Soft-photon emission in collisions in an external electromagnetic field

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The cross sections and probabilities for multiphoton emission by a classical electron in a Coulomb field are calculated. Both monochromatic and thermal (Planck) radiation are considered. The case of sufficiently large external-field strengths in a single-mode field, which leads to nonlinear effects in the radiation, is considered. The character of the low-frequency (infrared) divergence of the radiation probabilities in a Planck field is investigated and the probabilities of the multiphoton processes are found to have a non-Poisson distribution (in contrast to spontaneous emission).

§1. INTRODUCTION

Photon emission and absorption in collisions between electrons and ions in an external magnetic field is of interest for numerous applications. This pertains to an external monochromatic electromagnetic field[1,2] and to a wide-spectrum field, such as that of thermal (Planck) radiation.[3–5]

The cross sections for multiphoton stimulated direct and inverse bremsstrahlung (SDIB) of an electron scattered by a Coulomb center having a charge Ze in the field of a strong monochromatic $\mathscr{E}_0 \cos \omega t$ were calculated in a number of papers.[1,2,6,7] The principal parameters in this process are

$$\eta = Ze^2/\hbar v, \quad \gamma = e\mathscr{E}_0 v/\hbar\omega^2, \quad \xi = \omega a/v, \tag{1.1}$$

where v is the electron velocity, ω the field frequency, and $a = Ze^2/mv^2$ the Coulomb length.

The first of the parameters in (1.1), η, indicates the extent to which the electron motion is classical, the second the intensity of the monochromatic electromagnetic field, and the third the region of emitted (absorbed) frequencies.

Most preceding studies, including the seminal one by Bunkin and Fedorov,[1] were carried out in the framework of the Born approximation corresponding to the condition $\eta \ll 1$. The Born approximation can be extended to arbitrarily strong field (large parameters γ). The values of the parameter ξ of practical interest are as a rule small ($\xi \ll 1$). A detailed investigation of the dependence of the multiphoton cross sections σ_n on the number of emitted photons in a strong electromagnetic field was carried out on the basis of the Born approximation by Elyutin.[6]

A classical approach corresponding to the condition $\eta \gg 1$ was developed by Berson[7] for the investigation of multiphoton radiation processes in a Coulomb field. His method is based on the use of an approximation in which the photon trajectory in the Coulomb field is given. The ensuing classical current excites multiphoton transitions between electromagnetic-field states that can be regarded as quantum-oscillator levels. For classical motion, the parameter ξ can be arbitrary. For the case $\xi \gg 1$ corresponding to high radiation frequencies, Berson[7] obtained a generalization of the known (single-photon) Kramers equations to the case of multiphoton transitions ($n > 1$). His calculations, however, pertain to the case of small $\gamma \ll 1$, although the classical approach itself is applicable for larger γ.

The investigation of the case of large $\gamma \gg 1$ in the classical-trajectory method is one of the aims of the present paper. It must be borne in mind here that the classical approach requires that the electromagnetic field $\mathscr{E}_0$ be small compared with the Coulomb field e^2/ρ_{eff}^2 at the effective distances ρ_{eff} governing the process considered. Within the framework of the classical method, therefore, in contrast to the Born method, the value of $\mathscr{E}_0$ is bounded from above, and large values of the parameter γ correspond predominantly to the region of low frequencies ω.

The method of given classical electron trajectories (classical current) is suitable for the investigation of creation of low-frequency (soft) photons, since their emission does not alter the electron trajectory. This method can be used also to analyze multiphoton induced emission or absorption processes in a nonmonochromatic external field, for example in the field of thermal (Planck) radiation of temperatures T_{P}. An example of multiple production of photons with classical motion of the radiated particle is the known "infrared catastrophe," first considered by Bloch and Nordsieck[8] (see also Ref. 9). In an external Planck field, the "infrared catastrophe" is enhanced as a result of the increase, with decrease of frequency, of the average number $\bar{n}_\omega$ ($\hbar\omega \ll T_{\text{P}}$) of the Planck-field photons that cause the induced emission or absorption of the soft photons.

Analysis of the "infrared catastrophe" in a Planck radiation field is of interest for at least two reasons. First, any emission or absorption process takes place against the background of the existing relict radiation with temperature $T_{\text{P}} \sim 3$ K. Second, the presence of an external thermal field of sufficiently high temperature T_{P} moves the "infrared catastrophe" question from the region of "academically" low frequencies

$$\omega \sim m\hbar^{-1}v^2 \exp(-\hbar c/e^2)$$

into the region of actually observable frequencies $\omega \sim T_{\text{P}}/\hbar$.

Section 2 below contains an exposition of the classical method; see also Refs. 5 and 7. In §3 is analyzed the case of monochromatic radiation at parameter values $\gamma \gg 1$, see (1.1). In §4 are considered multiphoton processes in a Planck radiation field. The electron energy losses are calculated in §5.

§2. FUNDAMENTAL APPROXIMATION OF THE SEMICLASSICAL METHOD

The main approximation of the semiclassical method is that of the given classical current produced by a charged

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particle moving along a given classical trajectory with velocity $\mathbf{v}(t)$. This approximation is the basis of the analysis of multiphoton processes in spontaneous production of soft photons, see Refs. 9 and 10. The Hamiltonian of the interaction between the particle and the magnetic field is of the form

$$\hat{V}=-\frac{e}{c}\hat{\mathbf{A}}\mathbf{v}(t),\quad \hat{\mathbf{A}}=c\left(\frac{2\pi\hbar}{\omega V}\right)^{1/2}\sum_{\mathbf{k},\lambda}\mathbf{e}_{\mathbf{k},\lambda}(\hat{a}_{\mathbf{k},\lambda}+\hat{a}^{+}_{\mathbf{k},\lambda}),\tag{2.1}$$

where $\mathbf{A}$ is the second-quantized vector potential of the field, expressed in terms of the creation or annihilation operator of the various field modes characterized by wave vectors $\mathbf{k}$ (frequencies $\omega=c|\mathbf{k}|$) and polarization vectors $\mathbf{e}_{\mathbf{k},\lambda}$.

