Diffractive focusing due to almost-Bragg processess in a medium with periodic inhomogeneities

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We consider wave propagation in a medium with periodic variations in the permittivity, and we calculate changes in the shape of the wave vector surface. Conditions are found under which virtual Bragg rereflections give rise to flattening of this surface, i.e., $k_z = k_0 + Bk_x^4$. We discuss the suppression of diffraction or the focusing effect for propagation in the direction of flattening.

I. INTRODUCTION

It is well known that when the Bragg condition is satisfied in a medium with periodic inhomogeneities, an effective pumping of energy takes place from one wave to another, and Bragg scattering thereby changes the mean direction of energy flow. If the Bragg condition is satisfied approximately, then only virtual scattering takes place, and the relative amplitude of the scattered wave does not increase with distance, but instead remains at a constant (low) level. It is important, however, that even this admixture changes the direction of the Poynting vector, and thus the direction of the group velocity.

For a monochromatic wave, the direction of the group velocity $\mathbf{v}/|\mathbf{v}|$ is normal to the wave vector surface specified by the equation $\Phi(\mathbf{k},\omega) = \text{const}$. In the isotropic two-dimensional case, this equation gives a curvature

$$k_z=(k_0^2-k_x^2)^{1/2}\approx k_0-k_x^2/2k_0+\dots \text{ and } \mathbf{v}/|\mathbf{v}|\equiv\mathbf{k}/|\mathbf{k}|,$$

i.e., the group velocity is directed along the wave vector. According to the uncertainty principle, the spread in wave vectors for a wave packet propagating in the z-direction and bounded in the transverse x-direction is $\Delta k_x \sim 1/\Delta x$. Such a packet will then have a spread in group velocity direction $\Delta\theta_v = \Delta v_x/v = \Delta k_x/k \sim (k_0\Delta x)^{-1}$, resulting in the well known diffractive spreading of the packet.

The focusing of a sound wave in a homogeneous medium is a well known effect in crystal acoustics,[1–4] with the surface formed by the wave vectors containing a point of flattening, $k_z = k_0 + Ak_x^3 + Bk_x^4 + \dots$. When a wave packet propagates in this direction, variations in the group velocity vector are of high order in the small quantity k_x. Diffractive spreading of the packet is then suppressed; this is known as the focusing effect.

In general, the presence of Bragg reflections, even if they are virtual, causes the wave vector and group velocity vector to be noncollinear. In the present paper, we investigate the possibility of obtaining a point of flattening on the wave-vector surface for light waves, thereby realizing the focusing effect, through virtual Bragg reflections from specially prepared volume permittivity gratings.

We give the following physical interpretation of diffractive focusing or the suppression of diffraction (see Fig. 1). A transversely localized wave packet is made up of a set of angular components. When the packet propagates in free space, energy outflow into $x > 0$ during spreading is due to Fourier components of the field with $k_x > 0$.

The virtual reflection of these Fourier components into waves with $\tilde{k}_x < 0$ by the grating 1 results in their being "squeezed back in the z-direction. The same remark also applies to the original components with $k_x < 0$, which lie closer to the Bragg condition for scattering into waves with $\tilde{k} > 0$ by grating 2. The rate of diffractive beam spreading is thus retarded.

The influence of virtual (almost-Bragg) reflections on birefringence and gyrotropy has been examined in Ref. 5. X-ray diffractive focusing under multiwave diffraction conditions has been treated theoretically and detected experimentally by Kohn and coworkers.[6–8] In contrast to the case of current interest, real Bragg scattering processes played an important role in these studies, in addition to virtual processes.

2. DERIVATION OF EQUATION FOR THE WAVE VECTOR SURFACE (DISPERSION RELATION)

We consider a monochromatic wave of frequency ω in the two-dimensional (x,z) space. We start with the Helmholtz wave equation for the scalar field $E(x,z)$ with a spatially inhomogeneous permittivity $\varepsilon(x,z)$, or equivalently, with the wave number $(\omega/c)\cdot(\varepsilon(x,z))^{1/2}$:

$$\frac{\partial^2 E}{\partial x^2}+\frac{\partial^2 E}{\partial z^2}+k^2[1+\alpha(x,z)]E(x,z)=0, \qquad (1)$$

where $\alpha(x,z) = \delta\varepsilon(x,z)/\varepsilon_0$, $\alpha = \alpha^*$, $|\alpha| \ll 1$. The quantity $k = (w/c)\varepsilon_0^{1/2}$ is the wave number in a medium with uniform permittivity $\varepsilon_0 = \varepsilon_0^*$.

