# Antiferromagnetic spin glass in the Ising model

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The properties of the nonergodic state of a frustrated Ising antiferromagnet (antiferromagnetic spin glass) and the peculiarities of the reentrant transition to this state are investigated. Phase diagrams in a magnetic field are plotted for various degrees of disorder. Distinctive properties of the phase diagram are the rise of the reentrant transition temperature $T_g$ with increase of the magnetic field, and the feasibility of a field or temperature phase transition when long order sets in or vanishes in the nonergodic state. The temperature dependences of the magnetic susceptibility and of the sublattice magnetization are investigated in an approximation linear in $\tau = (T_g - T)/T_g$.

One of the most interesting results of the infinite-radius theory for strongly disordered magnets is the predicted existence of nonergodic systems with long-range magnetic order (called the mixed phase). In this state, the spin-glass order parameter [the Parisi function $q(x)$] and the usual ferro- or antiferromagnetic order parameter are simultaneously nonero.

A temperature transition from an ergodic ferromagnetic state into a nonergodic one was observed and investigated many times (see the reviews by Fischer[1] and Huang[2]). In recent studies[3–13] a similar transition was observed both in Heisenberg and in Ising antiferromagnets. It is shown in Ref. 6 that in the frustrated Ising antiferromagnet $Fe_xMg_{1-x}Cl_2$ the mixed state actually exists: the long-range magnetic-order parameter differs from zero in the nonergodic phase. The question of the existence of a mixed state in frustrated ferromagnets has not yet been answered, possibly because their investigation is made complicated in this case by domain effects.[14,15]

The standard Sherrington-Kirkpatrick model[16] describes transitions into a mixed state only from a ferromagnetic phase (these are customarily called reentrant transitions). In our preceding studies[17,18] we propose a two-sublattice variant of the Sherrington-Kirkpatrick model, which describes transitions not only from the ferromagnetic but also from the antiferromagnetic phase into a mixed state. We shall hereafter call this state antiferromagnetic spin glass (AFSG).

The distinctive nature of frustrated antiferromagnet is manifeted in their behavior in an external field. In a transition from the ferromagnetic phase into ferromagnetic spin glass (FSG), a magnetic field increases the magnetization and suppresses thereby the glass so that the reentrant transition decreases monotonically with increasing field. In AFSG, on the contrary, a magnetic field suppresses both the antiferromagnetic and the spin-glass order parameters. The phase separation boundary has thus a nontrivial dependence on the field. Thus, it was shown in Ref. 17 that for strongly and weakly frustrated antiferromagnets there exists a range of fields in which the temperature $T_g(H)$ of the transition into the nonergodic state increases with the field.

We show in the present paper that the magnetic feld can raise the transition temperature $T_g(H)$ at any degree of frustration of an antiferromagnet.

In Refs. 17 and 18 were investigated the properties of only the ergodic state. We pay here principal attention to the magnetic properties and to the phase diagram in the nonergodic state. Equations of state are derived for the Parisi functions $q_{1,2}(x)$, the sublattice magnetization $m_{1,2}$, and the magnetic-field distribution functions. These equations were solved near the $T_g(H)$ line, making it possible to plot the phase diagram in a nonergodic phase near the multicritical point of intersection of the $T_g(H)$ line and the line $T_N(H)$ of transition to the magnetically ordered state, and find the temperature dependences of the long-range magnetic order and of the susceptibility below $T_g(H)$.

It was found that in the nonergodic state a temperature or magnetic-field transition is possible from the antiferromagnetic phases into a phase without long-range order. At the point of phase transition from the ergodic into the nonergodic state, $m(T)$ is a continuous function ($m(T)$ is the magnetization of the ferromagnet or the moment of the antiferromagnet sublattices): the derivative $dm/dT$ is continuous at the point $T = T_g$.

It is known[19] thae equilibrium susceptibility (i.e., the susseptibility average over all the states of the nonergodic ensemble) of a "pure"[1)] spin glass is independent of temperature down to $T = 0$. This is one of the few Parisi-theory results that can be directly compared with experiment. Numerous experiments show indeed that the equilibrium is independent or almost independent of temperature below $T_g$. The question of the temperature in the mixed phase, FSG or AFSG, remains open. We show in the present paper that in the mixed phase the susceptibility depends on temperature, but weakly. In AFSG the susceptibility decreases with decreasing temperature. In the transition through $T_g$ from AFSG to AF derivative $\partial \ln\chi/\partial \ln T$ increases jumpwise, although the susceptibility depends weakly on the temperature near $T_g$ even above the transition. With further temperature rise, the susceptibility in the AF phase begins to increase rapidly and can have in a strongly frustrated antiferromagnet a maximum below the Néel temperature. The susceptibility in the FSG phase increases weakly with decrease of temperature.