The Schrödinger equation for the wave function Φ of one mode with wave vector $\mathbf{k}$ takes in the interaction representation the form

$$i\hbar\frac{\partial\Phi}{\partial t}=-e\left(\frac{2\pi\hbar}{\omega V}\right)^{1/2}(\mathbf{e}_{\mathbf{k},\lambda}\mathbf{v}(t))(\hat{a}_{\mathbf{k},\lambda}+\hat{a}^{+}_{\mathbf{k},\lambda})\Phi.\tag{2.2}$$

The scattering matrix $\hat{S}$ that connects the initial and final states of the system is

$$\hat{S}(-\infty,+\infty)=\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{+\infty}\hat{V}\,dt\right).\tag{2.3}$$

In the classical-current representation, the action of the matrix $\hat{S}$ on the operators $\hat{a}_{\mathbf{k},\lambda}$, $\hat{a}^{+}_{\mathbf{k},\lambda}$ is given by the relations

$$\hat{S}^{-1}\hat{a}_{\mathbf{k},\lambda}\hat{S}=\hat{a}_{\mathbf{k},\lambda}+ie\left(\frac{2\pi}{\hbar\omega V}\right)^{1/2}\int_{-\infty}^{+\infty}(\mathbf{e}_{\mathbf{k},\lambda}\mathbf{v}(t))e^{-i\omega t}\,dt,\tag{2.4}$$

$$\hat{S}^{-1}\hat{a}^{+}_{\mathbf{k},\lambda}\hat{S}=\hat{a}^{+}_{\mathbf{k},\lambda}-ie\left(\frac{2\pi}{\hbar\omega V}\right)^{1/2}\int_{-\infty}^{+\infty}(\mathbf{e}_{\mathbf{k},\lambda}\mathbf{v}(t))e^{i\omega t}\,dt.\tag{2.5}$$

An exact solution of (2.2) is known for one mode, see Ref. 11. The expression for the probability W_{ls} of the transition from an initial state containing s photons (i.e., from a state corresponding to the sth level of the oscillator) to the final one, containing l photons, is

$$W_{ls}=\begin{cases}x_{\mathbf{k},\lambda}^{s-l}\dfrac{l!}{s!}[L_l^{s-l}(x_{\mathbf{k},\lambda})]^2\exp(-x_{\mathbf{k},\lambda}), & s\geqslant l,\\[2mm] x_{\mathbf{k},\lambda}^{l-s}\dfrac{s!}{l!}[L_s^{l-s}(x_{\mathbf{k},\lambda})]^2\exp(-x_{\mathbf{k},\lambda}), & s<l,\end{cases}\tag{2.6}$$

$$x_{\mathbf{k},\lambda}=\frac{2\pi e^2}{\hbar\omega V}\left|\int_{-\infty}^{+\infty}(\mathbf{e}_{\mathbf{k},\lambda}\mathbf{v}(t))e^{-i\omega t}\,dt\right|^2,\tag{2.7}$$

where L_k^p is a Legendre polynomial of degree k.

In the case of a monochromatic field, the number s of the photons can be expressed in terms of the electromagnetic-field strength $\mathscr{E}_0$:

$$\mathscr{E}_0^2V/8\pi=s\hbar\omega\quad(s,\ V\to\infty),\tag{2.8}$$

after which the parameter $x_{\mathbf{k},l}$, which we shall denote simply by x_ω, takes the form

$$x_\omega=\frac{\mathscr{E}_0^2e^2}{4s(\hbar\omega)^2}\left|\int_{-\infty}^{\infty}(\mathbf{e}_{\mathbf{k},\lambda}\mathbf{v}(t))e^{-i\omega t}\,dt\right|^2.\tag{2.9}$$

In the case of a classical electromagnetic field (as $s\to\infty$) we obtain from (2.6), by taking the known limit

$$\lim_{s\to\infty}\{s^{-n}L_s^{\,n}(y/s)\}=y^{-\frac{n}{2}}J_n((4y)^{1/2}),$$

where J_n is Bessel function of integer index, the basic single-mode approximation formula obtained by Berson[7]:

$$W_{ls}=J_n^{\,2}\left(\frac{e\mathscr{E}_0}{\hbar\omega}\left|\int_{-\infty}^{+\infty}(\mathbf{e}\mathbf{v}(t))e^{-i\omega t}\,dt\right|^2\right),\tag{2.10}$$

where $n=s-l$ is the number of emitted ($n<0$) or absorbed ($n>0$) photons.

Note that in the semiclassical method the emission and absorption probabilities are equal.

For multimode spontaneous emission, the occupation numbers $n_{\mathbf{k},\lambda}\equiv l$ of each of the modes are small, and the transition probability W_{ls} ($s=0$, $l=n_{\mathbf{k},\lambda}$) for each of the oscillators of the field is greatly simplified and takes the form of a Poisson distribution:

$$W_{n_{\mathbf{k},\lambda}}=\frac{1}{n_{\mathbf{k},\lambda}!}(x_{\mathbf{k},\lambda})^{n_{\mathbf{k},\lambda}}\exp(-x_{\mathbf{k},\lambda}).\tag{2.11}$$

The probability that N field oscillators with frequencies $\omega_1,\omega_2,\ldots\omega_N$ will emit simultaneously a definite number

$$n=\sum_{i=1}^{N}n_{\mathbf{k}_i\lambda_i}$$

of photons is

$$W_n=\sum_{n=n_{\mathbf{k}_1,\lambda_1}+\cdots+n_{\mathbf{k}_N,\lambda_N}}\ \prod_{i=1}^{N}W_{n_{\mathbf{k}_i\lambda_i}}.\tag{2.12}$$

From the known addition theorem for Poisson probabilities we have

$$W_n=\frac{1}{n!}\left(\sum_{\mathbf{k},\lambda}x_{\mathbf{k},\lambda}\right)^n\exp\left(-\sum_{\mathbf{k},\lambda}x_{\mathbf{k},\lambda}\right).\tag{2.13}$$

Changing in the usual manner from summation over the oscillators to integration over the frequencies and angles, we obtain a known expression for the probability of multiphoton spontaneous emission in a finite frequency interval (ω_1,ω_2) (Refs. 8, 9)

$$W_n=\frac{1}{n!}\left(\int_{\omega_1}^{\omega_2}y_\omega\,d\omega\right)^n\exp\left(-\int_{\omega_1}^{\omega_2}y_\omega\,d\omega\right),\tag{2.14}$$

$$y_\omega=\frac{1}{2\pi^2c^3}\int_{-\pi}^{\pi}x_{\mathbf{k}}\omega^2\sin^2\theta\,d\theta.\tag{2.15}$$

In the case of a Coulomb field the parameter $x_{\mathbf{k}}$ that determines the probabilities of multiphoton processes can be calculated in explicit analytic form.

