S matrix and resonance states in the Kerr-Newman geometry

A. B. Gaina and F. G. Kochorbé

Kishinev Polytechnic Institute
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A unified approach to the description of scattering processes and resonance states for the Klein-Gordon equation in the field of a black hole is developed on the basis of Jost's method. Some general definitions and properties of the *S* matrix, amplitudes, and differential and total absorption and scattering cross sections are obtained for the axisymmetric Kerr-Newman configuration. Concrete results (scattering and absorption cross sections and the characteristics of resonance levels of massive particles) are obtained for the long-wave case.

The problem of the scattering of particles and fields of various spins by a black hole with phenomenological specification of the boundary conditions on the event horizon has been considered in many studies (see, for example, Refs. 1–3, and also the bibliography of Ref. 3). In particular, the partial-wave (and, in some cases, the differential and total) cross sections for absorption and scattering have been calculated in the long-wave approximation ($R_G \ll \lambda$) for massless and massive particles. The studies of Refs. 4–9 proved the existence of quasibound resonance (including superradiant) states for massive particles, and their characteristics—energy spectrum, damping, and excitation—were calculated in some important special cases. The method of Refs. 4–9 is based on the use of analytic and WKB solutions of the wave equations for $E<\mu c^2$.

In the present paper, we consider the scattering problem for a massive scalar field in the field of a rotating and charged Kerr-Newman black hole. We generalize the approach of de Alfaro and Regge,[10] based on Jost's method, for the case of curved space-time. We construct the *S* matrix for states $E>\mu c^2$; its analytic continuation into the region $E<\mu c^2$ makes it possible to find the resonance states as its poles. Thus, we develop a unified formalism for describing the problems of scattering and resonance states in black hole fields. The concrete results (scattering and absorption cross sections and the characteristics of the resonance states) generalize all the results obtained earlier in Refs. 1–3, 5, 7, and 8 in the approximation

$$GEM/\hbar c^3 \sim |eQ|/\hbar c \ll 1, \quad J \leqslant GM^2/c^2 \tag{1}$$

(M and Q are the mass and charge of the black hole, $J=Mac$ is its intrinsic angular momentum, and E, μ, and e are the energy, mass, and charge of the particles).

1. The behavior of a spinless particle in the Kerr-Newman geometry[11] is described by the Klein-Gordon-Fock equation

$$(D_\mu D^\mu+\mu^2)\Phi=0, \tag{2}$$

where

$$D_\mu=\partial_\mu+ieA_\mu, \quad A_\mu=(-Qr/\Sigma,\ 0,\ 0,\ Qra\sin^2\theta/\Sigma). \tag{3}$$

Here and in what follows, we simplify the expressions by using a system of units in which $\hbar=C=G=1$. In separated variables, the solution of Eq. (2) is the wave function of a particle in a stationary state with energy ω, angular momentum projection m, and square of the generalized angular momentum λ^2:

$$\Phi_{\omega l m}(t,r,\theta,\varphi)=\frac{1}{r}R_{\omega l m}(r)Z_l^m(\theta,\varphi)e^{-i\omega t}, \tag{4}$$

where $Z_l^m(\theta,\varphi)$ are spheroidal harmonics,

$$Z_l^m(\theta,\varphi)=\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}S_l^m(a(\omega^2-\mu^2)^{1/2},\cos\theta)e^{im\varphi}, \tag{5}$$

normalized as follows:

$$\int Z_l^m(\theta,\varphi)Z_{l'}^{m'}(\theta,\varphi)\,d\Omega=\delta_{ll'}\delta_{mm'}, \tag{6}$$

and $S_l^m(a(\omega^2-\mu^2)^{1/2},\cos\theta)$ are spheroidal functions (see Ref. 12), satisfying the equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left(\lambda^2-\frac{m^2}{\sin^2\theta}-a^2(\omega^2-\mu^2)\sin^2\theta\right)S=0. \tag{7}$$

The eigenvalues $\lambda^2=\lambda^2_{lm}(a)$ cannot be expressed analytically in terms of the numbers l and m.[12] If $a(\omega^2-\mu^2)^{1/2}=0$, we obtain

$$\lambda^2=l(l+1),\quad S_l^m=P_l^m(\cos\theta),\quad Z_l^m(\theta,\varphi)=Y_l^m(\theta,\varphi).$$

