# Inhomogeneous superconductivity in disordered metals

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A parameter $\tau_D$ is introduced to describe the temperature region near $T_c$, in which the statistical spatial fluctuations of the order parameter are strong. It is shown on the basis of the Ginzburg–Landau functional, with the aid of the replica method, that two temperature superconductivity regimes are realized, depending on the degree of the disorder. At $\tau_D > \tau_D^* = 2.49\tau_G$, where $\tau_G$ is the Ginzburg parameter that characterizes the size of the region of strong thermodynamic fluctuations, the superconductivity is produced in spatially inhomogeneous fashion with droplike seeds. The drop density and their contribution to the free energy and to the diamagnetic susceptibility are obtained in a model of non-interacting drops. If $\tau_D < \tau_D^*$, superconductivity sets in below $T_c$ simultaneously in the entire volume, i.e., the usual second-order transition is realized.

## INTRODUCTION

The theory of dirty superconductors, developed by Abrikosov and Gor'kov[1,2] and by Anderson,[3] is the basis of the quantitative description of the superconducting properties of a large number of disordered alloys. As the theory of strongly disordered system progressed, however, it became clear that the main results of Refs. 1–3 must be modified to fit mean free paths $l$ of the order of the Fermi wave number $k_F^{-1}$ (of the order of the interatomic distance). A growth of disorder in three-dimensional systems causes the electron diffusion to stop at mean free path $l$ shorter than a certain value $l_c \approx k_F^{-1}$, the electron diffusion ceases, the electronic states near the Fermi level become localized, and the system goes over into the state of an Anderson dielectric.[4,5] This metal-insulator transition manifests itself in a continuous vanishing of the metallic conductivity (at $T=0$) as $l\, l_c$. At $l \gg l_c$ the conductivity is determined by the standard Drude formula and $\sigma \sim l$, whereas at $l \gtrsim l_c$ it decreases like $\sigma \sim (l-l_c)^\nu$, where $\nu$ is a certain critical exponent. The transition from diffusion to localization takes place at conductivities $\sigma$ on the other of the so-called minimal metallic conductivity $\sigma_c \approx (e^2k_F/\pi^3\hbar) \approx (2\text{–}5)\cdot 10^2\ \Omega^{-1}\cdot\text{cm}^{-1}$. The theory of dirty superconductors does not take localization effects into account and is valid for conductivities in the interval $(E_F/T_c)\sigma_c \gg \sigma \gg \sigma_c$.

The data known so far on the behavior of superconductors near the localization threshold are the following.

1. Assuming the density of states $N(E_F)$ to be independent of the Fermi level and the dimensionless electron-photon interaction parameter $\lambda_{e,ph}$ to be independent of $l$, it can be shown that $T_c$ decreases with decrease of $l$, owing to the corresponding growth of the Coulomb pseudopotential $\mu^*$. This effect is due to the increase of the delay of the Coulomb repulsion in the Cooper pair as the diffusion coefficient decreases on approaching the Anderson transition. The decrease of the superconducting transition temperature $T_c$ begins in the region $\sigma \gg \sigma_c$ and becomes rapid at $\sigma \lesssim \sigma_c$ (Refs. 6–8). Belitz[9] calculated the decrease of $T_c$ due to the decrease of the effective density of the electronic states on the Fermi level under the influence of the Coulomb repulsion in the presence of impurities (the Al'tshuler-Aronov effect). The enhancement of the spin fluctuations with increase of disorder, and the appearance of localied magnetic moments near the localization threshold, due to the electron repulsion,[10] can also cause a decrease of $T_c$ in ultradirty superconductors,[11,12] but there is still no consistent quantitative theory of this effect. We note that the decrease of $T_c$ due to the mutual influence of the disorder and of the Coulomb effect was first considered by Ovchinnikov[13] and by Maekawa and Fukuyama (see Refs. 4 and 13) within the framework of the BCS model with allowance for the lowest localized corrections.

2. Bulaevskiĭ and Sadovskiĭ[7] and later Kapitulnik and Kotliar[14] found the superconducting coherence length $\xi$ (at $T=0$) in the region $\sigma \ll \sigma_c$, and also in the localization region ($l<l_c$). At the mobility threshold itself, where $l=l_c\approx k_F^{-1}$ and $\sigma=0$, we have

$$\xi \approx (\xi_0 k_F^{-2})^{1/3}, \qquad \xi_0 = 0.18\hbar v_F/T_c.$$

In contrast to the standard theory of dirty superconductors with $l \gg l_c$ (Refs. 1 and 2), in which $\xi^2 \approx \xi_0 l$ is proportional to $\sigma$, as $l \to l_c$ we have $\sigma \to 0$ whereas $\xi^2$ remains different from zero both at the mobility threshold ($l=l_c$) and in the localization region, i.e., in an Anderson dielectric. The same result was obtained recently by Ma and Li[15] who used another method. Obviously, there results are valid only if $T_c$ does not vanish all the way to the Anderson transition, a situation possible only if rather strigent conditions imposed by the effects noted in Sec. 1 are met. The fact that $\xi^2$ differs from zero when $\sigma$ vanishes at $l \ll l_c$ means conservation of the superconducting response in the phase of an Anderson dielectric.

3. As the disorder increases, the region of thermodynamic fluctuations near $T_c$ increases. The width of this region is defined as $\tau_G T_c$, where the characteristic Ginzburg parameter for dirty superconductors is equal to $\tau_G = [\pi^2 T_c N(E_F)\xi^3]^{-2}$. Kapitulnik and Kotliar[14] noted that near the mobility threshold, where $\xi \approx (\xi_0 k_F^{-2})^{1/3}$, the parameter $\tau_G$ does not contain a small quantity such as $T_c/E_F$ (is not excluded, of course, that $\tau_G$ remains small because of a small numerical factor). Therefore a superconducting transition near the location threshold can in princi-

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ple not become an analog of a $\lambda$ transition in $He^4$. Allowance for the fluctuations would lead in fact to a change of the critical exponents in the temperature dependence of the thermodynamic quantities near $T_c$ compared with the corresponding exponents of the molecular-field theory.

All the cited theoretical analysis of the influence of disorder on superconductivity were made under the assumption that the superconducting order parameter is self-averaging. This remarks pertains both the the classical papers on dirty superconductors[1-3] and to all recent studies of superconductivity near and in the Anderson localization state.[6-16]. It is assumed here that the spatial fluctuations of the superconducting order parameter $\Delta(\mathbf{r})$ are small, and the use of the parameter $\langle\Delta(\mathbf{r})\rangle$ is justified. It seems natural for such a procedure to be valid at $\sigma \gg \sigma_c$, but there are no grounds for believing it to be correct near the localization threshold.[1)]. In such a system the electronic characteristics fluctuate strongly, and we shall in Sec. I below that these fluctuations actually lead to substantial spatial fluctuations of the parameter $\Delta(\mathbf{r})$ (a brief summary of this section is given in Ref. 17).

In Sec. II we consider superconductors with spatial fluctuation of the local "temperature" of the superconducting transition. We shall show that if the amplitude of such statistical fluctuations exceeds a critical value, the superconductivity manifests itself with decrease of temperature in a spatially inhomogeneous manner, in the form of superconducting drops. We shall find the density of these drops as a function of temperature. In the model of noninteracting drops, we shall obtain also their contribution to the free energy of the system, and the diamagnetic susceptibility.

## I. ESTIMATE OF THE REGION OF STRONG STATISTICAL FLUCTUATIONS OF THE SUPERCONDUCTING ORDER PARAMETER.

As a starting point, we consider the usual BCS Hamiltonian

$$\mathscr{H}=\mathscr{H}_0+\mathscr{H}_{int}+\mathscr{H}_M, \quad \mathscr{H}_M=\int d\mathbf{r}\, B^2(\mathbf{r})/8\pi,$$

$$\mathscr{H}_0=\int d\mathbf{r}\, \psi^+(\mathbf{r})\left[\frac{1}{2m}\left(i\hbar\nabla-\frac{e}{c}\mathbf{A}(\mathbf{r})\right)^2+U(\mathbf{r})\right]\psi(\mathbf{r}), \quad (1)$$

$$\mathscr{H}_{int}=\frac{1}{N(E_F)}\int d\mathbf{r}\, \lambda_{e,ph}(\mathbf{r})\psi^+(\mathbf{r})\psi^+(\mathbf{r})\psi(\mathbf{r})\psi(\mathbf{r}),$$

where $\mathbf{B}=\operatorname{curl}\mathbf{A}$ and $u(\mathbf{r})$ is a random potential in the disordered system. The pairing interaction can also fluctuate in space, but we assume hereafter that this interaction is weak, $\lambda_{e,ph}(\mathbf{r}) \ll 1$.