In fact, using the Coulomb-trajectory equation

$$x=a(\varepsilon-\operatorname{ch}u),\quad y=a(\varepsilon^2-1)^{1/2}\operatorname{sh}u,\quad t=av^{-1}(\varepsilon\operatorname{sh}u-u),\tag{2.16}$$

where $\varepsilon=\sin^{-1}\theta/2$ is the eccentricity of the orbit and θ the scattering angle, we obtain, following Refs. 7 and 12, the Fourier component of the velocity $\mathbf{v}(t)$:

$$(\mathbf{e}\mathbf{v}(\omega))=ae^{\pi\xi/2}[\varepsilon^2(\mathbf{n}_i\mathbf{e}-\mathbf{n}_f\mathbf{e})^2K_{i\xi}'^2(\xi\varepsilon)+(\mathbf{n}_i\mathbf{e}+\mathbf{n}_f\mathbf{e})^2K_{i\xi}^2(\xi\varepsilon)]^{1/2},\tag{2.17}$$

where $K_{i\xi}$ is a modified Bessel function of second kind and imaginary argument, and the prime denotes differentiation with respect to the argument.

We shall find it more convenient to use a coordinate frame in which the vector **e** and the velocity direction $\mathbf{n}_i$ and $\mathbf{n}_f$ of the incident and scattered electrons have the coordinates

$$\mathbf{n}_i=(0,\ 0,\ 1),\qquad \mathbf{n}_f=(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta), \tag{2.18}$$

$$\mathbf{e}=(\sin\alpha,\ 0,\ \cos\alpha), \tag{2.19}$$

where α is the angle between the vectors $\mathbf{n}_i$ and **e**.

According to (2.7), the expression for x_k in the case of scattering by a Coulomb center can be written with the aid of (2.17) in the form

$$x_k=\frac{2\pi e^2 a}{\hbar\omega V}\exp\left(\frac{\pi\xi}{2}\right)[\varepsilon^2(\mathbf{n}_i\mathbf{e}-\mathbf{n}_f\mathbf{e})^2 K_{i\xi}'^2(\xi\varepsilon)+(\mathbf{n}_i\mathbf{e}+\mathbf{n}_f\mathbf{e})^2K_{i\xi}^2(\xi\varepsilon)]^{1/2}. \tag{2.20}$$

§3. CALCULATION OF MULTIPHOTON SDIB CROSS SECTIONS IN SCATTERING BY A COULOMB CENTER IN A MONOCHROMATIC FIELD OF A WAVE

The differential cross section $d\sigma_n = W_{ls}d\sigma_{\text{res}}$ of an n-photon process, where $d\sigma_{\text{res}} = a^2\varepsilon d\varepsilon d\varphi$ is the differential Rutherford cross section for elastic scattering in the case of a monochromatic external wave field, can be written according to (2.10) and (2.20) in the form

$$d\sigma_n(\alpha)=a^2\varepsilon d\varepsilon d\varphi J_n^2(\gamma\xi e^{\pi\xi/2}[(\mathbf{n}_i\mathbf{e}-\mathbf{n}_f\mathbf{e})^2\varepsilon^2K_{i\xi}'^2(\xi\varepsilon)+(\mathbf{n}_i\mathbf{e}+\mathbf{n}_f\mathbf{e})^2K_{i\xi}^2(\xi\varepsilon)]^{1/2}). \tag{3.1}$$

This is the basic equation from which we obtain below specific results. It was used in Ref. 7 to calculate the cross sections at $\gamma\ll1$ in the cases of low ($\xi\ll1$) and high ($\xi\gg1$) frequencies, by expanding J_n^2 in power of γ. We obtain the integral (in φ and ω) cross section from (3.1) in two opposite cases: 1) $\xi\varepsilon\ll1$ and 2) $\xi\varepsilon\gg1$, where ε is the effective eccentricity, i.e., the one making the contribution to the cross section. This eccentricity will be determined below. In case (1) we are interested in the limit $\omega\to0$, so that $\xi\ll1$. Since $\xi\varepsilon$ has the meaning of the product of the frequency by the collision time, the field oscillator is excited in the first and second cases by the low-frequency and high-frequency "tail" of the Fourier spectrum of the paricle trajectory, respectively.

1. Limit $\xi\varepsilon\ll1$, fast collisions

Using (2.18), (2.19), and the fact that under the assumed condition

$$K_{i\xi}(\xi\varepsilon)\sim(\xi\varepsilon)^{-1},$$

and $K_{i\xi}(\xi\varepsilon)$ can be neglected, we have from (3.1):

$$d\sigma_n(\alpha)=a^2J_n^2\left(\frac{2\gamma}{\varepsilon^2}\,|\cos\alpha-(\varepsilon^2-1)^{1/2}\sin\alpha\cos\varphi|\right)\varepsilon\,d\varepsilon. \tag{3.2}$$

Note that this expression is valid not only in the case of a Coulomb field of the scatterer, since it can be obtained from the initial expression (2.10) for the emission probability. If the condition $\xi\varepsilon\sim\omega\tau_{\text{collis}}\ll1$ is met, the integral in (2.10) depends only in the initial and final directions of the electron motion on scattering:

$$\left|\int_{-\infty}^{+\infty}(\mathbf{ev})e^{-i\omega t}\,dt\right|=\frac{1}{\omega}\left|\int_{-\infty}^{+\infty}e^{-i\omega t}\left(\mathbf{e}\frac{d\mathbf{v}}{dt}\right)dt\right|\xrightarrow{\omega\tau_{\text{collis}}\ll1}\frac{v}{\omega}|(\mathbf{n}_i\mathbf{e}-\mathbf{n}_f\mathbf{e})|.$$

It can be verified using (2.18) and (2.19) that this expression corresponds to (3.2).

The main contribution to the result of the integration with respect to ε in (3.2) is made at $n>1$ (the case $n=1$ will be considered separately) by the region $\varepsilon\lesssim2\gamma n^{-1}\sin\alpha\times\cos\varphi$ at not too small $\sin\alpha$, namely, at

$$\operatorname{tg}\alpha\gg(n/2\gamma)^{1/2}. \tag{3.3}$$

We have then approximately

$$\sigma_n(\alpha)=a^2\int_0^{2\pi}\int_1^{1/\xi}J_n^2\left(\frac{|2\gamma\sin\alpha\cos\varphi|}{\varepsilon}\right)\varepsilon\,d\varepsilon\,d\varphi. \tag{3.4}$$

We put $\tilde\varepsilon=2\gamma n^{-1}\sin\alpha$ ($\tilde\varepsilon$ has the meaning of the effective eccentricity at $\varphi=0$) and stipulate satisfaction of the condition

$$\xi\tilde\varepsilon=2e\mathscr{E}_0a/n\hbar\omega\ll1. \tag{3.5}$$

Recognizing that the region $\varepsilon\sim1$ is of little importance, we can represent (3.4), by virtue of (3.5), in the form

$$\sigma_n(\alpha)=a^2\int_0^{2\pi}\cos^2\varphi\int_0^{+\infty}J_n^2\left(\frac{2\gamma\sin\alpha}{x}\right)x\,dx\,d\varphi=\pi a^2\int_0^{+\infty}J_n^2\left(\frac{2\gamma\sin\alpha}{x}\right)x\,dx,$$

or

$$\sigma_n(\alpha)=4\pi a^2\gamma^2\sin^2\alpha\int_0^{+\infty}[J_n^2(t)]''\frac{dt}{t} \tag{3.6}$$