The equation for the radial part of the wave function (4) takes the form of the one-dimensional Schrödinger equation

$$d^2R/dr^{*2}+W_l^m(r^*)R=0, \tag{8}$$

where r^* is the "tortoise" coordinate:

$$\frac{dr^*}{dr}=\frac{r^2}{\Delta},$$

$$r^*=r+\frac{1}{r_+-r_-}\left\{r_+^2\ln\left(\frac{r}{r_+}-1\right)-r_-^2\ln\left(\frac{r}{r_-}-1\right)\right\}, \tag{9}$$

and the function W_l^m has the form

$$W_l^m=\left(\frac{\mathcal{K}}{r^2}\right)^2-\frac{\Delta}{r^2}\left[\mu^2+\frac{\lambda^2-2am\omega+\mu^2a^2}{r^2}+\frac{1}{r}\frac{d}{dr}\left(\frac{\Delta}{r^2}\right)\right],$$

$$\mathcal{K}(r)=(r^2+a^2)\omega-am+eQr,\quad r_\pm=M\pm(M^2-a^2-Q^2)^{1/2}. \tag{10}$$

We consider three types of exact solutions of the radial equation (8):

$$R_{lm}^{(1)}(\tau,k,r^*)=A_{lm}(\tau,k,r^*)e^{-i\tau r^*}+B_{lm}(\tau,k,r^*)e^{i\tau r^*}, \tag{11a}$$

$$R_{lm}^{(2)}(\tau,k,r^*)=C_{lm}(\tau,k,r^*)e^{-ikr^*}+D_{lm}(\tau,k,r^*)e^{ikr^*}, \tag{11b}$$

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$$R_{lm}^{(3)}(\tau,k,r^*)=2E_{lm}(\tau,k,r^*)\sin[kr^*-\pi l/2+\delta_{lm}(\tau,k)], \quad (11c)$$

where

$$\tau=\frac{r_+^2+a^2}{r_+^2}(\omega-\omega_0), \quad k=(\omega^2-\mu^2)^{1/2}=\omega v, \quad \omega_0=m\Omega_H-eV_H,$$

$$\Omega_H=a/(r_+^2+a^2), \quad V_H=Qr_+/(r_+^2+a^2), \quad (12)$$

and Ω_H and V_H are the angular velocity of the rotation and the electric potential of the black hole.

Using the constancy of the Wronskian for the solutions (11a) and (11c), we calculate the Wronskian in the limits $r^*\to\pm\infty$:

$$R_{lm}^{(1)}(\tau,k,r^*)\underset{r^*\to-\infty}{\longrightarrow}A_{lm}(\tau,k)e^{-i\tau r^*}+B_{lm}(\tau,k)e^{i\tau r^*},$$

$$R_{lm}^{(3)}(\tau,k,r^*)\underset{r^*\to+\infty}{\longrightarrow}2\sin(kr^*-\pi l/2+\delta_{lm}(\tau,k)).$$

We obtain the relation

$$2k\sinh(2\,\mathrm{Im}\,\delta_{lm}(\tau,k))=\tau(|A_{lm}|^2-|B_{lm}|^2). \quad (13)$$

It can be seen that the phase shift is real only under the condition

$$|A_{lm}(\tau,k)|=|B_{lm}(\tau,k)|. \quad (14)$$

But if we require fulfillment in the limit $r^*\to-\infty$ of the condition of capture of the wave by the trapped surface of the event horizon, we obtain

$$2\sinh[2\,\mathrm{Im}\,\delta_{lm}(\tau,k)]=\tau k^{-1}|A_{lm}(\tau,k)|^2. \quad (15)$$

Thus, allowance for the absorption of particles by the black hole leads to a complex phase shift, and the behavior of the coefficient functions will be as follows:

$$\lim_{r^*\to-\infty}A_{lm}=1, \quad \lim_{r^*\to-\infty}B_{lm}=0, \quad \lim_{r^*\to+\infty}C_{lm}=1,$$

$$\lim_{r^*\to+\infty}D_{lm}=0, \quad \lim_{r^*\to+\infty}E_{lm}=1. \quad (16)$$

It can be seen from the general form of the function (10) that

$$R_{lm}^{(1)}(\tau,-k,r^*)=R_{lm}^{(1)}(\tau,k,r^*),$$
$$R_{lm}^{(2)}(-\tau,k,r^*)=R_{lm}^{(2)}(\tau,k,r^*).$$

