-excited modes on an active mode.

Under quite general conditions the change in time of the dimensionless amplitude A and of the phase φ of the passive mode can be described in the framework of the assumptions made here by the following set of averaged equations:[7]

$$\begin{aligned} dA/d\tau &= -\alpha A + B_1 \sin(\varphi+\Omega\tau) + B_2 \sin(\varphi-\Omega\tau), \\ A\,d\varphi/d\tau &= -\Delta A + \beta A^3 + B_1 \cos(\varphi+\Omega\tau) + B_2 \cos(\varphi-\Omega\tau). \end{aligned} \tag{1}$$

We have used here the following dimensionless quantities: τ is the "slow" time, α a parameter characterizing the dissipation in the system, β a parameter of the non-isochronism of the passive-mode oscillations, $\Delta = (\omega - \omega_0)/\omega$ the frequency mismatch parameter, $\omega = (\omega_1 + \omega_2)/2$, $\Omega = (\omega_2 - \omega_1)/2\omega$, B_1 and B_2 the dimensionless amplitudes of the active modes with respective frequencies ω_1 and ω_2. We have written down Eqs. (1) assuming for the sake of argument a cubic non-linearity of the medium. Since we assume that the passive mode is far from its threshold for a Hopf bifurcation, we confine ourselves in the equation for the amplitude to the linear term describing the dissipation in the system; its non-linearity appears only in the non-isochronism of the oscillations.

In the general case the set of Eqs. (1) was studied numerically. As the criterion for the transition of the passive mode to chaotic motion we used the maximum characteristic Lyapunov exponent (Ref. 8, Ch. 5) that determines the local instability of the phase trajectories. Moreover, we calculated the power spectra, the phase trajectories, the time realizations, and the Poincaré mappings.

3. Our studies showed that a necessary condition for the stochastization of the oscillations is the presence of a singular saddle point in the phase plane of the set (1) when one excited mode acts on the passive mode. For instance, when $B_2 = 0$ this is possible when the following inequalities are satisfied:

$$B_1 > \{2(\Delta-\Omega)\,[\alpha^2 + ((\Delta-\Omega)/3]^2]/3\beta\}^{1/2}, \tag{2}$$

$$\Delta-\Omega > 3^{1/2}\alpha. \tag{3}$$

These conditions indicate that the amplitude of one of the active modes must exceed a minimum value and then there exists a well defined upper bound on the magnitude of the dissipation in the system. The stochastization of the oscillations when a second mode is excited is caused by the fact that a homoclinic structure is formed for well defined values of Δ and Ω in the vicinity of the hyperbolic singular point (Ref. 8, Ch.7).

We consider the nature of the transition of the system to a turbulent motion when the amplitude of the second mode increases. It is well known that when the characteristic bifurcation parameter of the system R (e.g., the Reynolds or Rayleigh number) increases the amplitude of the oscillations of the excited mode increases in proportion to $(R - R_{cr})^{1/2}$, where R_{cr} is the value of the parameter R for which a Hopf bifurcation occurs.[6] In the vicinity of the second Hopf bifurcation, of most importance is the change in the amplitude of the second mode, and the amplitude of the

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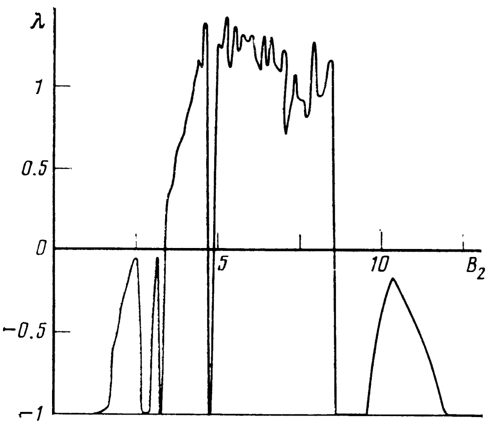

FIG. 1. The Lyapunov exponent λ as function of the amplitude B_2 of the active mode for $\alpha = 1, \beta = 1, \Delta = 11, \Omega = 4.9, B_1 = 6$.