## 1. EQUATIONS OF STATE OF ANTIFERROMAGNETIC SPIN GLASS

In the simplest mode that describes AFSG, the Hamiltonian is of the form

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$$\mathcal{H}=\sum_{i,j}J_{ij}S_{1i}S_{2j}-H\sum_{p=1,2}\sum_{i}S_{pi}, \tag{1}$$

where $S_{pi}$ are the spins of subsystems 1 and 2, $J_{ij}$ are the energies of the exchange interaction of spins of different subsystems; these, as always in the Sherrington-Kirpatrick model,[16] are independent of the distance $r_{ij}$ and have a normal distribution with variance $N^{-1/2}$ and mean value $J_0/N$, where $N$ is the total number of spins in each subsystem. The exchange interaction within a subsystem is neglected.

Using the Parisi parameterization[20,21] for the matrices $q^{\alpha\beta}$ that determine the spin-glass order parameter, we obtain in the Appendix expressions for the free energy $F$ and for the equation of state of antiferromagnetic spin glass:

$$F=-\frac{1}{4T}-\frac{J_0}{2}m_1m_2$$
$$-\frac{1}{4T}\left[\int_0^1 dx\, q_1(x)q_2(x)-q_1(1)-q_2(1)\right]-f_1-f_2. \tag{2}$$

Here $q_{1,2}$ are Parisi functions, and $f_1$ and $f_2$ are equal to

$$f_{1,2}=\tfrac{1}{2}\int dy\,\varphi_{1,2}(0,y)P_{1,2}(0,y), \tag{3}$$

where $\varphi(x,y)$ and $P(x,y)$ satisfy the equations

$$\dot\varphi_{1,2}=-\frac{\dot q_{2,1}}{2}\varphi''_{1,2}-\frac{x}{2T}\dot q_{2,1}(\varphi'_{1,2})^2, \tag{4}$$

$$\dot P_{1,2}=\frac{\dot q_{2,1}}{2}P''_{1,2}-\frac{\dot q_{2,1}}{T}(\varphi'_{1,2}P_{1,2})' \tag{5}$$

with boundary conditions

$$\varphi_{1,2}(1,\,y)=T\ln 2\,\mathrm{ch}(y/T), \tag{6}$$

$$P_{1,2}(0,y)=\frac{1}{(2\pi q_{2,1}(0))^{1/2}}\exp\left[-\frac{(y-(H-J_0m_{2,1}))^2}{2q_{2,1}(0)}\right], \tag{7}$$

and

$$m_{1,2}=\int P_{1,2}(x,y)\varphi'_{1,2}(x,y)\,dy, \tag{8}$$

$$q_{1,2}(x)=\int P_{1,2}(x,y)\,(\varphi'_{1,2}(x,y))^2\,dy. \tag{9}$$

A dot denotes differentiation with respect to $x$, and a prime with respect to $y$. $P_{1,2}(x,y)$ are distribution functions of the molecular fields $y$ for valleys whose overlap is less than $q(x)$.[22]

We shall use frequently in lieu of $\varphi_{1,2}(x,y)$ the function $m_{1,2}(x,y)=\varphi'_{1,2}(x,y)$, given by the equation

$$\dot m_{1,2}(x,y)=-\frac{\dot q_{2,1}}{2}\left(m''_{1,2}(x,y)+\frac{2x}{T}m'_{1,2}(x,y)m(x,y)\right) \tag{10}$$

with the boundary condition

$$m_{1,2}(1,\,y)=\mathrm{th}(y/T). \tag{11}$$

Differentiating (9) with respect and using (8) and (10), we obtain a system of equations for $\dot q_1$ and $\dot q_2$:

$$\dot q_1=\dot q_2\int dy\,P_1(x,y)\,(m_1'(x,y))^2,$$

$$\dot q_2=\dot q_1\int dy\,P_2(x,y)\,(m_2'(x,y))^2.$$

It is seen from this that in the interval $x_0<x<x_1$, where $\dot q_1$ and $\dot q_2$ differ from zero, the following relation should hold:

$$1=\int dy\,P_1(x,y)\,(m_1'(x,y))^2\int dy'\,P_2(x,y')\,(m_2'(x,y'))^2, \tag{12}$$

meaning marginal stability of the Parisi parameteriation at all temperatures.

From this relation, just as in the case of pure spin glass,[22] it follows that at low temperature the AFSG entropy, and with it the heat capacity, is proportional to $T^2$.