For the radial equation (8) there exist at least two pairs of linearly independent solutions,

$$R_{lm}^{(1)}(\tau,k,r^*), \quad R_{lm}^{(1)*}(\tau,k,r^*);$$

$$R_{lm}^{(2)}(\tau,k,r^*), \quad R_{lm}^{(2)*}(\tau,k,r^*),$$

for which the relations

$$R_{lm}^{(1)*}(\tau,k,r^*)=R_{lm}^{(1)}(-\tau,k,r^*),$$
$$R_{lm}^{(2)*}(\tau,k,r^*)=R_{lm}^{(2)}(\tau,-k,r^*), \quad (17)$$

hold. Among these four solutions, any three are linearly dependent:

$$R_{lm}^{(1)}(\pm\tau,k,r^*)=\alpha^{(\pm)}R_{lm}^{(2)}(\tau,k,r^*)+\beta^{(\pm)}R_{lm}^{(2)}(\tau,-k,r^*),$$

$$R_{lm}^{(2)}(\tau,\pm k,r^*)=\gamma^{(\pm)}R_{lm}^{(1)}(\tau,k,r^*)+\Delta^{(\pm)}R_{lm}^{(1)}(-\tau,k,r^*). \quad (18)$$

The Jost function

$$f_{lm}(\tau,k)=W[R_{lm}^{(1)}(\tau,k,r^*),\ R_{lm}^{(2)}(\tau,k,r^*)]$$
$$=R_{lm}^{(1)}(\tau,k,r^*)\frac{d}{dr^*}R_{lm}^{(2)}(\tau,k,r^*)$$
$$-\frac{d}{dr^*}R_{lm}^{(1)}(\tau,k,r^*)R_{lm}^{(2)}(\tau,k,r^*) \quad (19)$$

does not depend on r^*, since the Wronskian of two solutions of the radial equation (8) does not depend on r^*. Hence, we find the coefficients

$$\alpha^{(\pm)}=\frac{1}{2ik}f_{lm}(\pm\tau,-k), \quad \beta^{(\pm)}=-\frac{1}{2ik}f_{lm}(\pm\tau,k),$$

$$\gamma^{(\pm)}=-\frac{1}{2i\tau}f_{lm}(-\tau,\pm k), \quad \Delta^{(\pm)}=\frac{1}{2i\tau}f_{lm}(\tau,\pm k). \quad (20)$$

Equations (18) describe a number of scattering processes phenomenologically. For the solution that has the form of an incident wave on the black hole horizon and a superposition of incoming and outgoing waves at infinity, we obtain

$$R_{lm}^{(1)}(\tau,k,r^*)$$
$$=\frac{1}{2ik}[f_{lm}(\tau,-k)R_{lm}^{(2)}(\tau,k,r^*)-f_{lm}(\tau,k)R_{lm}^{(2)}(\tau,-k,r^*)]. \quad (21)$$

In the limit $r^*\to+\infty$, we have

$$R_{lm}^{(1)}(\tau,k,r^*)\sim-\frac{1}{2ik}f_{lm}(\tau,-k)[S_{lm}(\tau,k)e^{ikr^*}-e^{-ikr^*}], \quad (22)$$

where

$$S_{lm}(\tau,k)=\frac{f_{lm}(\tau,k)}{f_{lm}(\tau,-k)}e^{i\pi l}. \quad (23)$$

We choose the Jost function in the form

$$f_{lm}(\tau,k)=\exp\left\{i\delta_{lm}(\tau,k)-i\frac{\pi l}{2}\right\},$$

$$f_{lm}(\tau,-k)=\exp\left\{-i\delta_{lm}(\tau,k)+i\frac{\pi l}{2}\right\}; \quad (24)$$

then

$$S_{lm}(\tau,k)=\exp[2i\delta_{lm}(\tau,k)]. \quad (25)$$

It follows from the theory of second-order differential equations[10] that the Jost function is an analytic function, while $S_{lm}(\tau,k)$ is a meromorphic function.