Let us write down Ginzburg-Landau functional for the non-averaged order parameter $\Delta(\mathbf{r})$. We introduce to this end the exact energy eigenvalues $\varepsilon_\mu$ and the exact eigenfunctions $\varphi\mu(\mathbf{r})$ of the electrons, corresponding to the Hamiltonian $\mathscr{H}_0$. We obtain with their aid a superconducting functional in the form[18]

$$\mathscr{F}_s\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\}=\int d\mathbf{r}\left\{\frac{B^2(\mathbf{r})}{8\pi}+N(E_F)\int d\mathbf{r}'\right.$$
$$\left.\times\left[\frac{\delta(\mathbf{r}-\mathbf{r}')}{\lambda_{e,ph}(\mathbf{r})}-K(\mathbf{r},\mathbf{r}')\right]\Delta(\mathbf{r})\Delta^*(\mathbf{r}')+\frac{1}{2}\lambda N(E_F)|\Delta(\mathbf{r})|^4\right\},$$

$$K(\mathbf{r},\mathbf{r}')=\frac{T}{N(E_F)}\sum_{\varepsilon_n,\mu,\nu}\frac{\varphi_\mu^*(\mathbf{r})\varphi_\mu(\mathbf{r}')\varphi_\nu^*(\mathbf{r})\varphi_\nu(\mathbf{r}')}{(i\varepsilon_n-\varepsilon_\mu)(-i\varepsilon_n-\varepsilon_\nu)}, \quad (2)$$

$$\varepsilon_n=\pi T(2n+1), \quad \lambda=7\zeta(3)/8\pi^2T^2.$$

The statistical fluctuations of $\lambda_{e,ph}(\mathbf{r})$ and of the kernel $K(\mathbf{r},\mathbf{r}')$, (in view of the random character of the values of $\varphi_\nu(\mathbf{r})$ and $\varepsilon_\nu$) cause spatial fluctuations of the superconducting order parameter $\Delta(\mathbf{r})$. We have neglected in (2) the fluctuations of the parameter $\lambda$; it will be seen from the analysis that follows that they are less substantial than the fluctuations of the kernel $K(\mathbf{r},\mathbf{r}')$. Assuming the fluctuations of the kernel $K(\mathbf{r},\mathbf{r}')$ and of the parameter $\Delta(\mathbf{r})$ to be small, we estimate the temperature region in which this assumption turns out to be incorrect and where a description with the aid of the averaged order parameter is inadequate. It will be shown below that the variance is determined mainly by the long-wave variation of $\Delta(\mathbf{r})$. We can herefore transform from (2) to the GL functional for the order parameter:

$$\mathscr{F}_{GL}\{\mathbf{A}(\mathbf{r}),\Delta(\mathbf{r})\}=\int d\mathbf{r}\left\{\frac{B^2(\mathbf{r})}{8\pi}+N(E_F)\Big[(\tau+t(\mathbf{r}))|\Delta(\mathbf{r})|^2\right.$$
$$\left.+\xi^2\left|\left(\nabla-\frac{2ie}{\hbar c}\mathbf{A}(\mathbf{r})\right)\Delta(\mathbf{r})\right|^2+\frac{1}{2}\lambda|\Delta(\mathbf{r})|^4\Big]\right\}, \quad (3a)$$

$$\xi^2=\frac{1}{6}\int K_0(r)r^2\,dr, \quad K_0(\mathbf{r}-\mathbf{r}')=\langle K(\mathbf{r},\mathbf{r}')\rangle,$$

$$\tau=\frac{T}{T_{c0}}-1, \quad (3b)$$

where $T_{c0}$ is the transition temperature determined by the averaged kernel $K_0(\mathbf{r}-\mathbf{r}')$ with allowance for the short-wave fluctuations of the kernel $K(\mathbf{r},\mathbf{r}')$. In the derivation of (3) we neglected the fluctuations of the coefficient $\xi^2$. The function $t(\mathbf{r})$ plays here the role of the fluctuation local critical "temperature," It takes into account the fluctuations of the pairing interaction, for which $t(\mathbf{r})=\lambda_{e,ph}^{-1}(\mathbf{r})-\langle\lambda_{e,ph}^{-1}(\mathbf{r})\rangle$, and also the fluctuations of the dipole density of the electronic states $N(\mathbf{r},E_F)$:

$$t(\mathbf{r})=\int_0^{\omega_D}\frac{dE}{E}\operatorname{th}\frac{E}{2T_{c0}}\left[\frac{N(\mathbf{r},E)}{N(E_F)}-1\right],$$
$$N(\mathbf{r},E)=\sum_\nu|\varphi_\nu(\mathbf{r})|^2\delta(E-\varepsilon_\nu), \quad N(E_F)=\langle N(\mathbf{r},E_F)\rangle. \quad (4)$$

The functional for the fluctuations of the pairing interaction was investigated by Larkin and Ovchinnikov[19] in connection with a study of the influence of structure inhomogeneities of the samples on their superconducting properties; the analysis that follows will be similar. Within the framework of perturbation theory in the fluctuation $\Delta(\mathbf{r})$ we obtain from (3) the renormalized temperature $T_c$ and the variance of $\Delta(\mathbf{r})$:

$$\frac{T_c-T_{c0}}{T_{c0}}=\frac{1}{(2\pi)^3}\int\frac{\gamma(\mathbf{q})\,d\mathbf{q}}{\xi^2q^2}, \quad \gamma(\mathbf{q})=\int d\mathbf{r}\, e^{i\mathbf{q}\mathbf{r}}\langle t(\mathbf{r})t(0)\rangle, \quad (5a)$$

$$\frac{\langle\Delta^2\rangle-\langle\Delta\rangle^2}{\langle\Delta\rangle^2}=\frac{1}{(2\pi)^3}\int\frac{\gamma(\mathbf{q})\,d\mathbf{q}}{(\xi^2q^2+2|\tau|)^2}. \quad (5b)$$

It follows from (5a) that the fluctuation-induced shift of $T_c$ is positive and the contribution made to it by the short-wave fluctuations is generally speaking not small. According to (5b), the fluctuations of $\Delta(\mathbf{r})$ are determined mainly by the behavior of the correlation function $\gamma(\mathbf{q})$ at small $q$.

The value of $\gamma(\mathbf{q})$ neglecting the fluctuations of the pairing interactions, was obtained in Ref. 17. In dirty superconductors with $\sigma \gg \sigma_c$ (i.e., $l \gg k_F^{-1}$) we have $\gamma(0) \approx \xi N^{-2}(E_F) D_0^{-2}$, where $D_0 = v_F l/3$ is the classical diffusion coefficient. We then obtain from (5)

$$\langle\Delta^2\rangle/\langle\Delta\rangle^2 - 1 \approx (\tau_D/\tau)^{1/2}, \quad \tau_D = \gamma^2(0)\xi^{-6}, \tag{6}$$

where $\xi \approx (\xi_0 l)^{1/2}$. The parameter $\tau_D$ introduced by us defines the region in which statistical (spatial) fluctuations of the order parameter are significant.[1)] It can be seen from (6) that $\tau_D \approx \tau_G^2 \ll \tau_G \ll 1$, in dirty superconductors, i.e., the statistical fluctuations are unimportant even in region where thermodynamic flucturations are noticable enough.