(the prime denotes differentiation with respect to the argument). With the aid of the known expressions for the derivative of the Bessel function, the calculation of (3.6) reduces to calculation of an integral of the form

$$\int_0^{+\infty}t^{-1}J_{\nu+2k+1}(t)J_{\nu+2m+1}(t)\,dt=\begin{cases}(2\nu+k+2)^{-1}, & k=m,\quad \nu>-1\\ 0, & k\neq m,\quad \nu>-1\end{cases}. \tag{3.7}$$

The result is the sought expression for the cross section

$$\sigma_n(\alpha)=\frac{2\pi a^2\gamma^2\sin^2\alpha}{n(n^2-1)} \tag{3.8}$$

with condition (3.3) for α.

The case (3.3) of α that are not small is typical of an isotropic electron distribution in velocity, when scattering at small fixed angles α has low probability. The case $\alpha\to0$, the opposite of (3.3), can be of interest for experiments with electron beams. An estimate of the cross sections for this limit yields the relation

$$\sigma_n(\alpha=0) \sim a^2\gamma n^{-2}. \tag{3.9}$$

The cross sections (3.8) and (3.9), in contrast to the case of weak fields (see Ref. 7), are of the same order in the light intensity $\mathscr{E}_0^2$ or $\mathscr{E}_0$ for processes corresponding to different numbers n of the absorbed (emitted) photons. An increase of n leads only to a relatively weak (power-law) decrease of their probabilities. A similar result was obtained earlier by Elyutin[6] in the Born approximation. This agreement is a reflection of the fact that at low frequencies, as already noted, the probability of a transition between the field-oscillator states does not depend on the character of the scattered-particle motion. The nature of the current that excites the field (quantum or classical) is thus unimportant. This simple argument gives grounds for extending the region of validity of expressions (3.8) and (3.9) to arbitrary values of the Born parameter $\eta = Ze^2/\hbar v$, including also to $\eta \sim 1$, where it is difficult to calculate the cross section by using (3.1).

In the derivation of (3.8) and (3.9) we used the condition that the radiation field is weak compared with the atomic one

$$\mathscr{E}_0 \ll Ze\rho^{-2}, \tag{3.10}$$

where $\rho = a|\cot(\theta/2)|$ is the impact parameter, as well as the condition $\gamma \gg n/2|\sin\alpha| \gtrsim 1$ that ensures a predominant contribution to the cross section from the region $\varepsilon \gg 1$. Substituting in (3.10) the effective values of ρ, expressed in terms of $\varepsilon = 2\gamma n^{-1}|\sin\alpha| \gg 1$, and taking (3.5) into account, we obtain the final result for the region of validity of expression (3.9):

$$\frac{n\hbar\omega^2}{2ev|\sin\alpha|} \ll \mathscr{E}_0 \ll \begin{cases} \dfrac{n\hbar\omega}{2ea|\sin\alpha|} \\ (4Z)^{-1/3}e^{-5/3}\left(\dfrac{nmv\hbar\omega^2}{\sin\alpha}\right)^{1/3} \end{cases}. \tag{3.11}$$

It can be seen from (3.11) that at low frequecies ω and at not too low velocities v the width of the region in which (3.8) is realized is determined by the parameter $\omega a/v$.

A similar criterion is easily obtained also for the case $\alpha \to 0$ (3.9). The criterion (3.11) does not permit the theory to be exended to the case of arbitrarily strong fields $\mathscr{E}_0$, therefore the law governing the growth of the cross sections (3.8) and (3.9) with increase of $\mathscr{E}_0$ is valid only in a limited region. The condition (3.11), in particular, prevents a transition to the q-strong fields $2e\mathscr{E}_0/2m\omega v \equiv q \gg 1$ considered in Ref. 6 in the Born approximation, where $\sigma_n(\mathscr{E}_0) \sim \mathscr{E}_0^{-1}$. The reason this result was obtained in Ref. 6 is the inelasticity of the scattering of the radiating (absorbing) electron, an inelasticity not accounted for in the semiclassical calculation used in that reference. A possible way of taking into account the scattering inelasticity for classical motion is by the procedure of symmetrizing relative to the initial and final momenta of the electron, used, for example, in the theory of Coulomb excitation of nuclei (see the discussion of this question in Ref. 7).

We consider now one-photon emission ($n = 1$). In the case (3.3) of greatest practical interest, the integrand in (3.4) decreases like $1/\varepsilon$ at $\varepsilon > \bar{\varepsilon}$ and the integral with respect to ε diverges logarithmically. It can be cut off at $\varepsilon_{\text{eff}} \sim \xi^{-1}$, beyond which the expression in the argument of the Bessel function in (3.1) begins to decrease exponentially.

Thus

$$\sigma_1(\alpha) = a^2 \int_0^{2\pi}\int_1^{\xi^{-1}} J_1{}^2\left(\frac{|2\gamma\sin\alpha\cos\varphi|}{\varepsilon}\right)\varepsilon\, d\varepsilon\, d\varphi, \tag{3.12}$$

$$\operatorname{tg}\alpha \gg (2\gamma)^{-1/2}.$$

An approximate calculation can be carried out by breaking up the integral with respect to ε into two and using in each the appropriate approximations

$$\sigma_1(\alpha) = a^2 \int_0^{2\pi}\Big[\frac{1}{\pi}|2\gamma\sin\alpha\cos\varphi|^{-1}\int_1^{2\gamma|\sin\alpha\cos\varphi|}\varepsilon^2\, d\varepsilon + (\gamma\sin\alpha\cos\varphi)^2\int_{2\gamma|\sin\alpha\cos\varphi|}^{\xi^{-1}}\frac{d\varepsilon}{\varepsilon}\Big]d\varphi. \tag{3.13}$$

An estimate yields

$$\sigma_1(\alpha) = a^2\gamma^2\sin^2\alpha\left(\ln|2\gamma\xi\sin\alpha|^{-1} + \frac{\pi}{2}\right). \tag{3.14}$$

The region of validity of (3.12) follows from (3.10) in which we put $\varepsilon_{\text{eff}} \approx \sim \xi^{-1}$, and also from allowance for the conditions that the field be strong, $\gamma \gg |2\sin\alpha|^{-1}$ and that the collisions be rapid, $2\gamma\xi\sin\alpha \ll 1$:

$$\frac{\hbar\omega^2}{ev|\sin\alpha|} \ll \mathscr{E}_0 \ll \begin{cases} Ze\left(\dfrac{\omega}{v}\right)^2, \\ \left(\dfrac{m\omega v}{e}\right)\tilde{\eta}^{-1}, \end{cases} \tag{3.15}$$

where $\tilde{\eta} = |2\sin\alpha|^{-1}Ze^2/\hbar v$.