The asymptotic formula (22) shows that the function $S_{lm}(\tau,k)$ transforms an individual incident wave into an outgoing wave, i.e., it is a diagonal element of the S matrix in the $|lm\rangle$ representation. From (17), (23), and (24) we readily obtain for the Jost functions and S matrix the formal relations

$$f_{lm}^*(\tau,k)=f_{lm}(-\tau,-k), \quad S_{lm}^+(\tau,k)=S_{lm}^{-1}(-\tau,k). \quad (26)$$

These relations give for the phase shifts the properties

$$\delta_{lm}^*(\tau,k)=\delta_{lm}(-\tau,k), \quad \delta_{lm}(\tau,k)=-\delta_{lm}(\tau,-k)+\pi l.$$

2. We consider the interpretation by means of the S matrix of the bound, resonance, and virtual states of a particle in the field of the Kerr-Newman black hole. If k is real ($\omega>\mu$), the normalization integral $\langle\Phi|\Phi\rangle$ diverges, i.e., there are no bound states. If k is complex ($k''=\mathrm{Im}\,k'>0$), then it follows from the constancy of the Wronskian that $\mathrm{Re}\, k = 0$. Therefore, bound states exist only under the condition $f_{lm}(\tau, ik'') = 0$, this corresponding to a pole of the S matrix:

$$S_{lm}(\tau, ik'') = \infty. \tag{27}$$

We consider the poles of the S matrix for arbitrary k ($k = k' + ik''$, $k'' < 0$). If $k' = 0$, we have a virtual state; if $k' \ll k''$, then we have a resonance state for which

$$S_{lm}(\tau, k) = \exp[2i\delta_{lm}(\tau, k)] = \frac{k - k' - ik''}{k - k' + ik''}$$

or

$$\sin^2\delta_{lm}(\tau, k) = \frac{k''^2}{(k-k')^2 + k''^2}. \tag{28}$$

Since $k = k(\omega)$, we obtain accordingly the Breit-Wigner formula for the resonance states of the quantum particle in the field of the Kerr-Newman black hole:

$$\sin^2\delta_{lm}(\tau, k) = \frac{\gamma^2}{(\omega-\omega_0)^2 + \gamma^2}, \tag{28a}$$

where

$$\omega = \omega_0 - i\gamma \quad (\gamma \ll \mu - \omega_0).$$

3. We find explicitly an expansion for the total scattering amplitude with respect to the partial-wave components and of the cross sections of elastic and inelastic scattering processes. The wave function (4) of any stationary state has the asymptotic behavior

$$\Phi \approx e^{-i\omega t} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{2A_{lm}}{r} \sin\left(kr - \frac{\pi l}{2} + \delta_{lm}(\tau, k)\right) Z_l^m(\theta, \varphi). \tag{29}$$

The asymptotic form of the wave function (4) at large distances for elastic scattering in an arbitrary field has the form[13]

$$\Psi(t, r, \theta, \varphi) \approx e^{-i\omega t}(e^{i\mathbf{kr}} + f(\mathbf{n}, \mathbf{n}_0) e^{ikr}/r), \tag{30}$$

where $f(\mathbf{n}, \mathbf{n}_0)$ is the elastic scattering amplitude, which depends on the directions of two unit vectors: along the direction $\mathbf{n}_0$ of incidence of the particles and along the direction $\mathbf{n}$ of scattering (or observation), these vectors having the components

$$\mathbf{n} = \{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\},$$

$$\mathbf{n}_0 = \{\sin\theta_0\cos\varphi_0, \sin\theta_0\sin\varphi_0, \cos\theta_0\}.$$

Then

$$\mathbf{kr} = kz' = kr(\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi_0 - \varphi)). \tag{31}$$

We shall seek the elastic scattering amplitude $f(\mathbf{n}(\theta\varphi), \mathbf{n}_0(\theta_0\varphi_0))$ in the form of an expansion in a series with respect to the spheroidal harmonics (5):

$$f(\mathbf{n}, \mathbf{n}_0) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_{lm}(\theta_0\varphi_0) Z_l^m(\theta, \varphi). \tag{32}$$