The situation changes radically in the vicinity of the mobility threshold, where[3)] $\gamma(\mathbf{q}) \approx \xi^3 \ln(1/\xi q)$. From (5) we obtain for the variance of $\Delta(\mathbf{r})$ the expression

$$\frac{\langle\Delta^2\rangle}{\langle\Delta\rangle^2} - 1 \approx \frac{1}{(|\tau|)^{1/2}} \ln\frac{1}{|\tau|}. \tag{7}$$

According to (7), the statistical fluctuations near the mobility threshold turn out to be most substantial, and are stronger here than the thermodynamic fluctuations in view of the logarithmic factor in $\gamma(\mathbf{q})$. Thus, near the localization threshold we have $\tau_D \gtrsim \tau_G \approx 1$. The transition from the regime of weak statistical fluctuations ($\tau_D \ll \tau_G$) to the regime of strong ones ($\tau_D \gtrsim \tau_G$) takes place at the values $\sigma \approx \sigma^* \approx \sigma_c (k_F \xi_0)^{-1/3}$ of the conductivity, the physical meaning of which was discussed in Ref. 7. At this conductivity, a transition takes place from the usual theory of dirty superconductors to the relations typical of the vicinity of the localization threshold.

Below the localization threshold. The region of strong $\Delta(\mathbf{r})$ fluctuations expands even more. This is due to the appearance of an additional delta-function singularity in the correlator of the local density of states.[21] We obtain accordingly in $\gamma(\mathbf{q})$ an additional term $[N(E_F) T_c (1 + R_l^2 q^2)]^{-1}$, where $R_l$ is the localization radius of the electronic states on the Fermi level. The variance of $\Delta(\mathbf{r})$ acquires according to (5) at $R_l > \xi(T)$ another term in addition to (7)

$$\langle\Delta^2\rangle/\langle\Delta\rangle^2 - 1 \approx [N(E_F) T_c R_l^3 \tau^2]^{-1},$$

which increases rapidly with decrease of the localization radius $R_l$ ($R_l = \infty$ at the localization threshold). It is shown in Ref. 7 that if $T_c$ remains different from zero at the localization threshold, the Copper pairing survives with further increase of the disorder and with decrease of $R_l$ only to values $R_l \gg [N(E_F) T_c]^{-1/3}$. This inequality means that the energy interval $T_c$ spans many discreate levels whose centers are located inside a region with radius $R_l$ (see also Ref. 15). In addition, it guarantees that the localization radius exceeds substantially the characteristic dimension of the cooper pairs. We see now that under the same condition the relative variance of $\Delta(\mathbf{r})$ remains at a level on the order of unity in the entire temperal interval in which superconductivity exists in the dielectric phase.

If the statistical fluctuations of $t(\mathbf{r})$ are caused by randomly disposed regions with dimensions $a$, where $k_F^{-1} \ll a \ll \xi$, and with increased values of the electron-photon interaction parameter $\lambda_{e,ph} + \delta\lambda_{e,ph}$ (in view of the change of the structure of the dislocations, twinning planes, etc), we have for such a model

$$\gamma(0) = c(1-c) a^3 (\delta\lambda_{e,ph}/\lambda_{e,ph})^2, \quad \tau_D = \gamma^2(0)\xi^{-6}, \tag{8}$$

where $c$ is the relative total volume of the regions with altered parameter $\lambda_{e,ph}$. In this case, at $c \approx 1$ and $\delta\lambda_{e,ph}/\lambda_{e,ph} \approx 1$, the regime of strong statistical fluctuations $\tau_D \gtrsim \tau_G$ is realized at $a \gg k_F^{-1}(E_F/T_c)^{1/3}$. This condition is compatible with the restriction $a \ll \xi$ assumed above. Note that at $a \gg \xi$ the appearance of inhomogeneous superconductivity is not surprising: on cooling it is formed initially only in regions with increased transition "temperature" corresponding to the parameter $\lambda_{e,ph} + \delta\lambda_{e,ph}$. A much less trivial factor is that at $a \ll \xi$ there is likewise no averaging of the superconducting properties if the level of the fluctuations of $t(\mathbf{r})$ [due to the fluctuations of $N(\mathbf{r}, E_F)$ or of $\lambda_{e,ph}(\mathbf{r})$] is high enough.

## II. SUPERCONDUCTING TRANSITION IN SYSTEMS WITH STRONG DISORDER

### 1. Formulation of problem

We consider now superconductivity in systems with strong spatial statistical Gaussian fluctuations of the local transition "temperature" $T_c(\mathbf{r})$. We shall show that in this model, depending on the degree of disorder, i.e., on the ratio $\tau_D/\tau_G$, two types of superconducting transition are possible At $\tau_D < \tau_D^* = 2.49\tau_G$ the superconductivity is a second-order phase transition at the point $T_c$. The superconducting order parameter is in this case equal to zero at $T > T_c$ and is spatially homogeneous over scales exceeding the correlation length $\xi(T)$ below $T_c$. Statistical fluctuations lead only to a change of the critical exponents in the temperature dependence of the basic characteristics of the system $\xi(T)$, $\lambda_L(t)$, and others.[22,23]

At $\tau_D > \tau_D^*$ the superconducting state appears in inhomogeneous fashion even if the correlation radius $a$ of the disorder-induced fluctuations of the temperature $T_c(\mathbf{r})$ is small compared with the superconducting correlation length $\xi$ (we refer to disorder of this type, with $a \ll \xi$, as microscopic). The first to deduce the possibility of an inhomogeneous superconducting transition for microscopic disorder were Ioffe and Larkin.[24]. Investigating the case of extremely strong disorder (in fact $\tau_D \gg (\tau_G \tau)^{1/2}$), they have shown that as the temperature is lowered the normal phase aquires localized superconducting regions (drops) with characteristic dimension $\xi(T)$. Far from $T_c$ their density is low, but with further cooling the density and dimensions of the drops increase and they begin to overlap. The superconducting transition becomes percolative in this case.

The Ioffe-Larkin transition, valid in the limit of very strong disorder, did not take thermodynamic fluctuations into account and provided no criterion for the transition from the homogeneous superconductivity to the inhomogeneous ones. The corresponding criterion $\tau_D > \tau_D^* \approx 2.49\tau_G$ will be obtained below for a model with Gaussian fluctuations of $T_c(\mathbf{r})$.

According to the estimates given in Sec. I, if the impurities influence only the local density of states $N(\mathbf{r}, E_F)$ in the system, the parameter $\tau_D/\tau_G$ increases from a very small value to values greater than unity as the disorder increases and a transition takes place from the $l \gg k_F^{-1}$ regime to the electron localization regime ($l \approx k_F^{-1}$). An onset of an inhomogeneous superconducting regime is therefore to be expected as the localization threshold is approached. In a system that contain regions with increased value of the parameter $\lambda_{e,ph}$ under conditions $l \gg k_F^{-1}$, this regime can be realized also at parameter values $\tau_D \ll 1$, since $\tau_G \ll 1$ in such a system.

Our treatment of superconductors with large disorder will be based on the GL functional (3a) with a Gaussian distribution of the temperature $t(\mathbf{r})$. Given the distribution $t(\mathbf{r})$, the free energy of the system and the order-parameter correlator are equal to

$$F_s\{t(\mathbf{r})\} = -T \ln Z, \quad Z = \int D\{\mathbf{A}, \Delta\} \exp[-\mathscr{F}_{GL}\{\mathbf{A}, \Delta\}/T], \tag{9a}$$

$$\langle \Delta(\mathbf{r}) \Delta(\mathbf{r}') \rangle = Z^{-1} \int D\{\mathbf{A}, \Delta\} \Delta(\mathbf{r}) \Delta(\mathbf{r}') \exp[-\mathscr{F}_{GL}\{\mathbf{A}, \Delta\}/T], \tag{9b}$$

and they must be averaged, assuming that the correlator

$$\langle t(\mathbf{r}) t(\mathbf{r}') \rangle = \gamma \delta(\mathbf{r}-\mathbf{r}'), \quad \gamma = \tau_D^{1/2} \xi^3, \tag{10}$$

is known. For Gaussian fluctuations with a correlator (10), the probability of a configuration with a given $t(\mathbf{r})$ distribution is

$$P\{t(\mathbf{r})\} = \exp\left[-\frac{1}{2\gamma} \int d\mathbf{r}\, t^2(\mathbf{r})\right]. \tag{11}$$

The problem reduces thus to calculation of the functions $F_s\{t(\mathbf{r})\}$ and $\langle \Delta(\mathbf{r}) \Delta(\mathbf{r}') \rangle$ (9b) and their subsequent averaging with the aid of (11).