Since the functions $m(x,y)$ and $P(x,y)$ are independent of $x$ at $T=T_g$, relation (12) goes over, if (7) and (11) are taken into account, into the equation for the de Alameida-Thouless line

$$T^4=\int dy\,P_1(0,y)\,\mathrm{ch}^{-4}\frac{y}{T}\int dy'\,P_2(0,y')\,\mathrm{ch}^{-4}\frac{y'}{T}, \tag{13}$$

and from (6)–(9) follow at $T=T_g$ the equations of state of a frustrated antiferromagnet, which agree with those obtained in our earlier paper[17] in the replica-symmetric approximation:

$$m_{1,2}=\left\langle\mathrm{th}\frac{y}{T}\right\rangle_{1,2}\qquad q_{1,2}=\left\langle\mathrm{th}^2\frac{y}{T}\right\rangle_{1,2}, \tag{14}$$

where the angle brackets denote averaging over $y$ with the distribution function $P_{1,2}(0,y)$ from (7).

## 2. LONG-RANGE MAGNETIC-ORDER PARAMETER NEAR $T_g$

We check in this section whether the long-range order parameter has a kink at the transition into the nonergodic state.

Near the transition point, at small $\tau=(T_g-T)/T_g\ll 1$, the function $q_{1,2}$ depends on $x$ only in the narrow interal $x_0<x<x_1$, whose width will be shown to be proportional to $\tau$. This allows us to expand the thermodynamic quantities in powers of $\tau$. At small $\tau$ the function $q_{1,2}(x)$ can be regarded as linear in the interval $x_0<x<x_1$, so that

$$q(x)=\begin{cases}q(0), & 0<x<x_0\\ q(0)+a\tau(x-x_0), & x_0<x<x_1.\\ q(0)+a\tau(x_1-x_0), & x_1<x<1\end{cases} \tag{15}$$

We confine ourselves here to the case $H=0$, so that

$$q_1(x,\,y)=q_2(x,\,y)\equiv q(x,\,y),$$

$$m_1(x,\,y)=-m_2(x,\,y)\equiv m(x,\,y).$$

Representing $m(x,y)$ in similar form, we obtain from (10)

$$m(x,y)=\begin{cases}\mathrm{th}\dfrac{y}{T}+\Delta\tau\dfrac{\mathrm{th}(y/T)}{\mathrm{ch}^2(y/T)}, & 0<x<x_0\\ \mathrm{th}\dfrac{y}{T}+\dfrac{\Delta}{x_0-x_1}\tau\dfrac{\mathrm{th}(y/T)}{\mathrm{ch}^2(y/T)}(x-x_1), & x_0<x<x_1,\\ \mathrm{th}\dfrac{y}{T} & x_1<x<1\end{cases} \tag{16}$$

where

$$\tau\Delta=a(1-x_1)(x_1-x_0)T^{-2}.$$

Substituting (15) and (16) in relation (12) at $x=x_0$ and the

self-consistency conditions (8) and (9) for $m$ and $q$ at $x = 0$, and expanding in terms of $\tau$ and $\tau\Delta$,

$$\tau\delta_q=(q(0)-q(0)|_{T=T_g}),\quad \tau\delta_m=(m-m|_{T=T_g}),$$

we obtain a set of equations for the quantities $\Delta,\delta_q$, and $\delta_m$ at $T<T_g$:

$$\delta_m\left(1-\frac{J_0}{T_g}(1-q)\right)+K_{1,2}(T_g^{-2}\delta_q+\Delta)=\frac{J_0}{T_g}m(1-q)-\frac{2}{T_g^2}qK_{1,2}, \tag{17}$$

$$-\delta_m\frac{J_0}{T_g}K_{1,2}+(1-q-T_g^2)(T_g^{-2}\delta_q+\Delta)=\frac{J_0}{T_g}K_{1,2}+3q-2qT_g^{-2}(1-q), \tag{18}$$

$$\delta_m\frac{J_0}{T_g}K_{1,4}+\delta_q\left(\frac{5}{T_g^2}K_{0,6}-4\right)+\Delta(3K_{0,6}-2T_g^2)=T_g^2-\frac{2J_0}{T_g}mK_{1,4}-2q\left(\frac{5}{T_g^2}K_{0,6}-4\right). \tag{19}$$

Here

$$K_{l,n}=\langle \mathrm{th}^l(y/T_g)\,\mathrm{ch}^{-n}(y/T_g)\rangle.$$

The first and second equations are obtained from the self-consistency of (8) and (9), and the third from (12).

The last equation does not apply at $T\geqslant T_g$, and $\Delta=0$. Since $\Delta$ and $\delta_q$ enter the first two equations only in the form of the combination $T_g^{-2}\delta q+\Delta$, it follows that $\delta_m$ is determined both above and below the transition point by one and the same expression. The long-range order parameter has therefore no kink at any $J_0$ at the transition to the mixed phase.

We show in the next section that the susceptibility does have a kink at the transition point. It can apparently be concluded from these two facts that the inverse phase transition, just as the transition from the parmagnetic phase into spin glass,[23] is of third order.