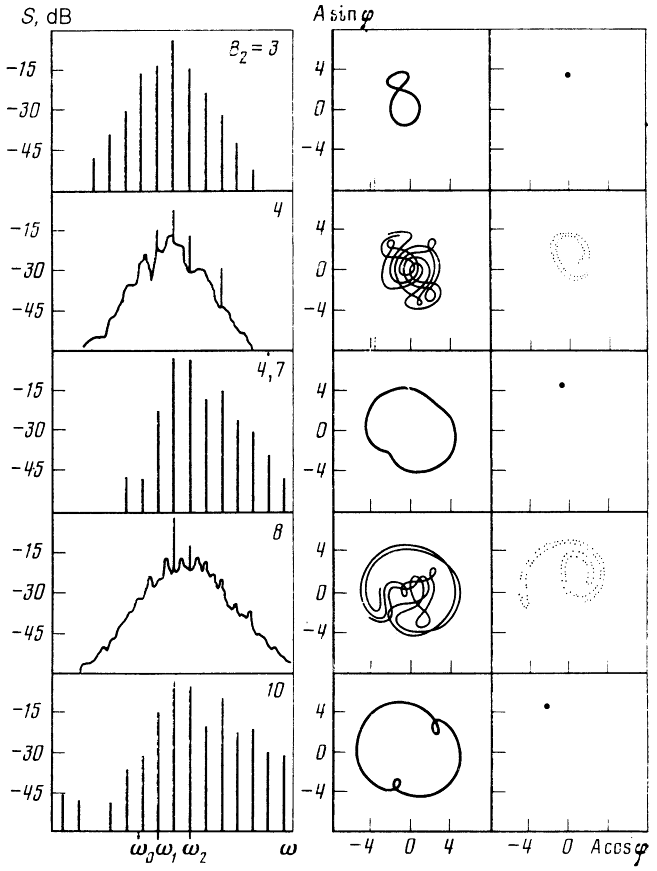

FIG. 2. Power spectra, phase trajectories and Poincaré mappings for different values of the amplitude B_2 for the system parameters corresponding to Fig. 1.

first active mode can then be assumed to be approximately constant. Typical results of the evaluations of the maximum characteristic Lyapunov exponent λ of the amplitude B_2 are given for that case in Fig. 1. In carrying out the calculations the value of α was specified to be equal to unity, which corresponds to normalizing all the parameters of Eqs. (1) to the parameter determining the dissipation in the system with an appropriate change in the time scale. The parameters of the passive mode and of the first active mode were chosen starting from the conditions (2), (3). Starting from some values of the amplitude B_2 one observes in the system under well defined conditions the excitation of stochastic oscillations. The transition from the regular oscillations to the stochastic ones and inversely then has a "rigid" character as one observes it for comparatively small relative changes in B_2. In the stochasticity region the value of the maximum Lyapunov exponent $\lambda \gtrsim 1$, i.e., the characteristic time for the dispersal of trajectories at the strange attractor of the system, is comparable to its characteristic relaxation time. The numerical studies made here did not exhibit any appreciable change in the behavior of the system when the initial values of the amplitude A and of the phase φ within a wide range of changes of the parameters, which enables us to make assumptions about the ergodicity of the processes considered.

The main properties of the excited oscillations are illustrated by the power spectra, sections of phase trajectories, and Poincaré mappings, constructed with a period $2\pi/\Omega$, given in Fig. 2 for a number of characteristic points of the function of Fig. 1. In the pre-turbulence state the oscillation spectrum is nearly symmetric and discrete. When B_2 increases there appears a complex spectrum, against whose background is preserved a series of discrete components, as is in general typical of spectra observed in real experiments.[3,4] For sufficiently large amplitudes B_2 the complex spectrum changes to a discrete one with a significant asymmetry relative to the frequencies of the modes considered. One observes here the excitation of more intensive spectral components in the high frequency region. The Poincaré mappings have a Cantor structure and their dimensionality is close to unity,[2] which is characteristic for systems for which an important role in the formation of the oscillation dynamics is played by the dissipation. This is also indicated by the fact that the characteristic time for changes in the amplitude of the passive mode is of the order of unity, i.e., of the order of the characteristic relaxation time of the system.