We confine ourselves in this article to consideration of noninteracting drops. We can then disregard the presence of vortices in the sample, and in each drop the phase of the order parameter $\Delta(\mathbf{r})$ can be regarded as nonsingular.[4)] Following the gauge transformation

$$\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + (c\hbar/2e) \nabla \varphi(\mathbf{r}),$$

$$\Delta(\mathbf{r}) \to \Delta(\mathbf{r}) \exp[-i\varphi(\mathbf{r})],$$

where $\varphi(\mathbf{r})$ is the phase of the order parameter, the quantity $\Delta(\mathbf{r})$ in (9b) is real and the GL functional becomes

$$\mathscr{F}_{GL}\{\mathbf{A}(\mathbf{r}), \Delta(\mathbf{r})\} = \int d\mathbf{r} \left\{ \frac{B^2(\mathbf{r})}{8\pi} + N(E_F) \left[ (\tau + t(\mathbf{r})) \Delta^2(\mathbf{r}) + \frac{4e^2\xi^2}{c^2\hbar^2} A^2(\mathbf{r}) \Delta^2(\mathbf{r}) + \xi^2 (\nabla \Delta(\mathbf{r}))^2 + \frac{\lambda \Delta^4(\mathbf{r})}{2} \right] \right\}. \tag{12}$$

Integration over the phase in (9) adds to the partition function an inessential constant factor which we shall disregard hereafter. To calculate the free energy of a system of noninteracting drops we shall use an approach similar to the fluctuation theory of nucleation of a new phase in first-order transitions, and also the replica method.

## 2. Fluctuation theory of drops

Superconducting drops can appear in a specified $t(\mathbf{r})$ configuration only in regions with locally higher superconducting-transition temperatures. We shall number these regions by the subscript $i$. The order parameter in each region is determined by a nontrivial localized solution $\Delta_d^{(i)}(\mathbf{r}) \neq 0$ of the GL equation, and the contribution of such a drop to the partition function of the system is

$$N^{(i)}\{t(\mathbf{r})\} \exp\left(-\frac{E_d^{(i)}\{t(\mathbf{r})\}}{T}\right),$$

$$E_d^{(i)}\{t(\mathbf{r})\} = \mathscr{F}_{GL}\{0, \Delta_d^{(i)}(\mathbf{r})\},$$

where $E_d^{(i)}$ is the drop energy, and the factor $N^{(i)}$ is determined by the contribution of the $\Delta(\mathbf{r})$ configurations that are close to the classical solution $\Delta_d^{(i)}(\mathbf{r})$. Summing the contribution of configurations containing an arbitrary number of drops and neglecting their interaction with one another, we obtain the partition function (9a) of the system,

$$Z = Z_0 \left[ 1 + \sum_i N^{(i)} \exp\left(-\frac{E_d^{(i)}}{T}\right) + \frac{1}{2!} \sum_{i,j} N^{(i)} N^{(j)} \exp\left(-\frac{E_d^{(i)} + E_d^{(j)}}{T}\right) + \dots \right]$$
$$= Z_0 \exp\left[\sum_i N^{(i)} \exp\left(-\frac{E_d^{(i)}}{T}\right)\right]. \tag{13}$$

Here $Z_0$ is the partition function of the system in the absence of drops. Substituting (13) in (9a) and averaging the free energy of the system over the $t(\mathbf{r})$ configurations, we get

$$F_s = -\frac{T}{N} \int D\{t(\mathbf{r})\} \sum_i N^{(i)}\{t(\mathbf{r})\} \exp\left(-\frac{\mathscr{F}_d^{(i)}\{t(\mathbf{r})\}}{T}\right), \tag{14}$$

where $N$ is a normalization factor and $\mathscr{F}_d$ assumes the role of the free energy of the drop:

$$\mathscr{F}_d\{t(\mathbf{r})\} = E_d\{t(\mathbf{r})\} - T \ln P\{t(\mathbf{r})\}. \tag{15}$$

The main contribution to the functional integral (14) is made by the configurations $t_0(\mathbf{r})$ that realize an extremum of the functional (15):

$$t_0(\mathbf{r}) = -\tilde{\gamma} \Delta_d^2(\mathbf{r}), \quad \tilde{\gamma} = \gamma N(E_F)/T_c. \tag{16}$$

Note that $t_0(\mathbf{r})$ is negative, since the drops appear in regions of higher superconducting-transition temperatures. Substitution of (16) in the GL equation that corresponds to the functional (3a) leads to a nonlinear equation for the order parameter $\Delta(\mathbf{r})$ in the superconducting drop. In dimensionless variables, this equation is

$$\Delta_d(r) = \left(\frac{\tau}{\tilde{\gamma} - \lambda}\right)^{1/2} \chi\left[\frac{r}{\xi(T)}\right], \quad \xi(T) = \frac{\xi}{\tau^{1/2}}, \tag{17}$$

$$\frac{1}{x} \frac{d^2}{dx^2} (x\chi(x)) - \chi(x) + \chi^3(x) = 0, \quad \chi(x \to \infty) = 0. \tag{18}$$

The asymptote of the function $\chi(x)$ at $x \gg 1$ is determined from the linearized form of Eq. (18), and $\chi(x) \sim x^{-1} e^{-x}$. The superconducting nuclei are thus localized over a scale of the order of the correlation radius $\xi(T)$. The quantity $\mathscr{F}_{\min}$ is obtained by substituting (16) and (17) in (15):

$$S_0(\tau) \equiv \frac{\mathscr{F}_{min}}{T} = \frac{A\xi^3 \tau^{1/2}}{\gamma - \lambda T/N(E_F)} = \frac{A(\tau/\tau_D)^{1/2}}{1 - (\tau_G/\tau_D)^{1/2}}, \quad \lambda < \tilde{\gamma}. \tag{19a}$$

It determines, with exponential accuracy, the free energy (14) of the drops. The constant $A$ in (19a) is equal to[25]

$$A = 4\pi \int_0^\infty dx\, x^2 \left[\chi^2(x) + \left(\frac{d\chi}{dx}\right)^2 - \frac{1}{2}\chi^4(x)\right] = 37.8. \tag{19b}$$

Note that the energy $E_d\{t_0(\mathbf{r})\}$ of superconducting drops is negative, and their production is energywise favored compared with the case of the spatially homogeneous solution

$\Delta(\mathbf{r})=0$. According to (19a), superconducting drops can exist only in the presence of sufficiently strong statistical fluctuations $\tau_D>\tau_G$; a rigorous restriction will be obtained below.

To determine the pre-exponential factor in (14) one must turn to the solution of the complete problem (11), (13). Neglecting its thermodynamic fluctuations, the order parameter can be obtained within the framework of the Ioffe–Larkin method.[24] We obtain for the free energy of the system and for the drop density $\rho_s$ the expressions

$$F_s(\tau)\approx-T\xi^{-3}(T)\,(\tau_D/\tau_G)^{1/2}\exp\left[-S_0(\tau)\right], \tag{20a}$$

$$\rho_s(\tau)\approx\xi^{-3}(T)S_0(\tau)\exp\left[-S_0(\tau)\right]. \tag{20b}$$

The exponent $S_0(\tau)$ is defined here by Eq. (19a) with $\lambda=0$. Note that the pre-exponential factor in (20a) differs from that obtained in Ref. 24, which contains an inaccurate expression for the free energy of one drop. It is seen from (19a) that at $\lambda\ll\tilde{\gamma}S_0^{-1}(\tau)$ we obtain for $S_0(\tau)$ the result of the Ioffe-Larkin theory of weak thermodynamic fluctuations. This means that their approach is valid if the inequality $\tau_D\ll\tau\ll\tau_D^2/\tau_G$ holds, and this is possible only if $\tau_D\gg\tau_G$. It follows from (20) that in the region where these expressions are valid the average energy $F_s/\rho_s$ of each drop is large compared with the temperature, and the two become comparable at $\lambda\approx\tilde{\gamma}S_0^{-1}(\tau)$. We confine ourselves hereafter to the region $\lambda\gg\tilde{\gamma}S_0^{-1}(\tau)$ i.e., $\tau\gg\tau_D^2/\tau_G$, where the contribution of the thermal fluctuations becomes substantial. It will be shown below that its precisely in this limit that the fluctuations of the order parameter are small relative to the most probable configuration (17). This enables us to use standard field-theoretical methods to find the free energy of the system and the order-parameter correlator in the region of strong thermodynamic fluctuations.