### 3. MAGNETIC PERMEABILITY

A direct consequence of the marginal-stability condition is that the equilibrium magnetic susceptibility of pure spin glass is independent of temperature.[19] The temperature dependence of the susceptibility in the mixed phase has not been studied for either the heretofore uninvestigated antiferromagnetic spin glasses or the ferromagnetic spin glasses.

In the calculation of the susceptibility it is convenient to use Eqs. (8) and (9) taken at the point $x = 0$. Representing (8) and (9) in the form

$$m_{1,2}=\int\frac{dz}{(2\pi)^{1/2}}e^{-z^2/2}m_{1,2}(0,H-J_0m_{2,1}+q_{2,1}^{1/2}(0)z),$$
$$q_{1,2}(0)=\int\frac{dz}{(2\pi)^{1/2}}e^{-z^2/2}m_{1,2}^2(0,H-J_0m_{2,1}+q_{2,1}^{1/2}(0)z), \tag{20}$$

differentiating with respect to $H$, and recognizing that $m_1(0,y)=-m_2(0,y)=m$, $q_1(x)=q_2(x)=q$, $dm_1/dH=dm_2/dH$ and $dq_1/dH=-dq_2/dH$ as $H\,0$, we get

$$\chi=\mathrm{P}/(1+J_0\Pi), \tag{21}$$

$$\mathrm{P}=\langle m'(0,y)\rangle-\frac{\langle m(0,y)m'(0,y)\rangle\langle m''(0,y)\rangle}{1+\langle m'^2(0,y)\rangle+\langle m(0,y)m''(0,y)\rangle}. \tag{22}$$

Equations (21) and (22) are valid in both the ergodic and nonergodic states. We find first the temperature dependence of the susceptibility in an ergodic antiferromagnet, substituting in (22) the solution of replica-symmetric equations (14).

Figure 1 shows a plot of $\chi(T)$ for $J_0=1.2$ and 1.5. It was shown in Ref. 18 that the susceptibility increases with decrease of temperature at $T<T_N$ if $J_0<2^{1/2}$. It is seen from Fig. 1 that in this case the susceptibility has a gently sloping maximum in the ergodic state and decreases with temperature as $T_g$ is approached.

Note that frustrations of the intrasublattice interaction suppress the increase of the susceptibility in a transition to the AF phase. If the intrasublattice interaction is large enough and is strongly frustrated, the susceptibility has at all $J_0$ a maximum at the Néel temperature. We have calculated the susceptibility in the nonergodic state ($T<T_g$) in an approximation linear in $\tau$. It suffices for this purpose to substitute in the expressions for the mean values in (22) the solutions of the system (17)–(19). The derivative $d\chi/dT$ is expressed in terms of various combinations of the quantities $\delta_m$, $\delta_q$, and $\Delta$. Since the last two quantities are discontinous at the point $T=T_g$, the derivative $d\chi/dT$ also has a discontinuity at this point.

Figure 2 shows a plot of $d\ln\chi/d\ln T$ vs. $J_0$ above and below $T_g$. It can be seen that in the mixed state the susceptibility depends on the temperature, in contrast to pure spin glass. But this dependence is weak. On going from the nonergodic to the ergodic state, the derivative $d\ln\chi/d\ln T$ increases abruptly, but remains much less than unity.

A similar calculation for FSG shows that in this case the susceptibility in the nonergodic state depends on temperature. The susceptibility has a kink at $T_g$, but, in contrast to AFSG, in the nonergodic state it increases weakly with decrease of temperature.

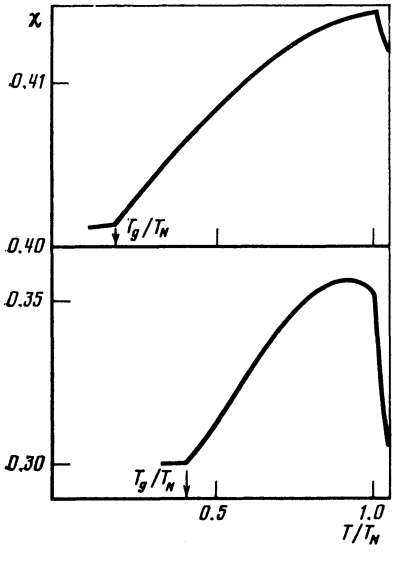


FIG. 1. Magnetic susceptibility vs. temperature at $J_0=1.5$ (upper curve) and $J_0=1.2$ (lower curve).