We take the perturbation $\alpha(x,z)$ to be a sum of two

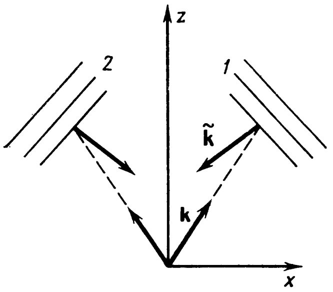

FIG. 1. Schematic representation, for diffractive focusing, of wave vectors $\mathbf{k}$ and $\tilde{\mathbf{k}}$ for an incident wave and virtual reflection and periodic inhomogeneities in a medium.

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sinusoidal gratings:

$$\alpha(x,z)=\sum_{j=1}^{2}\alpha_j\exp(i\mathbf{Q}_j\mathbf{r})+\sum_{j=1}^{2}\alpha_j^*\exp(-i\mathbf{Q}_j\mathbf{r}), \qquad (2)$$

where the grating wave vectors are $\mathbf{Q}_j=(Q_{jx},Q_{jz})$, $\mathbf{r}=(x,z)$. In the zeroth approximation, the wave of interest is of the form

$$E(x,z)=E_0\exp(i\mathbf{pr}), \quad |\mathbf{p}|=k. \qquad (3)$$

In first-order perturbation theory, the scattering of the wave (3) by the inhomogeneities (2) leads to excitation of the virtual waves

$$E(x,z)=E_0\exp(i\mathbf{pr})+\sum_j\left\{E_j^{(+)}\exp[i(\mathbf{p}+\mathbf{Q}_j)\mathbf{r}]+E_j^{(-)}\exp[i(\mathbf{p}-\mathbf{Q}_j)\mathbf{r}]\right\}. \qquad (4)$$

From Eq. (1), we then have

$$E_j^{(+)}=\alpha_j\frac{k^2E_0}{(\mathbf{Q}_j+\mathbf{p})^2-k^2}, \quad E_j^{(-)}=\alpha_j^*\frac{k^2E_0}{(\mathbf{Q}_j-\mathbf{p})^2-k^2}. \qquad (5)$$

In next order, the waves (5) scattered by the perturbations (2) give components of the form $\exp[(\mathbf{p}\pm\mathbf{Q}_j\pm\mathbf{Q}_m)\mathbf{r}]$. When $j=m$, there is among these a coherent contribution to the original wave field. Requiring that there be no resonant build-up of the original wave,[9,10] we can derive an improved equation for the wave vector $\mathbf{p}$:

$$\left\{k^2-\mathbf{p}^2+k^4|\alpha|^2\times\sum_{j=1}^{2}\left[\frac{1}{(\mathbf{Q}_j-\mathbf{p})^2-k^2}+\frac{1}{(\mathbf{Q}_j+\mathbf{p})^2-k^2}\right]\right\}E_0\exp(i\mathbf{pr})=0, \qquad (6)$$

where we have assumed $|\alpha_1|=|\alpha_2|$ for definiteness. We now seek a solution of (6), expressing $\mathbf{p}$ in the form

$$\mathbf{p}=(k+\delta p_z)\mathbf{e}_z+p_x\mathbf{e}_x \qquad (7)$$

and assuming $|p_x|\ll k$, i.e., assuming that the beams propagate essentially in the z-direction. We solve Eq. (6) in the sense of finding a value of p_z for given p_x. Let the wave vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ be symmetric relative to the z-direction:

$$|\mathbf{Q}_1|=|\mathbf{Q}_2|=Q=2k(1+\Delta),$$
$$Q_{1z}=Q_{2z}=Q\cos\varphi, \quad Q_{1x}=-Q_{2x}=Q\sin\varphi \qquad (8)$$

(see Fig. 2). An almost-Bragg interaction can take place, in particular, if we simultaneously have $|\Delta|\ll 1$, $\varphi\ll 1$; this is in fact the case we consider. The second denominator in (6) can then be neglected (i.e., the $E^{(+)}$ waves are only weakly excited), and Eq. (6) takes the form

$$-2k\delta p_z-p_x^2+\sum_{j=1}^{2}\left\{k^4|\alpha|^2\Big/\left[4k^2\left(\Delta+\frac{\varphi^2}{2}\right)-2k\delta p_z-4k\varphi_j p_x+p_x^2\right]\right\}=0, \qquad (9)$$

where $\varphi_1=+\varphi$, $\varphi_2=-\varphi$, $\varphi>0$. We have assumed here that $\Delta+\varphi^2/2\neq 0$, so that there is no Bragg scattering when the wave interacts with a periodic medium. In writing (9), we have omitted all-high order terms in φ^2 and Δ and their derivatives, as well as terms of order $(\delta p_z)^2$.