In order to find the coefficients q_{lm}, it is necessary to expand the incident plane wave with respect to the spheroidal harmonics (5):

$$e^{i\mathbf{kr}} = e^{ikz'} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{a_{lm}(\theta_0\varphi_0)}{r} R_{\omega lm}(r) Z_l^m(\theta, \varphi). \tag{33}$$

Using the integral representation for the product of a spheroidal function and a radial spheroidal function,[14]

$$S_l^m(p, \eta) X_{lm}^{(1)}(p, i\xi) \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} m\varphi = \frac{1}{4\pi i^l} \int e^{ikz'} \times S_l^m(p, \cos\theta_0) \left\{ \begin{matrix} \cos \\ \sin \end{matrix} \right\} m\varphi_0 \, d\Omega_0 \tag{34}$$

and the orthogonality of the spheroidal harmonics (6) for the coefficient in (33), we obtain the asymptotic expression[1)]

$$\lim_{r\to\infty} \frac{a_{lm}(\theta_0\varphi_0)}{r} R_{\omega lm}(r) = \frac{4\pi i^l}{r} \left[\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} \times \lim_{r\to\infty} S_l^m(p, \eta_0) X_{lm}^{(1)}(p, i\xi_0) e^{-im\varphi_0}$$

$$= \left\langle X_{lm}^{(1)}(p, i\xi_0) \approx \frac{1}{p\xi_0} \cos\left(p\xi_0 - \frac{l+1}{2}\pi\right), \right.$$

$$\eta_0 = \cos\theta_0 + O\left(\frac{1}{r^2}\right),$$

$$\left. \xi_0 = \frac{kr}{p} + O\left(\frac{1}{r}\right) \right\rangle = 4\pi i^l Z_l^{m*}(\theta_0\varphi_0) \frac{1}{kr} \sin\left(kr - \frac{\pi l}{2}\right).$$

Then

$$\Psi(t, r, \theta, \varphi) \sim e^{-i\omega t} \left\{ \sum_{lm} \frac{4\pi i^l}{kr} Z_l^{m*}(\theta_0\varphi_0) \sin\left(kr - \frac{\pi l}{2}\right) Z_l^m(\theta\varphi) + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_{lm}(\theta_0\varphi_0) Z_l^m(\theta\varphi) \frac{e^{ikr}}{r} \right\}. \tag{35}$$

Comparing (29) and (35), we find the coefficients

$$A_{lm}(\theta_0\varphi_0) = \frac{2\pi i^l}{k} Z_l^{m*}(\theta_0\varphi_0) \exp[i\delta_{lm}(\tau, k)],$$

$$q_{lm}(\theta_0\varphi_0) = \frac{2\pi}{ik} Z_l^{m*}(\theta_0\varphi_0) \{\exp[2i\delta_{lm}(\tau, k)] - 1\}.$$

Then the expansion of the total amplitude $f(\mathbf{n}, \mathbf{n}_0)$ has the form

$$f(\mathbf{n}, \mathbf{n}_0) = \frac{2\pi}{ik} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (S_{lm}(\tau, k) - 1) Z_l^{m*}(\theta_0\varphi_0) Z_l^m(\theta\varphi). \tag{36}$$

If $a = 0$, then, applying the theorem for the composition of spherical harmonics,

$$\sum_{m=-l}^{l} Y_l^{m*}(\theta_0\varphi_0) Y_l^m(\theta\varphi) = \frac{2l+1}{4\pi} P_l(\cos\alpha) \tag{37}$$

(α is the angle between the directions of incidence, $\mathbf{n}_0$, and observation, $\mathbf{n}$), we obtain the elastic scattering amplitude for the case of central symmetry in the field of the Reissner-Nordström black hole (see Ref. 3). The differential cross section for scattering of the particle into the direction $\mathbf{n}$ from the direction $\mathbf{n}_0$ is

$$|f(\mathbf{n}, \mathbf{n}_0)|^2 = \frac{4\pi^2}{k^2} \sum_{lm, l'm'} (S_{l'm'} - 1)^+ (S_{lm} - 1) \times Z_{l'}^{m'}(\theta_0\varphi_0) Z_{l'}^{m'*}(\theta\varphi) Z_l^{m*}(\theta_0\varphi_0) Z_l^m(\theta\varphi). \tag{38}$$