### 3. Replica method and instantons

To average the logarithm of the partition function (9a) over $t(\mathbf{r})$ with weight (11) we use the replica method, which permits the averaging to be carried out in explicit form.[26]

We express the average free energy (9a) of the system in the form

$$F=-T\lim_{n\to 0}\frac{1}{n}\left[\langle Z^n\rangle-1\right]. \tag{21}$$

To calculate $\langle Z^n\rangle$ in accordance with the idea of the replica method, we assume first $n$ to be an arbitrary integer. Expressing $Z^n$ in terms of an $n$-fold functional integral over the the fields of the replicas $\mathbf{A}_\alpha(\mathbf{r})$, $\Delta_\alpha(\mathbf{r})$, $\alpha=1,\ldots,n$ and carrying out exact Gaussian averaging over $t(\mathbf{r})$, we get

$$\langle Z^n\rangle=\int D\{\mathbf{A}_\alpha,\Delta_\alpha\}\exp[-S_n\{\mathbf{A}_\alpha,\Delta_\alpha\}], \tag{22}$$

$$S_n\{\mathbf{A}_\alpha,\Delta_\alpha\}=\int dr\left\{\sum_{\alpha=1}^{n}\frac{B_\alpha^2(\mathbf{r})}{8\pi}+\frac{N(E_F)}{T}\sum_{\alpha=1}^{n}\left[\tau\Delta_\alpha^2(\mathbf{r})+\frac{4e^2\xi^2}{c^2\hbar^2}\mathbf{A}_\alpha^2(\mathbf{r})\Delta_\alpha^2(\mathbf{r})+\xi^2(\nabla\Delta_\alpha(\mathbf{r}))^2+\frac{1}{2}\lambda\Delta_\alpha^4(\mathbf{r})\right]-\frac{1}{2}N(E_F)T^{-1}\tilde{\gamma}\left[\sum_{\alpha=1}^{n}\Delta_\alpha^2(\mathbf{r})\right]^2\right\}.$$

Note that the random quantities $t(\mathbf{r})$ have already dropped out of these expressions, and that the action $S_n\{\mathbf{A}_\alpha,\Delta_\alpha\}$ is translationally invariant. For the mean value of the order-parameter correlator (9b) we get

$$\langle\Delta(\mathbf{r})\Delta(\mathbf{r}')\rangle=\lim_{n\to 0}\frac{1}{n}\int D\{\mathbf{A}_\alpha,\Delta_\alpha\}\exp[-S_n\{\mathbf{A}_\alpha,\Delta_\alpha\}]\sum_{\alpha=1}^{n}\Delta_\alpha(\mathbf{r})\Delta_\alpha(\mathbf{r}'), \tag{23}$$

where we have symmetrized over the replica indices.

Far from the region of strong fluctuations of the order parameter $|\tau|\gg\tau_D,\tau_G$ the functional integrals (22) and (23) can be calculated by the saddle-point method. The external trajectories are classical solutions for the action (22), and when calculating the functional integrals account must be taken of the Gaussian fluctuations about them. The extremal trajectories are defined by

$$\left[\tau-\xi^2\nabla^2+\lambda\Delta_\alpha^2(\mathbf{r})-\tilde{\gamma}\sum_{\beta=1}^{n}\Delta_\beta^2(\mathbf{r})\right]\Delta_\alpha(\mathbf{r})=0,\qquad A_\alpha(\mathbf{r})=0. \tag{24}$$

These equations for $\Delta_\alpha(\mathbf{r})$ have a spatially homogeneous solution and localized (instanton) solutions. The latter correspond at $\tau>0$ to superconducting drops. We confine ourselves in this article to considerations of non-interacting drops and consider only instanton solutions above $T_c$ (at $\tau>0$). We shall be interested hereafter only in those solutions that admit analytic continuation as $n\to 0$. We designate them $\Delta_\alpha^{(i)}(\mathbf{r})$, where the superscript $i$ labels the type of solution. To find their contribution we must expand the action (22) accurate to terms quadratic in the deviations $\varphi_\alpha(\mathbf{r})=\Delta_\alpha(\mathbf{r})-\Delta_\alpha^{(i)}(\mathbf{r})$. It is shown in the Appendix that the fluctuations of the fields $\mathbf{A}_\alpha(\mathbf{r})$ can be neglected when isolated seeds are considered. The action (22) takes then the form

$$S_n\{\Delta_\alpha\}=S_n\{\Delta_\alpha^{(i)}\}+\frac{1}{2}\int d\mathbf{r}\sum_{\alpha,\beta}(\varphi_\alpha\hat{M}_{\alpha\beta}^{(i)}\varphi_\beta). \tag{25}$$

To calculate the functional integral over the fields $\varphi_\alpha$ we expand them in terms of the normalized eigenfunctions of the operator $\hat{M}^{(i)}$:

$$\varphi_\alpha(\mathbf{r})=\sum_k c_k\varphi_{k\alpha}(\mathbf{r}),\qquad\sum_\beta M_{\alpha\beta}^{(i)}\varphi_{k\beta}=\varepsilon_k\varphi_{k\alpha}. \tag{26}$$

Substitution of (36) in (25) yields for the action the expression

$$S_n\{\Delta_\alpha\}=S_n\{\Delta_\alpha^{(i)}\}+\frac{1}{2}\sum_k c_k^2\varepsilon_k. \tag{27}$$

The Gaussian functional integral in (22) is calculated by replacing the integration variables

$$\int D\{\varphi_\alpha\}\ldots=\prod_k\int\frac{dc_k}{(2\pi)^{1/2}}\ldots, \tag{28}$$

and its value is determined by the eigenvalue spectrum of the operator $\hat{M}^{(i)}$.

At $\lambda=0$ Eqs. (24) are symmetric with respect to rotations in replica space, and admit of solutions of the form[5,27]

$$\Delta_\alpha^{(e)}(r)=\Delta_d(r)e_\alpha,\qquad\Delta_d(r)=\left(\frac{\tau}{\tilde{\gamma}}\right)^{1/2}\chi\left[\frac{r}{\xi(T)}\right],$$

$$\sum_{\alpha=1}^{n} e_\alpha^2=1, \quad (29)$$

where $e_\alpha$ is an arbitary unit vector in replica space, and the function $\chi(x)$ was defined earlier. Such instantons corresponds to the already considered limiting case of weak thermodynamic fluctuations, and the action on them is given by $S_0(\tau)$ from (19a) at $\lambda = 0$.

At $\lambda \neq 0$ this symmetry of the action (22) is violated by the term $\lambda\ \Delta_\alpha^4$ (it plays the role of cubic anisotropy in replica space), and there are $n$ types of instanton solutions of Eqs. (24):

$$\Delta_\alpha^{(i)}(r)=\Delta_d(r)\delta_{\alpha i}, \quad i=1,\dots,n. \quad (30)$$

The function $\Delta_d(r)$ is defined in (17) and the index $i$ characterizes the direction, in replica space, along which spontaneous symmetry breaking takes place.[5] A number of important relations between the integrals of the function $\chi(x)$ can be found by noting that Eq. (18) can be obtained from the condition that the functional $A\{\chi(x)\}$ (19b) have an extremum with respect to $\chi(x)$. To this end, we replace $\chi(x)$ in it by $\alpha\chi(\beta x)$. The minimum of the function $A(\alpha,\beta)$ with respect to $\alpha$ and $\beta$ should be reached at $\alpha=\beta=1$, so that

$$\int_0^\infty dx\, x^2\chi^2(x)=\frac{1}{3}\int_0^\infty dx\, x^2\left(\frac{d\chi}{dx}\right)^2=\frac{1}{4}\int_0^\infty dx\, x^2\chi^4(x)=\frac{A}{8\pi}. \quad (31)$$

The action (22) on the instanton solution (30) is equal to the value of $S_0(\tau)$ given in (19a). It follows from (22) that the instanton contribution to $\langle Z^n\rangle$ is proportional to $n \exp[-S_0(\tau)]$, where the factor $n$ is the result of summation of contributions of all $n$ types of solutions (30). Substituting this expression in (21), we get for the free energy of the seeds the result (14) and (19) of the fluctuation theory. Allowance for the fluctuations of the replica fields in the vicinity of the classical solution enables us to find the pre-exponential factor in (14).