The ratio of the first expression to the second in the right-hand side of (3.13) is equal to $\xi\tilde{\eta}$. At $\tilde{\eta} > \xi^{-1}$, the more stringent condition is the first:

$$\hbar\omega^2/ev|\sin\alpha| \ll \mathscr{E}_0 \ll Ze(\omega/v)^2,$$

which is incontrovertible by virtue of the initial criterion $\eta \gg 1$. At $\tilde{\eta} < \xi^{-1}$ the second condition of (3.13) is significant:

$$\frac{\hbar\omega^2}{ev|\sin\alpha|} \ll \mathscr{E}_0 \ll \left(\frac{m\omega v}{e}\right)\tilde{\eta}^{-1}.$$

The condition for its certainty reduces to the order of the initial criteria $\hbar\omega \ll mv^2/2$.

For weak fields corresponding to $\gamma \ll 1$ we get from (3.14) the known result[7,12]

$$\sigma_1(\alpha) = \pi a^2\gamma^2\sin^2\alpha\ln\xi^{-1}. \tag{3.16}$$

All the cross sections of the one-photon processes have a characteristic logarithmic structure and differ only in the forms of the logarithms. It is convenient to represent these results in tabular form (Table I).

It can be seen from Table I that in the case of a strong field the cross section has a logarithmic dependence on the field strength and differs from the Born cross section by a factor $Ze^2/\hbar v$ under the logarithm sign, just as in the case of a weak field.

A large contribution to the cross section was made, when calculating the single-photon SDIB process, by the

TABLE I. Logarithmic factors in the cross sections for single-photon processes.

η	$\nu \ll 1$	$\nu \gg 1$
$\eta \ll 1$	$\ln \eta\xi^{-1} = \ln (mv^2/\hbar\omega)$ Born logarithm	$\ln \gamma\eta\xi^{-1} = \ln (e\mathscr{E}_0 mv^3/\hbar^2\omega^3)$ [6]
$\eta \gg 1$	$\ln \xi^{-1} = \ln (mv^3/Ze^2\omega)$ classical logarithm	$\ln \gamma\xi^{-1} = \ln (e\mathscr{E}_0 mv^4/Ze^2\hbar\omega^3)$ present paper

region of small scattering angles, corresponding to $\varepsilon \sim \xi^{-1}$, i.e., angles at which $\omega\tau_{\text{collis}} \sim 1$. The transition probability in this region, and with it the cross section, now depends on the character of the current, and this is in fact the reason why logarithms are contained in the first and second (reading downward) lines of rows of Table I. As $\omega \to 0$ the upper limit in (3.12) increases, causing an increase in the contribution from the region of ε in which $\omega\tau_{\text{collis}} \ll 1$, and consequently, a decrease of the relative difference between the Born and classical logarithms, as seen from the table.

2. The limit $\xi\varepsilon \gg 1$—slow collisions

At $\varepsilon \gg 1$, expression (3.1) becomes

$$d\sigma_n(\alpha) = a^2 J_n^2 (2\gamma\xi e^{\pi\xi/2}(1-\sin^2\varphi\cos^2\alpha)^{1/2} K_{i\xi}(\xi\varepsilon))\,\varepsilon\, d\varepsilon\, d\varphi. \tag{3.17}$$

Using the known asymptotic expression for $K_{i\xi}$

$$K_{i\xi}(\xi\varepsilon) \sim \left(\frac{2}{\pi\xi\varepsilon}\right)^{1/2} \exp(-\xi\varepsilon) \quad \text{for}$$

$$\xi\varepsilon \gg 1, \quad \xi \lesssim 1 \quad \text{or} \quad \varepsilon \gg 1, \quad \xi \gtrsim 1, \tag{3.18}$$

and introducing the variable $x = be^{-\xi\varepsilon}/(\xi\varepsilon)^{1/2}$, where

$$b = \left(\frac{8}{\pi}\right)^{1/2} \gamma\xi e^{\pi\xi/2}(1-\sin^2\varphi\cos^2\alpha)^{1/2}$$
$$= \left(\frac{8}{\pi}\right)^{1/2} e^{\pi\xi/2} \frac{e\mathscr{E}_0 a}{\hbar\omega}(1-\sin^2\varphi\cos^2\alpha)^{1/2}, \tag{3.19}$$

we represent the integral cross section in the form

$$\sigma_n(\alpha) = \frac{a^2}{\xi^2}\int_0^{2\pi}\left[\int_0^{be^{-1}} J_n^2(x)\ln\left(\frac{b}{x}\right)\frac{dx}{x}\right] d\varphi. \tag{3.20}$$

This expression can in turn be further simplified if

$$x_{\text{eff}} \sim n \ll be^{-1}, \tag{3.21}$$

when the slowly varying logarithm can be taken outside the integral sign with the value $x \sim x_{\text{eff}}$. Calculation of the integral in accordance with (3.8) yields

$$\sigma_n(\alpha) = \frac{a^2}{2n\xi^2}\int_0^{2\pi} \ln\left(\frac{b}{n}\right) d\varphi.$$

From the weak dependence of the integrand on φ and α at the considered values of α that are not small, we obtain the estimate

$$\sigma_n(\alpha) = \frac{\pi v^2}{2n\omega^2}\left(\frac{\pi\xi}{2} + \ln\frac{e\mathscr{E}_0 a}{n\hbar\omega}\right). \tag{3.22}$$

This cross section depends very weakly on the field—saturation of sorts takes place in the continuous spectrum.