The differential cross section for elastic scattering of the particle into any direction from the direction $\mathbf{n}_0$ is determined by integration of (36):

$$\frac{d\sigma_e}{d\Omega_0}=\frac{4\pi^2}{k^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|S_{lm}-1|^2\,|Z_l^m(\theta_0\varphi_0)|^2. \tag{38a}$$

If there is no rotation, then, since $S_{lm}=S_l$, $Z_l^m=Y_l^m$, theorem (37) gives

$$\frac{d\sigma_e}{d\Omega_0}=\frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\,|S_l-1|^2, \tag{38b}$$

i.e., in this case the probability of elastic scattering into any direction does not depend on the direction of incidence.

Averaging over the rotations of the black hole, we obtain the cross section for elastic scattering of the particle into the direction $\mathbf{n}$ from any direction:

$$\frac{d\sigma_e}{d\Omega}=\frac{4\pi^2}{k^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|S_{lm}-1|^2\,|Z_l^m(\theta\varphi)|^2. \tag{38c}$$

Finally, the total cross section of elastic scattering into any direction from any direction of incidence is

$$\sigma_e=\frac{4\pi^2}{k^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}|S_{lm}-1|^2. \tag{38d}$$

We introduce the probability of inelastic scattering as the probability that a particle in the state $\Phi_{\omega lm}(t,r,\theta,\varphi)$ does not impinge on the infinitesimal area $r^2d\Omega$ from the direction of incidence $\mathbf{n}_0$:

$$\left(\frac{d\sigma_A}{d\Omega\,d\Omega_0}\right)_{lm} r^2\,d\Omega_0\,d\Omega=\frac{J_0}{J_1}\,d\Omega_0\,d\Omega, \tag{39}$$

where

$$J_0=-\frac{i}{2}g^{rr}(\theta_0)\left(\Phi_0^*\frac{\partial\Phi_0}{\partial r}-\Phi_0\frac{\partial\Phi_0^*}{\partial r}\right),$$

$$J_1=-\frac{i}{2}g^{rr}(\theta_0)\left(\Phi_1^*\frac{\partial\Phi_1}{\partial z'}-\Phi_1\frac{\partial\Phi_1^*}{\partial z'}\right); \tag{40}$$

$$\Phi_0=-\frac{f_{lm}(\tau,k)}{2ikr}[S_{lm}(\tau,k)e^{ikr}-e^{-ikr}]Z_l^m(\theta_0\varphi_0)e^{-i\omega t},$$

$$\Phi_1=e^{ikz'}e^{-i\omega t}, \tag{41}$$

J_0 is the flux of absorbed particles incident radially on the black hole from the direction $\mathbf{n}_0$, and J_1 is the flux of particles incident on the black hole from infinity from the direction $\mathbf{n}_0$.

Substituting (41) in (40) and then in (39), we obtain the cross section for inelastic scattering of the particle:

$$\frac{d\sigma_A}{d\Omega\,d\Omega_0}=\frac{1}{4k^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1-|S_{lm}|^2)\,|Z_l^m(\theta_0\varphi_0)|^2. \tag{42}$$

The total cross section for absorption of the particle from the direction $\mathbf{n}_0$ is

$$\frac{d\sigma_A}{d\Omega_0}=\frac{\pi}{k^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1-|S_{lm}|^2)\,|Z_l^m(\theta_0\varphi_0)|^2. \tag{42a}$$

Averaging over the rotations of the collapsed object, we obtain the total cross section for absorption of particles from any direction of incidence:

$$\sigma_A=\frac{\pi}{k^2}\sum_{lm}(1-|S_{lm}|^2)=\frac{\pi}{k^2}\sum_{lm}T_{lm}(\tau,k), \tag{42b}$$

where $T_{lm}(\tau,k)$ is the partial-wave absorption coefficient.