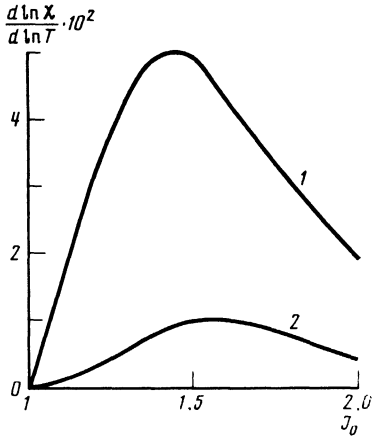


FIG. 2. Dependence of $(d\ln\chi/\partial\ln T)_{T=T_g}$ on $J_0$ in the ergodic ($T\to T_g+0$, curve 1) and nonergodic ($T\to T_g-0$, curve 2) phases.

## 4. PHASE DIAGRAM IN A MAGNETIC FIELD

As noted in the earlier paper,[17] frustration prevents the existence of the AF phase at sufficiently low temperatures, so that the $T_g(H)$ and $T_N(H)$ curves cross on the $(T,H)$ phase plane. The temperature of the transition from the AF phase into the AFSG state has a nontrivial dependence on the magnetic field, since the field suppresses these two phases unequally. It has shown $T_g$ has a nonmonotonic dependence on $H$ at $J_0\gg 1$ and $J_0\ll 1$, and that the temperature $T_g$ increases with the field at the point of intersection of $T_g(H)$ and $T_N(H)$. This analysis of the phase diagram, however, is of course incomplete. First, comparison with experiment requires an analysis at $J_0-1\approx 1$, where $T_N(0)$ and $T_g(0)$ are not too strongly different. At such $J_0$ the phase diagram differs substantially from that given in Ref. 17. Second, no study was made in Ref. 17 of the possibility of transitions between different phases of the nonergodic state. In particular, we do not know how the $T_N(H)$ line is continued into the nonergodic phase, and whether an AFSG-SG transition is produced accompanied by vanishing of the antiferromagnetic order parameter.

Phase diagrams for different values of $J_0$ are shown in Fig. 3. The principal features of the interface between the ergodic and inergodic states are the following:

1) There is always a field region in which $T_g$ increases with $H$. The increase of $T_g$ is stronger the larger $J_0$. At $J_0=3(T_N(0)/T_g(0)\approx 500)$ we have a ratio $T_g(H_0)/T_g(0)\approx 30$ ($H_0$ is the field in which the lines $T_g(H)$ and $T_N(H)$ cross). But also at $J_0=1.7$, when $T_N(0)/T_g(0)=9.4$, the temperature $T_g(H)$ is twice as high as $T_g(0)$. If, however, $J_0-1\ll 1$, then $T_g(H_0)<T_g(0)$.

2) At $J_0-1\ll 1$ and $J_0\gg 1$ the temperature $T_g(H)$ is a nonomotonic function of $H$ if $H<H_0$: initially $T_g(H)$ decreases with increases of the field, and then increases. For intermediate values of $J_0$ corresponding to the interval $T_N(0)/T_g(0)\approx 1.5$–$30$, the temperature $T_g(H)$ increases monotonically with $H$ all the way to $H=H_0$.

3) $T_g(H)$ decreases monotonically with the field at $H>H_0$. The $T_g(H)$ curve has a kink at the point $H=H_0$.

The maximum field at which an antiferromagnetic phase can exist is somewhat higher than $H_0$.

The ratio $H_0/T_N(0)$ has a nonmonotonic dependence on the parameter $J_0$. At $J-1\ll 1$ the field $H_0\approx T_N(0)(J_0-1)^{3/4}\ll T_N(0)$ (Ref. 17). If, however, $J_0\gg 1$ we can obtain for the field $H$ the estimate

$$\frac{H_0}{T_N(0)}=1+\frac{1}{J_0}\left(\ln\frac{2J_0^2}{\pi}\right)^{1/2}.$$

We consider the nonergodic state. To find the transition line from the AFSG phase, in which the antiferromagnetic order parameter $l$ differs from zero, to the SG phase, where $l=0$, we must find in the SG phase the generalized susceptibility relative to a field that is directed upwards for spins of one subsystem and downward for the other. This susceptibility has a pole when $J_0\,\mathrm{P}=1$, where P is determined by the same expression (22) as for the usual susceptibility, but the mean values are calculated in an external field.

The calculations can be carried out to conclusion only near $(H, T_0)$. Confining ourselves, as before, to the linear terms in the dependence of $q_{1,2}$ on $x$ in the interval $x_0(H)<x<x_1(H)$ and expanding all the quantities in powers of $\tau$ and $(H-H_0)/H_0$, we obtain the slope of the $T_N(H)$ curve in the nonergodic phase at the point $(H_0,T_0)$.