Let us write down the conditions for the validity of our result. Obtaining from x_{eff} in (3.21) the value $\varepsilon_{\text{eff}} \sim \xi^{-1}\ln(b/n)$ and substituting it in (3.18), we get

$$\ln(b/n) \gg 1, \quad \xi \lesssim 1 \quad \text{or} \quad \ln(b/n) \gg \xi, \quad \xi \gtrsim 1,$$

which can be combined into a single condition $b \gg ne^{\xi}$, or

$$\mathscr{E}_0 \gg \left(\frac{\pi}{8}\right)^{1/2} \frac{n\hbar\omega}{ea} \exp\left\{\xi\left(1-\frac{\pi}{2}\right)-1\right\}.$$

Knowledge of ε_{eff} enables us also to confirm the condition (3.10) that the external field be weak compared with the atomic field, so that the complete validity condition is

$$\left(\frac{\pi}{8}\right)^{1/2} \frac{n\hbar\omega}{ea} \exp\left\{\xi\left(1-\frac{\pi}{2}\right)-1\right\} \ll \mathscr{E}_0 \ll \frac{Ze\omega^2}{v^2(\pi\omega a/v + \ln(e\mathscr{E}_0 a/\hbar\omega))}, \tag{3.23}$$

and the condition that it be incontrovertible is

$$n \ll \frac{Ze^2}{\hbar v}\left(1 + \frac{v}{\pi\omega a}\ln\left(\frac{e\mathscr{E}_0 a}{n\hbar\omega}\right)\right)^{-1}. \tag{3.24}$$

§4. MULTIPHOTON PROCESSES IN A PLANCK RADIATION FIELD

We consider multiphoton processes in a non-single-mode monochromatic external radiation field. As an example of physical interest, we use a Planck thermal-radiation field.

Known results for spontaneous multiphoton processes were already cited in §2 above. They can set in because the smallness of the QED parameter $e^2/\hbar c$ is offset by the logarithmically large average number of emitted photons. This can be seen directly from the structure of the probability W_n of emission of low-frequency phonons. In fact, at $\omega\tau_{\text{collis}} \ll 1$ we have[9]:

$$W_n(\omega_1,\omega_2) = \frac{(\omega_2/\omega_1)^{-\beta}}{n!}\left(\beta\ln\left(\frac{\omega_2}{\omega_1}\right)\right)^n,$$

$$\beta \sim e^2/\hbar c, \quad \bar{n} = \sum_{n=1}^{\infty} nW_n(\omega_1,\omega_2) = \beta\ln\left(\frac{\omega_2}{\omega_1}\right). \tag{4.1}$$

Since the emission and absorption probabilities for an individual oscillator of the continuous spectrum are proportional to $(x_\omega/V)^n$, the emission (absorption) will be single-photon for this oscillator, since $W_{n+1}/W_n \to 0$ as $V \to \infty$. The emission, however, from the aggregate of all oscillators in a given frequency interval need not necessarily be single-photon. This is seen already with (2.14) as an example: depending on the number of field oscillators considered (on the width of the frequency interval), the value n_{max} at which W_n is a maximum can take on different values.

Equilibrium Planck radiation is characterized by a Gibbs probability distribution W_s^G of finding s photons (prior to the scattering) in an oscillator of frequency ω at an emission temperature T_P:

$$W_s^G=(1-\exp(-\hbar\omega/T_P))\exp(-s\hbar\omega/T_P).$$

The probability $\widetilde{W}_n^e$ of emission by an oscillator is obtained by weighting over this distribution the probabilities (26) of exciting this oscillator:

$$\widetilde{W}_n^e=\sum_{s=0}^{\infty}W_{s,n}^{l}W_s^G=(1-\exp(-\hbar\omega/T_P))\sum_{s=0}^{\infty}\frac{s!}{(s+n)!}$$
$$\times e^{-\hbar\omega/T_P}\left(\frac{x_k}{V}\right)^n\left[L_s^n\left(\frac{x_k}{V}\right)\right]^2\exp\left(-\frac{x_k}{V}\right)$$
$$=(x_k\exp(-\hbar\omega/T_P))^{-n}I_n\left(\frac{2x_k\exp(-\hbar\omega/T_P)}{V(1-\exp(\hbar\omega/T_P))}\right)$$
$$\times\exp\left(-\frac{1+\exp(-\hbar\omega/T_P)}{1-\exp(-\hbar\omega/T_P)}\frac{x_k}{V}\right)$$

where $W_{s,n}^e$ is the probability that the field oscillator will emit n photons if their initial number is s. For a distinct separation of the parts that are small as $V\to\infty$, we take x_k hereafter to mean Eq. (2.7) multiplied by V. The same transformation applies also to y defined earlier by (2.15).

As $V\to\infty$ we have

$$\widetilde{W}_n^e=\frac{1}{n!}\left(\frac{x_\omega}{V(1-\exp(-\hbar\omega/T_P))}\right)^n$$
$$\times\exp\left(-\frac{1-\exp(-\hbar\omega/T_P)}{1+\exp(-\hbar\omega/T_P)}\right),\qquad(4.2)$$

i.e., it is a Poisson distribution. In analogy with the derivation of (2.14), we obtain for the frequency interval (ω_1,ω_2) the photon-emission probability $W_n^e(\omega_1,\omega_2)$:

$$W_n^e(\omega_1,\omega_2)=\frac{1}{n!}\left(\int_{\omega_1}^{\omega_2}\frac{y_\omega\,d\omega}{1-\exp(-\hbar\omega/T_P)}\right)^n$$
$$\times\exp\left(-\int_{\omega_1}^{\omega_2}\frac{1+\exp(-\hbar\omega/T_P)}{1-\exp(-\hbar\omega/T_P)}y_\omega\,d\omega\right).\qquad(4.3)$$

For absorption, similarly,

$$\widetilde{W}_n^a=\sum_{s=n}^{\infty}W_{l,n}^aW_s^G=(1-e^{-\hbar\omega/T_P})\sum_{s=n}^{\infty}e^{-s\hbar\omega/T_P}\frac{(s-n)!}{s!}$$
$$\times\left(\frac{x_k}{V}\right)^n\left[L_{s-n}^n\left(\frac{x_k}{V}\right)\right]^2e^{-x_k/V}$$
$$=(1-\exp(-\hbar\omega/T_P))\,c^{-n\hbar\omega/T_P}\sum_{s=0}^{\infty}e^{-s\hbar\omega/T_P}\frac{s!}{(s+n)!}$$
$$\times\left(\frac{x_k}{V}\right)^n\left[L_s^n\left(\frac{x_k}{V}\right)\right]^2e^{-x_k/V}=e^{-n\hbar\omega/T_P}\widetilde{W}_n^e,$$
$$(4.4)$$

where $W_{l,n}^a$ is the probability that the oscillator will absorb n photons out of a finite number $l=s-n$.