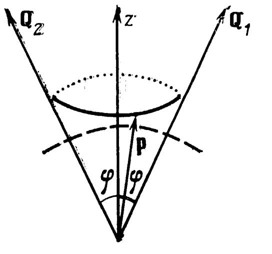

FIG. 2. The dashed line indicates the wave vector surface $p_x^2+p_z^2=k^2$ for a uniform medium, i.e., the Ewald sphere. The wave vector surfaces are modified in a medium with lattices $\exp(\pm i\mathbf{Q}_{1,2}\mathbf{r})$; the solid curve indicates the branch of the dispersion curve containing the point of flattening in the z-direction for the case of interest; the dotted curve shows the other branch of the dispersion curve.

3. SOLUTION OF THE DISPERSION EQUATION

Although we are interested in the dependence of δp_z on p_x, let us first consider Eq. (9) for δp_z with $p_x=0$:

$$-\delta p_z+\frac{1}{2}\frac{k^2|\alpha|^2}{2k(\varphi^2/2+\Delta)-\delta p_z}=0, \qquad (10)$$

the roots of which are

$$(\delta p_z)_{1,2}=k\left(\Delta+\frac{\varphi^2}{2}\right)\left\{1\pm\left[1-\frac{|\alpha|^2}{2(\Delta+\varphi^2/2)^2}\right]^{1/2}\right\}. \qquad (11)$$

We are interested in those solutions which in the absence of a grating ($|\alpha|\to 0$), give $\delta p_z=0$; this corresponds to choosing the minus sign preceding the square root in (11). It can be seen from (11) that when $|(\Delta+\varphi^2/2)|\ll 1$ and $|\alpha|^2\ll 1$ hold, terms involving $(\delta p_z)^2$ in the derivation of Eq. (9) can in fact be neglected. When the grating is strong enough, with $|\alpha|^2>2(\Delta+\varphi^2/2)^2$, Bragg reflection becomes so strong that wave propagation in the z-direction becomes impossible. The quantity δp_z becomes complex, corresponding to entry into the forbidden region. We consider the case $|\alpha|^2<2(\Delta+\varphi^2/2)^2$, i.e., the allowed region.

Starting with the root $(\delta p_z)_2$, we can find the correction for $p_x\neq 0$ from (11) with $p_x=0$. By summetry, δp_z is an even function of p_x. From (9), we have

$$\delta p_z=k\left(\Delta+\frac{\varphi^2}{2}\right)\times\left\{1-\left[1-\frac{|\alpha|^2}{2(\Delta+\varphi^2/2)^2}\right]^{1/2}\right\}-\gamma\frac{p_x^2}{2k}+O(p_x^4),$$

$$\gamma=-\frac{8t[1-(1-Y^2)^{1/2}]^2}{Y^2(1-Y^2)^{1/2}}+\frac{1}{(1-Y^2)^{1/2}}. \qquad (12)$$

We have introduced the following notation in (12) for the dimensionless quantities t and Y.

$$t=\frac{\varphi^2}{\varphi^2+2\Delta}, \quad t>0, \quad Y^2=\frac{|\alpha|^2}{2(\Delta+\varphi^2/2)^2}. \qquad (13)$$

It is important to notice that there are no terms of order p_x^3 in (12) because of the symmetry of the problem. For $|\alpha|\to 0$ and fixed Δ and φ, we find $\gamma\to 1$, which is responsible for the parabolic expansion $\delta p_z \approx -p_x^2/2k$ of the equation for the neighborhood $p_x^2 + p_z^2 = k^2$.

4. SUPPRESSION OF DIFFRACTION

We are interested in the possibility of obtaining a point of flattening on the wave vector surface with $p_x = 0$, corresponding to $\gamma(t,Y) = 0$. Figure 3 shows the plane in t, Y^2 coordinates. Below the solid curve, Bragg scattering is not strong enough to suppress diffraction. The curve correponds to the solution of the equation $\gamma(Y^2,t) = 0$. i.e.,

$$t=Y^2/8[1-(1-Y^2)^{1/2}]^2. \tag{14}$$

Above the solid curve, we have overcompensation, i.e., the x-components of the group velocity and wave vector have opposite signs.The forbidden region corresponds to $Y^2 > 1$ (to the right of the dashed line).