#### 4. Pre-exponential factor in the case of strong thermodynamic fluctuations

The pre-exponential factor in $F_s$ is determined by the replica-field configurations (26) near the external solution (30). The operator $\hat M^{(i)}$ on the solutions (30) is equal to

$$\hat M_{\alpha\beta}^{(i)}=[\hat M_L\delta_{\alpha i}+\hat M_T(1-\delta_{\alpha i})]\delta_{\alpha\beta},$$

$$\hat M_{L,T}=\frac{2N(E_F)}{T}[-\xi^2\nabla^2+\tau U_{L,T}(\mathbf r)],$$

$$U_L(\mathbf r)=1-3\chi^2[r/\xi(T)], \quad (32)$$

$$U_T(\mathbf r)=1-(1-\lambda/\tilde\gamma)^{-1}\chi^2[r/\xi(T)].$$

Its eigenfunctions are

$$\varphi_{k,\alpha}^L(\mathbf r)=\varphi_k^L(\mathbf r)\delta_{\alpha i}, \quad \varphi_{k,\alpha}^T(\mathbf r)=\varphi_k^T(\mathbf r)\delta_{\alpha\beta}, \quad \beta\neq i, \quad (33)$$

where the functions $\varphi_k^{L,T}(\mathbf r)$ are the solutions of the eigenvalue equations for the operators $\hat M_{L,T}$:

$$\hat M_{L,T}\varphi_k^{L,T}(\mathbf r)=\varepsilon_k^{L,T}\varphi_k^{L,T}(\mathbf r). \quad (34)$$

These equations have the form of Schrödinger equations with the potential $U_{L,T}(\mathbf r)$ shown schematically in Fig. 1. Let us examine the spectrum of these equations. The potential $U_L(r)$ always have a discrete level with zero eigenvalue $\varepsilon_i^L=0$. Its presence is connected with the translational symmetry of Eq. (22). A solution of (24), other than (30) and having the same action, is the function $\Delta_\alpha^{(i)}(\mathbf r+\mathbf r_0)$ with a shift of the localization center by an arbitrary vector $\mathbf r_0$. The corresponding deviation $\varphi_\alpha(\mathbf r)$ following a translation by an infinitely small vector $\delta\mathbf r_0$ takes the form

$$\varphi_\alpha(\mathbf r)=[\Delta_d(\mathbf r+\delta\mathbf r_0)-\Delta_d(\mathbf r)]\delta_{\alpha i}=(J_L^{1/2}\delta\mathbf r_0)\varphi_1^L(\mathbf r)\delta_{\alpha i}, \quad (35)$$

$$\varphi_1^L(\mathbf r)=J_L^{-1/2}\frac{\partial\Delta_d(r)}{\partial r}\frac{\mathbf r}{r},$$

$$J_L=\frac{1}{3}\int d\mathbf r\left(\frac{\partial\Delta_d}{\partial r}\right)^2=\frac{S_0(\tau)T}{2\xi^2N(E_F)}. \quad (36)$$

It can be verified by directly substituting (36) in (33) that the functions $\varphi_{1,x,y,z}^L(\mathbf r)$ are eigenfunctions of the operator $\hat M_L$ with zero eigenvalues. In (36) we have expressed with the aid of (17) and (31), in terms of the action (22), the integral that determines $J_L$. Comparison of (35) with the general expression (26) yields the differential of the coefficient $c_iL$ of the expansion (26): $dc_i^L=J_L^{1/2}d\mathbf r_0$. Since the eigenvalue is threefold degenerate, $J_L^{3/2}$ is the Jacobian of the transition from the coefficients $c_1^L$ to the collective variables $\mathbf r_0$ that determine the position of the superconducting drop. The integral with respect to $\mathbf r_0$ yields the volume of the system $V$. By calculating the remaining Gaussian integrals with respect to $c_k$ in (27) and (28), we obtain the contribution of the instanton configurations (30) to $\langle Z^n\rangle$ (22):

$$nV(J_L/2\pi)^{3/2}[\det{}'\hat M_L]^{-1/2}[\det\hat M_T]^{(1-n)/2}\exp[-S_0(\tau)]. \quad (37)$$

the determinant of the operator is equal to the product of all its eigenvalues, and the prime denotes exclusion of the zero eigenvalues from this product. Substituting (37) in (21), we obtain the contribution of the superconducting drops to the free energy of the system:

$$F_s=-\theta_s(\tau)T,$$

$$\theta_s(\tau)=\left[\frac{TS_0(\tau)}{4\pi N(E_F)}\right]^{3/2}\left[\frac{\det\hat M_T}{\det{}'\hat M_L}\right]^{1/2}\xi^{-3}\exp[-S_0(\tau)]. \quad (38)$$

To determine $\theta_s$ we must find the remaining eigenvalues of the operators $\hat M_L$ and $\hat M_T$ (32).

We consider first the operator $\hat M_L$. The angular dependence of the eigenfunction (36) obtained above corresponds to $p$-type state with orbital monmtun $l=1$. The min-

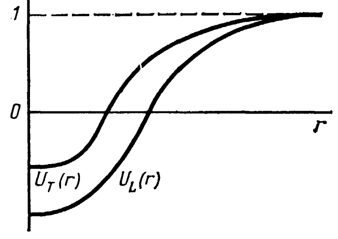


FIG. 1.

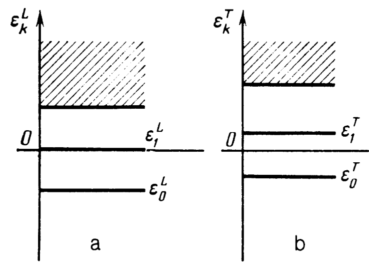


FIG. 2.

imum eigenvalue $\varepsilon_0^L$ should correspond to a nondegenerate $s$ state with $l=0$. The operator $\hat{M}_L$ should have thus at least one negative eigenvalue $\varepsilon_0^L<\varepsilon_1^L=0$. A more rigorous analysis (Ref. 28) shows that such an eigenvalues is unique. The remaining eigenvalues $\varepsilon_k^L$ with $k>1$ are positive. The described eigenvalue spectrum of the operator $\hat{M}_L$ is shown in Fig. 2 (the continuous section of the spectrum is shaded).

We consider now the eigenvalue spectrum of the operator $\hat{M}_T$. The quantity $\rho_s$ in (38) is positive only if the operator $\hat{M}_T$ has a single negative eigenvalue. We shall show below that this situation is realized if the condition $0<\lambda<\lambda^*=2\tilde{\gamma}/3$, is met, a condition that defines in fact that region of existence of superconducting drops. The spectrum of the eigenvalues of the operator $\hat{M}_T$ is shown in Fig. 2.