4. It turns out that by using rather simple qualitative considerations one can determine approximate values for the amplitudes and frequencies of the active modes for which involvement of the passive mode in the turbulent motion is observed. We consider the way the amplitude A_{st} of the stationary states of the passive mode depends on the amplitude of the single-frequency action B acting on it. When condition (3) is satisifed the typical form of this dependence is shown in Fig. 3. The solid line corresponds to stable states of the focus or node type, and the dotted one to saddle-point unstable states. It is well known that two harmonic oscillations with neighboring frequencies ω_1, ω_2 and amplitudes B_1, B_2 can be written as a single oscillation with a central frequency $\omega = (\omega_1 + \omega_2)/2$ and slowly changing amplitude B and phase; the amplitude B changes then with time between the limits $|B_1 - B_2|$ and $B_1 + B_2$. Numerical studies show that the transition to stochastic behavior is realized if the range of change of B is of the same order of magnitude as the range (B_L, B_R) (see Fig. 3) where B_L, B_R can be expressed in terms of the parameters of the model as follows:

$$B_{L,R}=\{6\Delta[\alpha^2+(\Delta/3)^2]\pm2[3(\Delta^2/3-\alpha^2)^3]^{1/2}/9\beta\}^{1/2}, \tag{4}$$

and, hence, the approximate values of the amplitudes of the active modes at which one may observe a transition to turbulent motion are given by the conditions

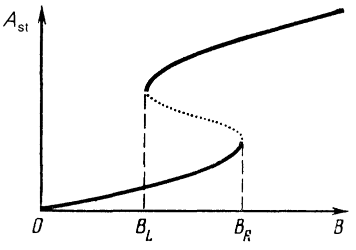

FIG. 3. Typical dependence of the amplitude A_{st} of stationary states of the passive mode on the amplitude from an external single-frequency action.

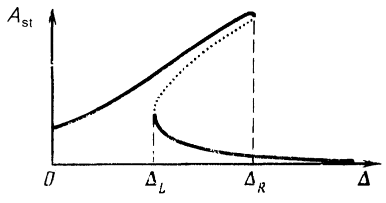

FIG. 5. Typical dependence of the amplitude A_{st} of stationary states of the passive mode on the normalized frequency mismatch Δ of an external single-frequency action.

$$\max(B_1,\ B_2)\approx(B_R+B_L)/2,\quad \min(B_1,\ B_2)\approx(B_R-B_L)/2. \quad (5)$$

These conditions are rather well satisfied in the numerical experiments. For instance, as applied to the results shown in Fig. 1 we see from (5) that the quantity $B_2\approx 8.8$ which agrees as to order of magnitude with the result of the numerical experiment.

We now determine the conditions, which must be satisfied by the frequencies of the active modes, for a chaotization of the oscillations of the passive mode. We show in Fig. 4 the stochasticity region for fixed values of the amplitudes of the active modes in the plane of the parameters $(\Delta - \Omega, \Delta + \Omega)$ which determine the normalized detunings of the frequencies ω_1 and ω_2 from the eigenfrequency of the oscillations of the passive mode ω_0. The numerical determination of the region of the existence of stochastic oscillations was carried out as follows. The $(\Delta - \Omega,\ \Delta + \Omega)$ parameter plane was split into squares of side length 0.3, in the center of each of which the value of the maximum characteristic Lyapunov exponent λ was evaluated. Positive values of λ in Fig. 4 correspond to points in the center of each square. To interpret these results we consider the dependence of the amplitude A_{st} of the stationary states of the passive mode on the detuning Δ when a single mode acts on it. The typical form of this dependence when condition (2) is satisfied is shown in Fig. 5. The values of $\Delta - \Omega = \Delta_L$, Δ_R where

$$\Delta_L\approx 2(\beta B_1^2)^{1/3}, \quad (6)$$

$$\Delta_R\approx\beta(B_1/\alpha)^2, \quad (7)$$

correspond to the presence in the phase plane of the system (1) of a singular saddle-point type point—a point which is unstable with respect to perturbations. The presence of a second active mode acting on the passive one can in this case lead to the "destruction" of the saddle-point type singular point and to stochastication of the oscillations. Numerical