We may also solve the analogous three-dimensional problem. We assume that there are two pairs of gratings (Q_1,Q_2) and (Q_3,Q_4), with wave vectors in the (x,z) and (y,z) planes respectively, and that these are symmetrically oriented about the z-axis; the amplitudes are pairwise equal: $|\alpha_1| = |\alpha_2|, |\alpha_3| = |\alpha_4|$. We also assume that the Bragg detuning is the same for both, $D = \varphi_1^2/2 + \Delta_1 = \varphi_3^2/2 + \Delta_3$. If we then introduce the quantities

$$Y_1^2=|\alpha_1|^2/2D^2,\quad Y_3^2=|\alpha_3|^2/2D^2,\quad t_1=\varphi_1^2/2D,\quad t_3=\varphi_3^2/2D, \tag{15}$$

we obtain for the wave vector surface

$$p_z(\mathbf{p}_\perp)=\text{const}+\frac{1}{2}(\beta_x p_x^2+\beta_y p_y^2),\qquad Y^2=Y_1^2+Y_3^2,$$

$$\beta_x=\frac{8[1-(1-Y^2)^{1/2}]^2 t_1 Y_1^2-Y^4}{kY^4(1-Y^2)^{1/2}}, \tag{16}$$

$$\beta_y=\frac{8[1-(1-Y^2)^{1/2}]^2 t_3 Y_3^2-Y^4}{kY^4(1-Y^2)^{1/2}}.$$

It is interesting to note that the presence of just one pair of gratings (for example, in the (x,z) plane: $Y_1 \neq 0$, $Y_3 = 0$) reduces $|\beta_x|$, i.e., it flattens the intersection of the wave vector surface with the plane $p_y = 0$, but it increases the curvature of its intersection with the orthogonal plane $p_x = 0$. The condition for surface flattening at the point $p_x = p_y = 0$ takes the form $\beta_x = \beta_y = 0$. In the totally symmetric case $t_1 = t_3$, $Y_1 = Y_3$, this condition takes the form

$$t_1=Y_1^2/8[1-(1-2Y_1^2)^{1/2}]^2 \tag{17}$$

(see Fig. 3b).

Let us make some numerical estimates. We take $\varphi \approx 0.28$ rad $\sim 14°$ and $\Delta \approx 10^{-2}$. Then for the two-dimensional problem, $t = 0.8$, $Y^2 \approx 0.5$, and $|\alpha| \approx 5\cdot 10^{-2}$. For beams with a total angular spread of $2\theta_0 \sim \varphi \sim 0.14$ rad, the Fresnel distance is $\lambda/(2\theta_0)^2 \sim 50$ at a wavelength of $\lambda \sim 1$

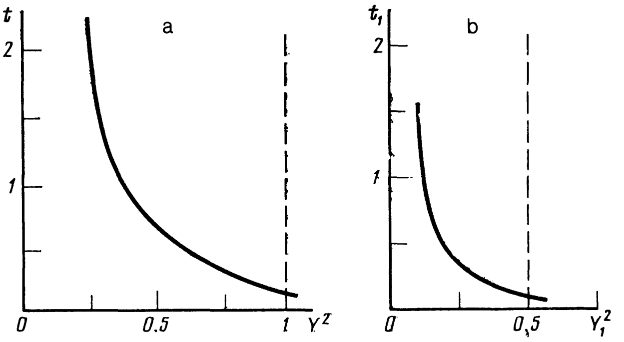

FIG. 3. Plots of parameter values Y,t for which a point of flattening occurs: a) two-dimensional problem; b) three-dimensional problem.

μm. Thus, in the example considered, diffraction suppression shows up at distances greater than or of order 50 μm.

These estimates indicate that near a Bragg resonance, even relatively weak amplitude gratings can markedly suppress diffraction, or give rise to diffractive focusing.

The calculation of fourth-order terms in $\mathbf{p}_\perp$ remains outside the scope of the present discussion. Moreover, calculations have shown that flattening takes place when the amplitude of the virtual waves is of the order of that of the main wave. Under these conditions, an initial incident wave generally excites both of the solutions corresponding to the two eigenvalues $(p_z)_{1,2}$ of (11). This is the basis for believing that diffraction will be enhanced for the second wave, and at sufficiently large distances, the amplitude of the first wave will dominate near the axis.

The temporal analog of the present problem is of great interest, specifically the inhibition of temporal pulse spreading in a single-mode fiber due to virtual Bragg scattering from a specially prepared grating.

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