In the case $\lambda\ll\lambda^*$ the minimum eigenvalue $\varepsilon_0^T<0$ can be obtained by perturbation theory in the small parameter $\lambda/\lambda^*$. At $\lambda=0$ the operator $\hat{M}_T$ (32) has a single zero eigenvalue $\varepsilon_0^T=0$. The corresponding Goldstone mode is connected with the isotropy of Eqs. (24) in replica space, and corresponds to rotation of the unit vector $e_a$ (29) in replica space

$$\varphi_\alpha(\mathbf{r})=\Delta_d(r)\,\delta e_\alpha=(J_T^{1/2}\delta e_\alpha)\,\varphi_0^T(\mathbf{r}), \tag{39}$$

where the normalization component $J_T$ and the function $\varphi_0^T$ are equal to

$$\varphi_0^T(\mathbf{r})=J_T^{-1/2}\Delta_d(r),\quad J_T=\int d\mathbf{r}\,\Delta_d^2(r)=S_0(\tau)\,T/2\tau N(E_F). \tag{40}$$

It is easy to verify that the function (40) at $\lambda=0$ is indeed a solution of Eq. (34) with zero eigenvalue $\varepsilon_0^T=0$. Comparing (39) with (26) we obtain the relation

$$c_{0\alpha}{}^T=J_T{}^{1/2}\delta e_\alpha. \tag{41}$$

At small $\lambda\ll\lambda^*$ we can neglect the change of the eigenfunction (40) of the operator $\hat{M}_T$. Its minimum eigenvalue $\varepsilon_0^T$ is obtained by multiplying both halves of Eq. (34) for $\varepsilon_0^T$ and by integrating with respect to the coordinate $\mathbf{r}$:

$$\varepsilon_0{}^T=-\frac{2\lambda N(E_F)}{T}\int d\mathbf{r}\,\Delta_d{}^4(r)\Big/\int d\mathbf{r}\,\Delta_d{}^2(r)=-\frac{8\lambda\tau}{\gamma}, \tag{42}$$

where we have used relations (17) and (30). The condition for the validity of the approach based on the instanton solutions (30) can be formulated in the form $\langle(\delta e_a)^2\rangle\ll1$. Since, as follows from (27), the characteristic values $(c_0^T)^2$ are proportional to $|\varepsilon_0^T|^{-1}$, this condition takes the form $\lambda\gg\tilde{\gamma}S_0^{-1}(\tau)$. The opposite case of small $\lambda$ was considered above using the Ioffe-Larkin approach. If $\tilde{\gamma}S_0^{-1}(\tau)\ll\lambda\ll\lambda^*=2\tilde{\gamma}/3$, all the eigenvalues of the operator $\hat{M}_T$ except $\varepsilon_0^T$ can be calculated under the assumption that $\lambda=0$, and the eigenvalue $\varepsilon_0^T$ is given by Eq. (42). It is easly seen that in this case all the eigenvalues of the operators $\hat{M}_T$ and $\hat{M}_L$ except $\varepsilon_0^T$ and $\varepsilon_1^L$ are proportional to $\tau N(E_F)/T$ and are independent of $\tilde{\gamma}$ and $\lambda$. A dimensional estimate of the ratio of their determinants yields therfore

$$|\det{}'\hat{M}_T/\det{}'\hat{M}_L|=[N(E_F)\,\tau/T]^2. \tag{43}$$

Substituting (42) and (43) in (38) we get

$$\theta_s(\tau)\approx\frac{1}{\xi^3(T)}\left(\frac{\lambda}{\tilde{\gamma}}\right)^{1/2}S_0{}^{5/2}(\tau)\exp[-S_0(\tau)]$$

$$\approx\xi^{-3}(T)\,\frac{\tau_G{}^{1/4}\tau_D{}^{5/4}}{\tau}\exp[-S_0(\tau)]. \tag{44}$$

When calculating the order-parameter correlator (23) it suffices to take into account in the pre-exponential factor only the fluctuations due to the translational mode with zero eigenvalue:

$$\Delta_\alpha(\mathbf{r})=\Delta_d(\mathbf{r})\,\delta_{\alpha i}+c_1{}^L\varphi_{1,\alpha}^L(\mathbf{r})=\Delta_d(\mathbf{r}+\mathbf{r}_0)\,\delta_{\alpha i}. \tag{45}$$

We obtain as a result

$$\langle\Delta(\mathbf{r})\,\Delta(\mathbf{r}')\rangle=\theta_s(\tau)\int d\mathbf{r}_0\,\Delta_d(\mathbf{r}+\mathbf{r}_0)\,\Delta_d(\mathbf{r}'+\mathbf{r}_0). \tag{46}$$

The integration with respect to the coordinate $\mathbf{r}_0$ in (46) means in fact averaging over different drop-localization positions. After averaging, the correlator (46) depends only on the coordinate difference. Note that in view of the possible scatter of the drop amplitudes the parameter does not determine their density. To find the latter we must obtain the distribution of the drop amplitudes. At $\lambda\approx\tilde{\gamma}S_0^{-1}(\tau)$ expressions (38) and (44) are transformed into (20a) and (20b).

At $\lambda=\lambda^*=2\tilde{\gamma}/3$ the operators $\hat{M}_T$ and $\hat{M}_L$ coincide. Accordingly, all their eigenvalues are equal and the operator $\hat{M}_T$ has a single negative eigenvalue $\varepsilon_0^T=0$. At small $\lambda^*-\lambda\ll\lambda^*$ we obtain the eigenvalue $\varepsilon_1^T$ by perturbation theory with the aid of the corresponding function (36):

$$\varepsilon_1{}^T=\frac{6N(E_F)}{T}(\lambda^*-\lambda)\int d\mathbf{r}\,\Delta_d{}^2(r)\left(\frac{\partial\Delta_d}{\partial r}\right)^2\Big/\int d\mathbf{r}\left(\frac{\partial\Delta_d}{\partial r}\right)^2$$

$$\approx\frac{N(E_F)\,\tau}{T}\left(\frac{\lambda^*}{\lambda}-1\right). \tag{47}$$

The remaining eigenvalues of the operator $\hat{M}_T$ are positive at $\lambda<\lambda^*$. Using the result (47) for $\varepsilon_1^T$ and setting the remaining $\varepsilon_k^T=\varepsilon_k^L$ for $k\neq0$, we obtain at $\lambda^*-\lambda\ll\lambda^*$

$$\theta_s(\tau)\approx\frac{1}{\xi^3(T)}\left(\frac{\lambda^*}{\lambda}-1\right)^{3/2}S_0{}^{3/2}(\tau)\exp[-S_0(\tau)],$$

$$\frac{\lambda^*}{\lambda}=0.64\left(\frac{\tau_D}{\tau_G}\right)^{1/2}. \tag{48}$$

As $\lambda\to\lambda^*$ the eigenvalue $\varepsilon_1^T\to0$ and account must be taken of the non-Gaussian character of the field fluctuations $\varphi_\alpha(\mathbf{r})$. These fluctuations can lead to a change of relation (48) in the region of small $\lambda^*-\lambda\lesssim\tilde{\gamma}S_0^{-1}(\tau)$. Thus, superconducting drops exist only if $\tau_D>\tau_D^*$, and their density vanishes as $\lambda\to\lambda^*$ because the superconductivity is destroyed in the drops by thermodynamic fluctuations.

In the calculation of the order-parameter correlator it is necessary, in the case $\lambda^*-\lambda\ll\lambda^*$, to take into account in (23), besides the zeroth translational mode, also the contribution of $n-1$ modes of the operator $\widehat{M}_T$, with eigenvalues $\varepsilon_1^T$ that tend to zero as $\lambda \to \lambda^*$. Neglecting the contribution of the remaining mode, we can, in analogy with the derivation of (45), replace in (23) the quantity

$$\sum_{\alpha=1}^{n} \Delta_\alpha(\mathbf{r})\Delta_\alpha(\mathbf{r}')$$

by

$$n\Delta_d\left(\mathbf{r}+J_L^{1/2}\sum_{\alpha=1}^{n}\mathbf{c}_{1,\alpha}\right)\Delta_d\left(\mathbf{r}'+J_L^{1/2}\sum_{\alpha=1}^{n}\mathbf{c}_{1,\alpha}\right). \tag{49}$$

Integrating over all the coefficients $c_k$ in (28) and (23), we obtain for the order parameter the results (46), where the factor $\theta_s(\tau)$ is defined in (38). Note that over large scales the function (43) decreases like $\exp[-|\mathbf{r}-\mathbf{r}'|/\xi(T)]$ and does not contain the Ornstein-Zernike factor $|\mathbf{r}'-\mathbf{r}|^{-1}$.