4. We consider the case of weak influence of the effects of rotation of the black hole on the picture of the scattering: $p=ka\ll 1$. In the first order of perturbation theory in p^2 we obtain for the spheroidal harmonics

$$Z_l^m(\theta,\varphi)=Y_l^m(\theta\varphi)+p^2[b^{l+2}Y_{l+2}^m(\theta\varphi)-b^{l-2}Y_{l-2}^m(\theta\varphi)], \tag{43}$$

where

$$b^{l+2}=\frac{1}{2(2l+3)}\left[\frac{((l+1)^2-m^2)((l+2)^2-m^2)}{(2l+1)(2l+3)^2(2l+5)}\right]^{1/2}$$

$$b^{l-2}=\frac{1}{2(2l-1)}\left[\frac{(l^2-m^2)((l-1)^2-m^2)}{(2l-3)(2l-1)^2(2l+1)}\right]^{1/2}. \tag{44}$$

With allowance for (43), the elastic scattering amplitude (36) takes the form

$$f(\mathbf{n},\mathbf{n}_0)=\frac{2\pi}{ik}\sum_{lm}(S_{lm}-1)\,Y_l^{m*}(\theta_0\varphi_0)\,Y_l^m(\theta\varphi)$$

$$-2\pi ika^2\sum_{lm}(S_{lm}-1)\,(b^{l+2}K_{l,l+2}(\mathbf{n}_0,\mathbf{n})-b^{l-2}K_{l,l-2}(\mathbf{n}_0,\mathbf{n})), \tag{45}$$

where

$$K_{r,q}(\mathbf{n}_0,\mathbf{n})=Y_r^{m*}(\theta_0\varphi_0)\,Y_q^m(\theta\varphi)+Y_q^{m*}(\theta_0\varphi_0)\,Y_r^m(\theta\varphi). \tag{46}$$

With allowance for (43), the differential cross section of inelastic scattering (42) takes the form

$$\frac{d\sigma_A}{d\Omega\,d\Omega_0}=\frac{1}{4k^2}\sum_{lm}(1-|S_{lm}|^2)\{|Y_l^m(\theta_0\varphi_0)|^2$$

$$+k^2a^2[b^{l+2}K_{l,l+2}(\mathbf{n}_0,\mathbf{n}_0)-b^{l-2}K_{l,l-2}(\mathbf{n}_0,\mathbf{n}_0)]\}. \tag{47}$$

The expressions (43), (45), and (47) show that the effects of rotation of the central body are quadratic in p for $p=ka\ll 1$ and can have an influence only in the case of an explicit dependence of the matrix elements S_{lm} on the rotation parameter a. In particular, in the long-wave approximation $p=\omega M(a/M)v\ll 1$, and therefore

$$\lambda^2=l(l+1),\quad Z_l^m(\theta\varphi)=Y_l^m(\theta\varphi).$$

Restricting ourselves also to the case of a weakly charged black hole, we can show that the matrix element S_{lm} has the explicit form

$$S_{lm}(\tau,k)=\frac{\Gamma(l+1-i\eta)}{\Gamma(l+1+i\eta)}\,\frac{1-y|\Gamma(l+1-i\eta)|^2e^{\pi\eta}}{1+y|\Gamma(l+1+i\eta)|^2e^{\pi\eta}}, \tag{48}$$

where

$$y=\Gamma_0\frac{l!^4 2^{2l}}{(2l)!^2(2l+1)!^2}[k(r_+-r_-)]^{2l+1}\prod_{q=1}^{l}\left(1+\frac{4\Gamma_0^2}{q^2}\right),$$

$$\Gamma_0=\mathscr{K}(r_+)/(r_+-r_-),\qquad \eta=[\omega M(1+v^2)+eQ]/v. \tag{49}$$

Knowing that the poles of the S matrix determine the energy spectrum of the particle for $\omega<\mu$, we find it in explicit form:

$$\omega/\mu=1-(\mu M+eQ)^2/2n^2-i\gamma_{nl}{}^m, \tag{50}$$

where

$$\gamma_{nl}^{m}=2\mu\omega r_+\left(1-\frac{a}{M}\frac{m}{2\omega r_+}+\frac{eQ}{2\omega M}\right)(\mu M)^{2l+1}(\mu M+eQ)^{2l+3}$$

$$\times\prod_{q=1}^{l}\left\{1-\left(\frac{a}{M}\right)^2+\frac{(2\omega r_+)^2}{q^2}\left(1-\frac{a}{M}\frac{m}{2\omega r_+}+\frac{eQ}{2\omega M}\right)^2\right\}$$

$$\times\frac{(n+l)!}{(n-l-1)!n^{4+2l}}\frac{(2l+1)^2}{(2l+1)!!^4}. \qquad (51)$$