Taking (4.2) into account, we obtain from (4.4)

$$\widetilde{W}_n^a=\frac{1}{n!}\left(\frac{x_k\exp(-\hbar\omega/T_P)}{V(1-\exp(-\hbar\omega/T_P))}\right)^n$$
$$\times\exp\left(-\frac{1+\exp(-\hbar\omega/T_P)}{1-\exp(-\hbar\omega/T_P)}\right).\qquad(4.5)$$

The absorption probability integrated over the interval (ω_1,ω_2) is

$$W_n^a(\omega_1,\omega_2)=\frac{1}{n!}\left(\int_{\omega_1}^{\omega_2}\frac{y_\omega\exp(-\hbar\omega/T_P)}{1-\exp(-\hbar\omega/T_P)}d\omega\right)^n$$
$$\times\exp\left(-\int_{\omega_1}^{\omega_2}\frac{1+\exp(-\hbar\omega/T_P)d\omega}{1-\exp(-\hbar\omega/T_P)}\right).\qquad(4.6)$$

Now, when the initial occupation numbers of the oscillators are not zero, both emission and absorption is possible. The observed n-photon process (we refer for the sake of argument to emission) is an aggregate consisting of emission of $n+k$ and absorption of k photons, where k assumes all possible values from zero to infinity. Since emission and absorption in a frequency interval are statistically independent, the actually observed probabilities w_n^e and w_n^a are determined by the set of two equations

$$w_n^e=\sum_{k=0}^{\infty}W_{n+k}^e(\omega_1,\omega_2)W_k^a(\omega_1,\omega_2)$$
$$\times\left(w_0+\sum_{p=1}^{\infty}(w_p^e+w_p^a)\right)^{-1},\qquad(4.7)$$

$$w_n^a=\sum_{k=0}^{\infty}W_{n+k}^a(\omega_1,\omega_2)W_k^e(\omega_1,\omega_2)\left(w_0+\sum_{p=1}^{\infty}(w_p^e+w_p^a)\right)^{-1}$$
$$(4.8)$$

with appropriate normalization, where $w_0=w_0^e=w_0^a$.

The number of terms that contribute to (4.7) and (4.8) depends at a given (ω_1,ω_2) on the ratio $\hbar\omega/T_P$. As $T_{P\to0}\to0$, contribution is made only by single-photon processes corresponding to the first terms of these sums. As $T_P\to\infty$ the contribution of the multistep induced emission or absorption processes that determine mainly the process increases sharply. This causes the resultant probability to deviate from a pure Poisson probability.

Indeed, solution of this system yields

$$w_n^e=Y^nI_n(X)\exp(-{}^1/_2X(Y+Y^{-1})),\qquad(4.9)$$
$$w_n^a=Y^{-n}I_n(X)\exp(-{}^1/_2X(Y+Y^{-1})),\qquad(4.10)$$

where I_n is a modified Bessel function of order n, and the arguments X and Y are given by

$$X=\left\{\int_{\omega_1}^{\omega_2}\frac{y_\omega\exp(-\hbar\omega/T_P)d\omega}{1-\exp(-\hbar\omega/T_P)}\int_{\omega_1}^{\omega_2}\frac{y_\omega\,d\omega}{1-\exp(-\hbar\omega/T_P)}\right\}^{1/2},$$
$$(4.11)$$
$$Y=\left\{\int_{\omega_1}^{\omega_2}\frac{y_\omega\exp(-\hbar\omega/T_P)d\omega}{1-\exp(-\hbar\omega/T_P)}\left[\int_{\omega_1}^{\omega_2}\frac{y_\omega\,d\omega}{1-\exp(-\hbar\omega/T_P)}\right]^{-1}\right\}^{1/2}.$$

These expressions are a new result, viz., generalization of the Poisson equation for spontaneous bremsstrahlung to include the case of induced emission and absorption in an external equilibrium thermal radiation field. The Poisson equation is obtained from (4.9) in the limit as $T_P \to 0$ (there are no thermal photons), while (4.10) yields zero in this case, as it should.

Using the known relation

$$\sum_{n=1}^{\infty} y^n I_n(x) = \frac{1}{2}\left(\exp\left\{\frac{x}{2}(y+y^{-1})\right\} - I_0(x)\right), \qquad (4.12)$$

we can calculate also such quantities as the mean number $\bar{n}$ and the mean squared number $\overline{n^2}$ of the emitted photons, and the variance that characterizes the given distibution

$$\bar{n} = \sum_{n=1}^{\infty} n w_n{}^e = \sum_{n=1}^{\infty} n Y^n I_n(X) \exp\left(-\frac{X}{2}(Y+Y^{-1})\right)$$

$$= Y e^{-X(Y+Y^{-1})} \frac{d}{dY}\left(\sum_{n=1}^{\infty} Y^n I_n(X)\right).$$

Using (4.12) we get

$$\bar{n} = \frac{X}{2}(Y-Y^{-1}). \qquad (4.13)$$

Similarly,

$$\overline{n^2} = \sum_{n=1}^{\infty} n^2 w_n{}^e = \left(\frac{X}{2}\right)^2 (Y-Y^{-1}) + \frac{X}{2}(Y+Y^{-1}). \qquad (4.14)$$

From this we get an expression for the variance $D(n) = \overline{(n-\bar{n})^2} = \overline{n^2} - \bar{n}^2$

$$D(n) = \frac{X}{2}(Y-Y^{-1}). \qquad (4.15)$$

We rewrite n, n^2, and $D(n)$ in standard notation

$$\bar{n} = D(n) = \frac{1}{2}\int_{\omega_1}^{\omega_2} \frac{1+\exp(-\hbar\omega/T_P)\,d\omega}{1-\exp(-\hbar\omega/T_P)}, \qquad (4.16)$$

$$\overline{n^2} = \frac{1}{2}\int_{\omega_1} \frac{1+\exp(-\hbar\omega/T_P)}{1-\exp(-\hbar\omega/T_P)}$$

$$\times y_\omega\, d\omega \left(1 + \frac{1}{2}\int_{\omega_1}^{\omega_2} \frac{1+\exp(-\hbar\omega/T_P)}{1-\exp(-\hbar\omega/T_P)} y_\omega\, d\omega\right). \qquad (4.17)$$

As $T_P \to 0$ the variance, and with it also n (4.16), go over into the usual results for spontaneous emission.[9] As $T_P \to \infty$, the distribution width increases strongly:

$$D(n) \approx T_P \int_{\omega_1}^{\omega_2} \frac{y_\omega\, d\omega}{\hbar\omega}, \qquad (4.18)$$

corresponding obviously to an increase of the probability of production of many photons. The normalization of the distribution, however, remains the same as before.

§5. CALCULATION OF THE ABSORBED ENERGY

We calculate the change, in one particle-scattering act, of the total energy of a field having a continuous spectrum. It is expedient to use for this calculation the S-matrix formalism, see §2.