The calculation results are shown in Figs. 3. It can be seen that in the nonergodic state there exist regions with antiferromagnetic order as well as with pure spin glass. There is an interval of fields weaker than $H_0$ in which the P–AF–AFSG–SS phase transitions take place in succession as the temperature is lowered.

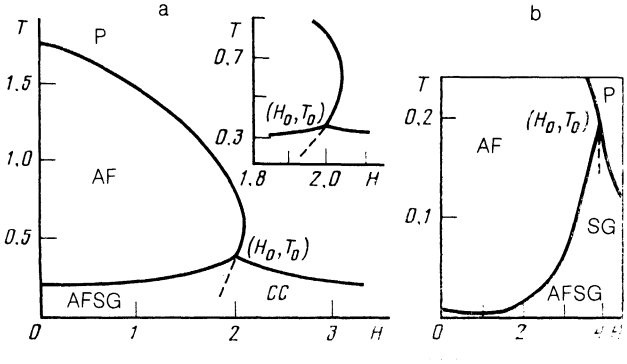


FIG. 3. Phase diagram in the $(T, H)$ plane at $J_0=1.7$ (Fig. 3a) and $J_0=3.0$ (Fig. 3b). The inset of Fig. 3a shows in enlarged scale a section of the phase diagram near the triple point. The dashed line shows the slope of the $T_N(H)$ curve in the nonergodic phase.

## CONCLUSION

We have shown that a temperature of field transition, in which long-range order vanishes (appears), is possible in the nonergodic state. Since it was found that the susceptibility in the mixed state depends little on the temperature, it is quite difficult to detect this transition by finding a kink an the $\chi(T)$ curve.

A more suitable method in this case is apparently neutron diffraction, which would permit detection of the vanishing of antiferromagnetic reflection. As to transitions from the antiferromagnetic phase into antiferromagnetic spin glass, they can be investigated by the usual macroscopic methods. Such investigations were carried out in Refs. 3–6 for the layered Ising metamagnet $Fe_{1-x}Mg_xCl_2$. The measurements were made at a magnesium concentration $x$ near the percolation threshold in a plane, so that the interaction

of the spins in the plane was greatly suppressed. In this sense, the situation accords with the model considered by us, in which no account was taken of interactions within the sublattices. A strong increase of $T_g$ with increase of the magnetic field, by almost a factor of two, was observed in Ref. 14 at a ratio $T_N/T_g(0)\approx 3$; this is in qualitative agreement with the result of the theory above.

The increase of $T_g$ with increase of $H$ was attributed in Ref. 4 to the influence of random magnetic fields on the antiferromagnetic phase. Such fields appear in a disordered antiferromagnet in an external magnetic field.[24] But these fields must suppress also the spin-glass state, so that the influence of the random field on $T_g$ is not clear beforehand. Since there are no random fields in our model with random bonds and with only intersublattice interactions,[25] and the transition temperature can increase with the field, it is clear that the $T_g(H)$ growth observed in Ref. 4 can be explained also without assuming an influence of the random field. Random-field effects can be easily taken into account by including intrasublattice interaction in the model.

Note that an increase of $T_g$ with $H$ was observed also in Heisenberg antiferromagnetic glass,[12] where the growth was stronger the larger $T_N(0)/T_g(0)$.

It was mentioned above that the temperature dependence of the sublattice magnetization in $Fe_xMg_{1-x}Cl_2$. was measured in Ref. 6. The magnetization varies smoothly with temperature, and in accordance with the results of Sec. 2 the $m_{1,2}(T)$ curve has no kink at $T_g$. It would be of interest to identify the singularity of $m_{1,2}$ at $T_g$. If the reentrant transition is, just as the transition into "pure" spin glass, of third order, a kink can appear already on $d^2m_{1,2}/dT^2$.

## APPENDIX

The replica method was used in Ref. 17 to obtain the following expression for the free energy

$$F=-\frac{1}{4T}-\frac{T}{2}\lim_{n\to 0}\frac{1}{n}\Phi(x_p{}^{\alpha}y_p{}^{\alpha\beta}), \tag{A.1}$$

where $p=1$ or 2, $n$ is the number of replicas,

$$\Phi(x_p{}^{\alpha}y_p{}^{\alpha\beta})=-\frac{1}{2}\sum_{\alpha}((x_3{}^{\alpha})^2+(x_1{}^{\alpha})^2+(x_2{}^{\alpha})^2)$$
$$-\frac{1}{2}\sum_{(\alpha,\beta)}\Big[(y_3{}^{\alpha\beta})^2+\sum_{p}(y_p{}^{\alpha\beta})^2\Big]-\ln \mathrm{Sp}\exp\Big[\frac{H}{T}\sum_{p,\alpha}S_p{}^{\alpha}$$
$$+i\Big(\frac{J_0}{T}\Big)^{1/2}\sum_{p,\alpha}S_p{}^{\alpha}x_3{}^{\alpha}+\Big(\frac{J_0}{T}\Big)^{1/2}\sum_{p,\alpha}S_p{}^{\alpha}x_p{}^{\alpha}$$
$$+\frac{1}{T}\sum_{(\alpha,\beta)p}S_p{}^{\alpha}S_p{}^{\beta}y_3{}^{\alpha\beta}+\frac{i}{T}\sum_{(\alpha,\beta)p}S_p{}^{\alpha}S_p{}^{\beta}y_p{}^{\alpha\beta}\Big], \tag{A.2}$$