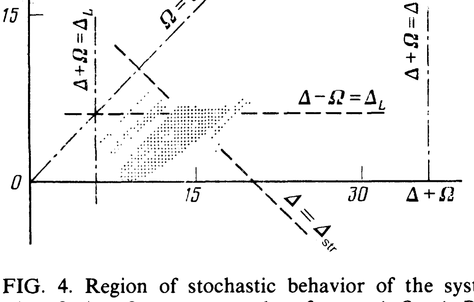

FIG. 4. Region of stochastic behavior of the system (dots) in the $(\Delta - \Omega, \Delta + \Omega)$ parameter plane for $\alpha = 1, \beta = 1, B_1 = 6, B_2 = 6$.

results shown in Fig. 4 show the validity of this assumption. The straight lines $\Delta - \Omega = \Delta_L$ and $\Delta = \Delta_{str}$, where Δ_{str} is the value of the parameter Δ evaluated from (4) and (5), are shown dashed in Fig. 4. The stochasticity region is close to the points where they intersect, i.e., when condition (5) and the condition

$$\Delta-\Omega\approx\Delta_L \quad (8)$$

are simultaneously satisfied.

The analytical conditions (5) and (8) found here enable us to localize with sufficient accuracy the region of the parameters of the dynamical system (1) for which one observes its stochastic behavior; this confirms the systematic numerical experiments which we carried out.

5. Our studies showed thus that already after the second Hopf bifurcation one can observe in systems with a bounded volume a transition to turbulence, which in general does not contradict well-known experimental results.[3,4] The transition to stochastic behavior of the system is caused by the formation in phase space of a strange attractor due to the "mixing" of the oscillations of two excited modes and a passive mode. On the basis of rather simple physical ideas we found analytical conditions establishing the values of the amplitudes and frequencies of the active modes for which one observes a transition to chaos. For the proposed scenario of the transition to a turbulent motion, the dissipation in the system must be less than some critical value determined by the distance between the eigenfrequencies of the modes (see (3)). Hence one can assume that such a scenario is the most probable one for large-scale fluctuations for which dissipation is relatively small,[6] although the process of formation of the stochastic dynamics of the oscillations in the model considered is typical of essentially non-Hamiltonian systems. The stochastization mechanism itself is crude in the sense that (1) it is described by averaged equations; (2) there are not imposed any rigid restrictions whatever on the relations between the frequencies of the interacting modes; (3) there exists a finite multidimensional region of the model parameters where the transition to turbulent motion is observed.

[1)]We call a mode passive when it is in a state before a Hopf bifurcation in contrast to the active modes which have undergone this bifurcation.

[2)]We note that a similar form of mappings was observed in the analysis of the transition to turbulent motion when an infinite number of active modes act upon an active mode[7] in the case of strong dissipation.

[1]L. D. Landau, Dokl. Akad. Nauk SSSR **44**, 339 (1944) [C. R. Acad. Sc. URSS **44**, 311 (1944)].

[2]D. Ruelle and F. Takens, Commun. Math. Phys. **20**, 167 (1971).

[3]J. P. Gollub and H. L. Swinney, Phys. Rev. Lett. **35**, 927 (1975).
[4]H. L. Swinney and J. P. Gollub, Phys. Today **31**, 41 (1978).
[5]A. S. Monin, Usp. Fiz. Nauk **125**, 97 (1978) [Sov. Phys. Usp. **21**, 429 (1978)].
[6]L. D. Landau and E. M. Lifshitz, Gidrodinamika (Hydrodynamics) Nauka, Moscow, 1986, Ch. 3 [English translation published by Pergamon Press].
[7]G. M. Zaslavskiĭ and Kh.-R. Ya. Rachko, Zh. Eksp. Teor. Fiz. **76**, 2052 (1979) [Sov. Phys. JETP **49**, 1039 (1979)].
[8]A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion,* Springer, New York, 1983.

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