## CONCLUSION

We have shown here that in the case of sufficiently strong statistical fluctuations of the order parameter $\tau_D > \tau_D^*$ superconductivity is produced in the form of isolated seeds-superconducting drops. We found the free energy of such an inhomogeneous superconducting state and the correlator of the order parameter in the temperature region $\tau \gg \tau_D$, where the function $\theta_s$ defined in (38) is exponentially small: $\theta_s \sim \exp[-A(\tau/\tau_D)^{1/2}]$. The drops can be regarded here as noninteracting. They make an exponentially small contribution to the heat capacity of the system, to the conductivity, and to the diamagnetic susceptibility. To calculate the latter, we find the changes induced in the exponents of (19), (15), and (20) by a change of the external field $H$:

$$\Delta S_0(\tau,H)=\frac{4e^2\xi^2N(E_F)}{3c^2\hbar^2T}H^2\int d\mathbf{r}\, r^2\Delta_d^2(\mathbf{r}). \tag{50}$$

Differentiating the free energy $F_s(\tau,H)$ with respect to $H$, we get

$$\chi_s=-F_s(\tau)S_0(\tau)\xi^4(T)/\Phi_0^2, \tag{51}$$

where $\Phi_0$ is the flux quantum.

The order parameter is locally small inside the drop in the region $\xi(T)=\xi\tau^{-1/2}$ only to the extent that $\tau^{1/2}$ is small, and local measurements (for example, with the aid of a tunnel-effect microscope) can reveal the appearance of the drops.

The theory predicts thus a strong enhancement of the thermodynamic and statistical fluctuations of the superconducting order parameter near the localization threshold. The thermodynamic fluctuations by themselves leave the system spatially homogeneous and therefore do not lead to a qualitatively new behavior. Statistical fluctuations alter the superconducting transition radically—it becomes percolative.[24] Although, there is as yet no quantitative theory of such a transition in the temperature region where the drop density is large, a number of qualitative conclusions that lend themselves to experimental verfication can be drawn.

A transition in an inhomogeneous superconductivity regime should be strongly smeared in temperature, and the degree of smearing should depend on the current flow in the measurements of $R$ and on the field in the measurements of the magnetic susceptibility $\chi_s$. In view of the strong fluctuations of $\Delta(\mathbf{r})$ there may be no BCS singularity in the density of states of the quasiparticles, and at $\tau_D \gtrsim 1$ it will have a zero-gap character down to zero temperature (the same result is produced also by an increase of the frequency of the electron inelastic collisions near the localization threshold, owing to the enhancement of the Coulomb repulsion of the electrons[29]). Finally, the inhomogeneous characterof the superconductivity (of the drop) can be observed with the aid of local measurements, e.g., by tunnel-effect microscopy.

A substantial broadening of the superconducting transition and a smearing of the singularity in the density of states of the quasiparticles was indeed observed in granulated aluminum as the conductivity was lowered below 1000 $\Omega^{-1}\cdot\text{cm}^{-1}$ (Ref. 29). These facts offer evidence of the increasing role of the fluctuations, although the only assumption made to intepret the zero-gap character of the spectrum at $\sigma = 10\ \Omega^{-1}\cdot\text{cm}^{-1}$ was that the frequency of the electron inelastic collisions increases near the localization threshold.

Similar peculiarities of the superconducting behavior should occur also in systems with strong statistical fluctuations of the pairing interaction, independently of their proximity to the localization threshold. Naturally, far from the Anderson transition, there are in this case no grounds whatever for enhancement of the inelastic scattering of the electron, and the zero gap in the quasiparticle spectrum can be due only to statistical fluctuations of the superconducting order parameter.

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## APPENDIX

Let us show that thermodynamic fluctuations of the magnetic field in superconducting drops, which we have neglected above, have no effect in dirty superconductors.

We expand the action (22) in the vicinity of the instanton solution (30) in the deviations of $A_{\mu\alpha}$ and $\varphi_\alpha$ accurate to quadratic terms ($A_{\mu\alpha}$ are the components of the vector $\mathbf{A}_\alpha$, and $\mu = 1,2,3$). The action (25) acquires then an additional term that describes the fluctuations of the magnetic field:

$$\frac{1}{2}\sum_{\mu,\nu,\alpha,\beta}\int d\mathbf{r}\,(A_{\mu\alpha}\hat{K}_{\mu\alpha,\nu\beta}A_{\nu\beta}), \tag{A.1}$$

where the quadratic-form operator $\hat{K}$ is equal to

$$\hat{K}^{(i)}_{\mu\alpha,\nu\beta}=\hat{K}_{L\mu\nu}\delta_{\alpha\beta}\delta_{\alpha i}+\hat{K}_{T\mu\nu}\delta_{\alpha\beta}(1-\delta_{\alpha i}), \tag{A.2}$$

$$K_{T\mu\nu}(\mathbf{r}-\mathbf{r}')=\frac{1}{T}D_{\mu\nu}^{-1}(\mathbf{r}-\mathbf{r}'), \tag{A.3}$$

$$K_{L\mu\nu}(\mathbf{r},\mathbf{r}')=\frac{1}{T}D_{\mu\nu}^{-1}(\mathbf{r}-\mathbf{r}')+\frac{8e^2N(E_F)\xi^2}{c^2T}\Delta_d^2(\mathbf{r})\delta_{\mu\nu}. \tag{A.4}$$

Here $D_{\mu\nu}(\mathbf{r})$ is the photon Green's function and is equal to $\delta_{\mu\nu}/r$ in the Coulomb gauge. Calculating the Gaussian integrals with respect to $\varphi_\alpha$ and $A_{\mu\alpha}$, we find that the magnetic-field fluctuations lead to the appearance of an additional multiplier $\Theta$ in the pre-exponential factor in (38). Regarding in (A.4) the term containing $\Delta_d^2$ as a perturbation, we obtain for the factor $\Theta$ the expression

$$\Theta = \exp\left[\frac{8e^2N(E_F)\xi^2}{c^2T}\sum_{\mu} D_{\mu\mu}(0)\int d\mathbf{r}\,\Delta_d^2(\mathbf{r}) + \frac{32e^4N^2(E_F)\xi^4}{c^4T} \times \sum_{\mu,\nu}\int d\mathbf{r}\int d\mathbf{r}'\,\Delta_d^2(\mathbf{r})\Delta_d^2(\mathbf{r}')D_{\mu\nu}^2(\mathbf{r}-\mathbf{r}')+\ldots\right]. \quad (A.5)$$

The first term in the exponential of (A.5) gives the renormalization of the superconducting-transition temperature. It is the same for both the spatially homogeneous state and for drops, and can hereafter be regarded as carried out. The second term in the exponential of (A.5) describes the influence of the screening of the fluctuating magnetic field on the form of superconducting seed. Substituting in (A.5) the instanton solutions for $\Delta_d(\mathbf{r})$ and integrating with respect to $\mathbf{r}$ and $\mathbf{r}'$, we obtain the condition under which this term in (A.4) is small and the inflence of the magnetic field on the drop is negligble, in the form

$$\lambda \ll \lambda^*(\lambda_L^2\xi_0^2/\xi^4(T)). \quad (A.6)$$

This condition is certainly met in type-II superconductors with $\lambda_L \gtrsim \xi_0$. In type-I superconductors it restricts the value of the critical disorder at which the magnetic-field thermodynamic fluctuations influence the properties of the seeds.

1)The question of the size of the statistical fluctuations in dirty superconductors was first raised in Ref. 16.

2)The parameter $\tau_G^{-1/2} \approx T_c N(E_F)\xi^3 = \langle\mathcal{N}\rangle$, where $\mathcal{N}$ is the number of levels in the system in the energy interval $T_c$ in a volume $\xi^3$. The a condition that the fluctuation region be narrow is $\langle\mathcal{N}\rangle \gg 1$. The parameter $\tau_D^{1/2} \approx \langle(\mathcal{N}-\langle\mathcal{N}\rangle)^2\rangle/\langle\mathcal{N}\rangle^2$, and determines the fluctuations of the relative number of levels.

3)This result was obtained using the scaling dependence of the correlation function $\langle N(\mathbf{r},E_F+\omega)N(0,E_F)\rangle$ near the mobility threshold.[20]

4)When the drop interaction is evaluated, it is necessary to take into account the vortices in the region between the drops; the vortices destroy the phase coherence of the different seeds. A similar situation is encountered in granulated superconductors.

5)At integer $n \geqslant 2$, Eqs. (24) have besides the solution (30) also solutions with spontaneously broken symmetry along two and more coordinate axes in replica space. Such solutions, however, do not admit analytic continuation $n \to 0$ and will not be considered further.


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