and $x_p^{\alpha}$, $x_3^{\alpha}$, $y_p^{\alpha\beta}$, $y_3^{\alpha\beta}$ the solutions of the saddle-point equations

$$x_p{}^{\alpha}=m_p{}^{\alpha}\Big(\frac{J_0}{T}\Big)^{1/2}=\Big(\frac{J_0}{T}\Big)^{1/2}\langle S_p{}^{\alpha}\rangle,\quad x_3{}^{\alpha}=2i\Big(\frac{J_0}{T}\Big)^{1/2}m_3{}^{\alpha},$$
$$y_p{}^{\alpha\beta}=\frac{1}{T}q_p{}^{\alpha\beta}=i\frac{1}{T}\langle S_p{}^{\alpha}S_p{}^{\beta}\rangle,\quad y_3{}^{\alpha\beta}=\frac{2}{T}q_3{}^{\alpha\beta}, \tag{A.3}$$
$$x_3{}^{\alpha}=\frac{i}{2}(x_1{}^{\alpha}+x_2{}^{\alpha}),\qquad y_3{}^{\alpha\beta}=-\frac{i}{2}(y_1{}^{\alpha\beta}+y_2{}^{\alpha\beta}).$$

The angle brackets denote averaging with the exponential under the trace sign in (A.2).

In the nonergodic phase, the replica symmetry of the matrices $q$ is violated. The quantities $m^{\alpha}$ do not depend on the replica indices and determine, as we shall show presently, the moments of the sublattices. We express next the trace in (A.2) as a product $L_1L_2$, where

$$L_p=\mathrm{Sp}\exp\Big[\frac{1}{T}\sum_{\alpha}S_p{}^{\alpha}H_p+\frac{1}{T^2}\sum_{(\alpha,\beta)}S_p{}^{\alpha}S_p{}^{\beta}(2q_3{}^{\alpha\beta}-q_p{}^{\alpha\beta})\Big],$$
$$H_p=H-J_0(2m_3+m_p),\quad p=1,2. \tag{A.4}$$

We use the Parisi parametrization for the matrices $q_p^{\alpha\beta}$. Then, using the identity

$$L_p=\exp\frac{1}{T}\sum_{(\alpha,\beta)}\frac{\partial^2}{\partial h_p{}^{\alpha}\,\partial h_p{}^{\beta}}(2q_3{}^{\alpha\beta}-q_p{}^{\alpha\beta})\prod_{\alpha}(2\operatorname{ch}h_p{}^{\alpha})\,|_{h_p{}^{\alpha}=H_p}, \tag{A.5}$$

we obtain, following the Duplantier method,[26] an expression for the free energy

$$F=-\frac{1}{4T}-\frac{1}{8T}\Big[4\Big(\int_0^1 dx\,q_3{}^2(x)-2q_3(1)\Big)$$
$$-\sum_{p=1,2}\Big(\int_0^1 dx\,q_p{}^2(x)-2q_p(1)\Big)\Big]-f_1-f_2$$
$$+\frac{J_0}{4}(m_1{}^2+m_2{}^2-4m_3{}^2). \tag{A.6}$$

Here $q_p(x)$ is the Parisi function for the parameter, while $f_1$ and $f_2$ are equal to

$$f_{1,2}=\frac{1}{2(2\pi q_{2,1}(0))^{1/2}}$$
$$\times\int dy\,\varphi_{1,2}(0,y)\exp\Big[-\frac{(y-(H-J_0m_{2,1}))^2}{2q_{2,1}(0)}\Big], \tag{A.7}$$

while the functions $\varphi(x,y)$ satisfy the equation

$$\dot{\varphi}_{1,2}=-\frac{2\dot{q}_3-\dot{q}_{1,2}}{2}\varphi_{1,2}''-\frac{1}{2T}(2\dot{q}_3-\dot{q}_{1,2})x(\varphi_{1,2}')^2 \tag{A.8}$$

with boundary conditions

$$\varphi_{1,2}(1,y)=T\ln 2\operatorname{ch}\frac{y}{T}. \tag{A.9}$$

A dot denotes differentiation with respect to $x$, a prime differentiation with respect to $y$, and $x$ varies in the interval [1,2].