It can be seen that (50) is the nonrelativistic hydrogenlike energy spectrum of the particle in the field of the rotating and charged black hole.[7] In this approximation, as (50) shows, a) the binding energy does not depend on the rotation of the central body and the projection of the total angular momentum onto the symmetry axis of the field, b) even in the case of a nonrotating black hole there is the degeneracy with respect to the quantum number l specific for nonrelativistic motion in fields with Coulomb potential. It can also be seen from (51) that the damping depends essentially on l. With increasing l, as with increasing radial quantum number n_r, the probability of capture decreases strongly. We emphasize that the expression (50) is valid for arbitrary rotation of the black hole ($a\leqslant M$), including the extremal case ($a\to M$). If the condition

$$ma/r_+>2\omega M+eQ$$

of superradiance is fulfilled, the damping of the corresponding level goes over into excitation in connection with the production and accumulation of particles in the given level. These processes are most intensive for the $2p$ states. In this case

$$\gamma_{2p}^{+1}=\frac{\mu}{48}\omega r_+\left(1-\frac{a}{M}\frac{1}{2\omega r_+}+\frac{eQ}{2\omega M}\right)(\mu M)^3(\mu M+eQ)^5\left\{1-\frac{4a}{M}\omega r_+\left(1+\frac{eQ}{2\omega M}\right)\right\}. \qquad (52)$$

In the long-wave approximation, the elastic scattering amplitude can be represented in the form

$$f(\mathbf{n},\mathbf{n}_0)=f_\eta{}^c(\mathbf{n},\mathbf{n}_0)-\frac{\pi}{ik}\sum_{lm}S_l{}^cT_{lm}Y_l{}^{m*}(\theta_0\varphi_0)Y_l{}^m(\theta\varphi), \quad (53)$$

where $f_\eta(\mathbf{n},\mathbf{n}_0)$ is the Coulomb elastic scattering amplitude,

$$f_\eta{}^c(\mathbf{n},\mathbf{n}_0)=\frac{\eta}{2k}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\frac{\exp[2i\eta\ln\sin(\alpha/2)]}{\sin^2(\alpha/2)}, \quad (54)$$

and T_{lm} is the partial-wave absorption coefficient:

$$T_{lm}(\tau,k)=2y|\Gamma(l+1-i\eta)|^2\cdot 2e^{\pi\eta}. \qquad (55)$$

If $(1+v^2)\omega M+eQ\neq 0$, then, restricting ourselves to s-wave absorption, we obtain the Rutherford distribution with respect to the angles with a correction,

$$\frac{d\sigma_e}{d\Omega\,d\Omega_0}=M^2\cdot\frac{(1+v^2+eQ/\omega M)^2}{4v^4\sin^4(\alpha/2)}$$

$$\times\left\{1+16\pi\frac{\omega^2Mr_+v}{1-\exp(-2\pi\eta)}\sin^2\frac{\alpha}{2}\sin\left[2\eta\ln\sin\frac{\alpha}{2}\right]\right\}. \qquad (56)$$

In this approximation, the absorption cross section (42a) is basically determined by

$$\frac{d\sigma_A}{d\Omega_0}=\frac{2Mr_+}{v^2}\left(1+\frac{eQ}{2\omega M}\right)\times 2\pi\omega M\frac{1+v^2+eQ/\omega M}{1-\exp\{-2\pi\omega M/v(1+v^2+eQ/\omega M)\}} \qquad (57)$$

i.e., in the long-wave approximation it does not depend on the orientation of the flux of particles with respect to the rotation axis of the black hole. The value of the cross section depends quadratically on the parameter a and is reduced by two times on the transition to the extremal Kerr-Newman configuration ($a\to M$, $Q\ll M$). It is interesting to note that for sufficiently large electric repulsion, when

$$|eQ|<2\omega M,$$

the absorption cross section changes sign in connection with the superradiant regime of scattering for the s waves.

We note finally that the expression (57) generalizes the results of Refs. 1–3 and the expressions (50) and (51) the results of Refs. 5, 7, and 8.

[1)]The intermediate relations in the angular brackets are taken from Ref. 12.

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