Using the operator $\hat{S}$, we express the field energy E_f after scattering in the form

$$E_f = \langle i|\hat{S}^{-1}\left(\sum_{\mathbf{k},\lambda} \hbar\omega \hat{a}^+_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}\right)\hat{S}\,|i\rangle. \qquad (5.1)$$

With the aid of relations (2.4) and (2.5) we get

$$E_f = \langle i|\sum_{\mathbf{k},\lambda}\hbar\omega\left[\hat{a}^+_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + ie\left(\frac{2\pi}{\hbar\omega V}\right)^{1/2}\int_{-\infty}^{+\infty}(\mathbf{ev}(t))e^{-i\omega t}\,dt\right.$$

$$-ie\left(\frac{2\pi}{\hbar\omega V}\right)^{1/2}\int_{-\infty}^{+\infty}(\mathbf{ev}(t))e^{i\omega t}\,dt$$

$$\left.+\frac{2\pi e^2}{\omega\hbar V}\left|\int_{-\infty}^{+\infty}(\mathbf{ev}(t))e^{-i\omega t}\,dt\right|^2\right]|i\rangle. \qquad (5.2)$$

Elementary operations with $\hat{a}^+_{\mathbf{k},\lambda}$ and $\hat{a}_{\mathbf{k},\lambda}$ yield the calculation result:

$$E_f = \sum_{\mathbf{k},\lambda}\hbar\omega n_{\mathbf{k},\lambda} + \sum_{\mathbf{k},\lambda}\frac{2\pi e^2}{V}\left|\int_{-\infty}^{+\infty}(\mathbf{ev}(t))e^{-i\omega t}\,dt\right|^2. \qquad (5.3)$$

The first term in (5.3) is, obviously, the initial energy E_i of the field, and the second is the sought difference in energy before and after the scattering, $\Delta E = E_f - E_i$, and equals the already known spontaneous-emission energy. In particular, this result holds also for a Planck radiation field, as can be verified directly by calculating the sum

$$\sum_{n=-\infty}^{n=+\infty} n\hbar\omega(w_n{}^e - w_n{}^a)$$

with the aid of (4.12).

To determine the energy acquired by the electron, we must take the kinetics of the processes into account. In the case of scattering of many classical electrons having a distribution function $f(\mathscr{E})$, the expression for the energy increment ΔE is

$$\Delta E = N\hbar\omega v\, d\mathscr{E}\, d\sigma_{\text{res}} \sum_{n=1}^{\infty} n(w_n{}^e f(\mathscr{E}-n\hbar\omega)$$

$$-w_n{}^a f(\mathscr{E}+n\hbar\omega))\Delta t, \qquad (5.4)$$

where $\mathscr{E} = \frac{mv^2}{2}$ and $\mathscr{E} \gg \hbar\omega$.

Here w^e_n and w^a_n are taken for that frequency interval in which ΔE is calculated. It must not be very large, viz., such that the integrands in the expressions for w^e_n and w^a_n not change too much in it, so that photon emission from even the opposite ends of the interval (ω_1,ω_2) has high probabilities, in accordance with the meaning of the differential description of ΔE.

To this end we must have

$$\omega_2 - \omega_1 \ll g(\omega)/g'(\omega)\,|_{\omega=\omega_1}, \qquad (5.5)$$

where g is one of the functions $y_\omega/\{1-\exp(-\hbar\omega/T_P)\}$ or $y_\omega\exp(-\hbar\omega/T_P)/\{1-\exp(-\hbar\omega/T_P)\}$, for which the ratio $g(\omega)/g'(\omega)|_{\omega=\omega_1}$ is a minimum.

At high temperatures the condition (5.5) ceases to depend on T_P, and the expression in the argument of the modified Bessel function becomes proportional to T_P. If the interval (ω_1,ω_2) that can be considered is fixed (is not too small), then w_n^e and w_n^a are no longer Poisson probabilities, i.e., in accordance with the foregoing, they do not describe processes that are multiphoton in (ω_1,ω_2).

Expanding $f(\mathscr{E}-n\hbar\omega)$ up to terms of first order of smallness, substituting the probabilities, and regrouping the corresponding terms, we have

$$dE=\Big[N\hbar\omega f(\mathscr{E})\exp\Big\{-\frac{X}{2}(Y+Y^{-1})\Big\}\sum_{n=1}^{\infty}(nY^nI_n(X)$$
$$-nY^{-n}I_n(X))$$
$$-Nf(\mathscr{E})\exp\Big\{-\frac{X}{2}(Y+Y^{-1})\Big\}(\hbar\omega)^2\frac{\partial f}{\partial\mathscr{E}}\sum_{n=1}^{\infty}n^2$$
$$\times I_n(X)(Y^n+Y^{-n})\Big]v\,d\mathscr{E}\,d\sigma\,\Delta t. \qquad (5.6)$$

Calculating the sums in (5.6) by elementary methods and returning to the original notation, we get

$$\Delta E=N\hbar\omega v\,d\mathscr{E}\,d\sigma_{\text{pe3}}\Big(f(\mathscr{E})\int_{\omega_1}^{\omega_2}y_\omega\,d\omega+\frac{\partial f}{\partial\mathscr{E}}\hbar\omega$$
$$\times\Big(\int_{\omega_1}^{\omega_2}\frac{1+e^{-\hbar\omega/T_\Pi}}{1-e^{-\hbar\omega/T_\Pi}}y_\omega\,d\omega+\frac{1}{2}\Big[\int_{\omega_1}^{\omega_2}y_\omega\,d\omega\Big]^2\Big)\Big)\Delta t. \qquad (5.7)$$

Changing to a small interval $d\omega$, we have

$$\frac{dE}{d\omega}=Nv\,d\mathscr{E}\,d\sigma_{\text{pe3}}\Big(f(\mathscr{E})\hbar\omega y_\omega+\hbar^2\frac{\partial f}{\partial\mathscr{E}}\frac{1+e^{-\hbar\omega/T_\Pi}}{1-e^{-\hbar\omega/T_\Pi}}y_\omega\omega^2\Big)dt. \qquad (5.8)$$

At large $T_P\gg\hbar\omega$ and also $\omega\tau_{\text{colliss}}\ll1$,

$$\frac{dE}{d\omega\,dt}=\Big(\frac{Ze^2}{mv^2}\Big)^2v\,d\mathscr{E}\,d\sigma\Big(f(\mathscr{E})+2T_\Pi\frac{\partial f}{\partial\mathscr{E}}\Big)\frac{2e^2v^2}{3\pi c^3}. \qquad (5.9)$$

This result agrees with the classical energy acquisition by an electron in a Planck field, see Ref. 5.

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