An equation for $q_p(x)$ can be obtained from (A.2) by varying and again using the Duplantier method. It is more convenient, however, to use the variational procedure proposed in Ref. 22. To this end we write down the variational functional $F$ in the form

$$F=-\frac{1}{4T}+\frac{J_0}{4}(m_1^2+m_2^2-4m_3^2)$$
$$+\frac{1}{8T}\sum_p\left\{\int_0^1 q_p^2(x)\,dx-2q_p(1)\right\}$$
$$-\frac{1}{2T}\left[\int_0^1 q_3^2(x)\,dx-2q_3(1)\right]-f_1-f_2+\frac{1}{2}\sum_p\int_{-\infty}^{\infty}dy\,P_p(1,y)$$
$$\times\left[\varphi_p(1,y)-T\ln 2\,\mathrm{ch}\frac{y}{T}\right]-\frac{1}{2}\sum_p\int_0^1 dx\int_{-\infty}^{\infty}dy\,P_p(x,y)$$
$$\times\Big[\dot\varphi_p(x,y)+\frac{2\dot q_3-\dot q_p}{2}\varphi_p''(x,y)$$
$$+\frac{x}{2T}(2\dot q_3-\dot q_p)\,(\varphi_p'(x,y))^2\Big]. \qquad (A.10)$$

The last two terms are zero by virtue of (A.8) and (A.9), and do not alter $F$. $P_0$ are indeterminate Lagrange multipliers.

Variation of the functional (A.10) with respect to the functions $q_p(x)$ and $m_p$ leads to the equations

$$\frac{1}{2T}q_{1,2}(x)=\int dy\,P_{1,2}(x,y)\,(\varphi_{1,2})^2$$
$$+\int\frac{\partial}{\partial x}(P_{1,2}\varphi_{1,2}'')\,dy-\frac{x}{2T}\int\frac{\partial}{\partial x}(\varphi_{1,2})^2\,dy, \qquad (A.11)$$

$$q_3(x)=\tfrac{1}{2}(q_1(x)+q_2(x)), \qquad (A.12)$$

$$m_{1,2}=\frac{1}{(2\pi q_{2,1}(0))^{1/2}}\int dy\,\varphi_{1,2}'(0,y)$$
$$\times\exp\left[-\frac{(y-(H-J_0m_{2,1}))^2}{2q_{2,1}(0)}\right], \qquad (A.13)$$

$$m_3=\tfrac{1}{2}(m_1+m_2). \qquad (A.14)$$

Varying with respect to $\varphi(x,y)$ and $\varphi(0,y)$ and using (A.12), we obtain equations for $P_{1,2}(x,y)$

$$\dot P_{1,2}=\frac{\dot q_{2,1}}{2}P_{1,2}''-\frac{\dot q_{2,1}}{T}(\varphi_{1,2}'P_{1,2})' \qquad (A.15)$$

with boundary conditions

$$P_{1,2}(0,y)=\frac{1}{(2\pi q_{2,1}(0))^{1/2}}\exp\left[-\frac{(y-(H-J_0m_{2,1}))^2}{2q_{2,1}(0)}\right]. \qquad (A.16)$$

Using (A.15) we can, following Ref. 22, simplify expressions (A.11) for $q_{1,2}(x)$. To this end, differentiating both halves of (A.8) with respect to $y$, we obtain equations for $m_{1,2}(x,y)=\varphi_{1,2}(x,y)$:

$$\hat\Omega_{1,2}m_{1,2}(x,y)=0, \qquad (A.17)$$

where we have introduced the operator

$$\hat\Omega_{1,2}=\frac{\partial}{\partial x}+\frac{\dot q_{2,1}}{2}\left(\frac{\partial^2}{\partial y^2}+\frac{2x}{T}m_{1,2}\frac{\partial}{\partial y}\right). \qquad (A.18)$$

It follows from (A.15) that any function satisfies the relation

$$\frac{d}{dx}\int dy\,P_{1,2}(x,y)f(x,y)=\int dy\,P_{1,2}(x,y)\hat\Omega_{1,2}f(x,y). \qquad (A.19)$$

Using (A.19), we see that the last two terms in (A.11) cancel out, so that

$$q_{1,2}(x)=\int dy\,P_{1,2}(x,y)\,m_{1,2}(x,y).$$

It follows from (A.17) that

$$\int dy\,P_{1,2}(x,y)\,m_{1,2}(x,y)=\int P_{1,2}(0,y)\,m_{1,2}(0,y)\,dy=m_{1,2}.$$

We have thus derived Eqs. (3)–(11) of the main text.

[1)]A spin glass is called "pure" if it has no long-range order, and only its spin-glass order parameter $q(x)$ differs from zero.


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